CHAPTER 7

Heat

Heat transport is an integral part of convection. Heat is transported in two principal ways in the mantle: by conduction and by advection. Advection means that heat is carried along with mass motion. Heat is also *generated* internally by radioactivity. Here we consider these processes in turn. A key feature of heat conduction is that there is a fundamental relationship between the time scale of conductive cooling (or heating) and the length scale over which the process is occurring. This is demonstrated in several ways and at different mathematical levels.

A key application of heat conduction theory is to the cooling oceanic lithosphere, and a key consequence is the subsidence of the sea floor with age. The lithosphere is a special case of a conductive thermal boundary layer, which is the source of convective motion (Chapter 8). The oceanic lithosphere and its subsidence play a central role in the discussions of Chapters 10 and 12 of what can be inferred about the form of mantle convection from observations. The role of the continents in the earth's thermal regime is considered separately, since continental lithosphere does not partake in subduction like oceanic lithosphere.

The advection of heat is a phenomenon that can be understood in quite simple terms. It is presented first in a simple way, and the idea is then used to derive a general equation that describes heat generation and transport, including both conduction and advection. Finally, thermal properties of materials are briefly considered, including their likely variations with pressure. This leads into the concept of adiabatic gradients of temperature and density.

7.1 Heat conduction and thermal diffusion

Let us start from Fourier's 'law' of heat conduction, that the rate of flow of heat is proportional to the temperature gradient:

$$q = -K\partial T/\partial x \tag{7.1.1}$$

where T is temperature, q is the rate of flow of heat per unit area (that is, the heat flux) in the positive x direction, and K is the conductivity of the material. The negative sign ensures that heat flows from hotter to cooler regions.

We need to be able to consider situations in which the temperature varies with time and in which heat is generated by radioactivity within the rocks. To do this, let us consider the thermal energy budget of the small block of material sketched in Figure 7.1. Suppose that the temperature T depends only on time and the x-coordinate. The change in heat content of the block during a time interval will be equal to the heat conducted in minus the heat conducted out plus the heat generated internally. Suppose the temperature changes by an amount $\mathrm{d}T$ within a time interval $\mathrm{d}t$. Then the change in heat content H is

$$\mathrm{d}H = \rho S \mathrm{d}x \cdot C_P \cdot \mathrm{d}T$$

where ρ is the density, S is the area of the end surfaces of the block (so that $\rho S\,\mathrm{d}x$ is the mass of the block), and C_P is the specific heat at constant pressure of the material. The specific heat measures the capacity of a material to hold heat, and for mantle minerals it has a value of the order of 1000 J/kg °C. (The subscript P is used because this is the specific heat at constant pressure. In other words it is the change in heat content, per unit mass per degree, with the pressure held constant so that thermal expansion is allowed to happen. It is possible to define the specific heat at constant volume, C_V, but we will not have any use for this here. Since the two quantities have significantly different values, it is usual to distinguish them.)

Again taking positive heat flow to be in the positive x direction, the heat added by conduction through the left side of the box during the interval $\mathrm{d}t$ is $qS\,\mathrm{d}t$, and the heat lost by conduction through the right side is $(q + \mathrm{d}q)S\,\mathrm{d}t$. If A is the rate of radioactive heat generation per unit volume, the heat generated during $\mathrm{d}t$ is $A \cdot S\mathrm{d}x \cdot \mathrm{d}t$. The total heat budget for the time interval $\mathrm{d}t$ is then

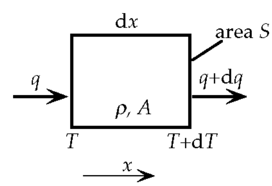

Figure 7.1. Heat budget of a small block of material with heat transported by thermal conduction and with internal heat generation (A).

$$\rho S \mathrm{d}x \cdot C_P \cdot \mathrm{d}T = qS\,\mathrm{d}t - (q + \mathrm{d}q)S\,\mathrm{d}t + A \cdot S\mathrm{d}x \cdot \mathrm{d}t$$

which yields, upon dividing by $S\,\mathrm{d}x\,\mathrm{d}t$ and taking limits,

$$\rho C_P \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} + A \tag{7.1.2}$$

If K is uniform (that is, independent of x), this can be written, using Equation (7.1.1),

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} + a \tag{7.1.3}$$

where

$$\kappa = K/\rho C_P$$
$$a = A/\rho C_P$$

κ is called the *thermal diffusivity* and a is the rate of increase of temperature due to the radioactive heating. Notice that the dimensions of κ are length^2/time. With $K = 3\,\mathrm{W/m\,^\circ C}$, $\rho = 3000\mathrm{kg/m^3}$ and $C_P = 1000\,\mathrm{J/kg\,^\circ C}$, a typical value of κ for rocks is $10^{-6}\ \mathrm{m^2/s}$.

Equation (7.1.3) governs conductive heat flow in situations where the material is not moving, so that conduction is the only method of heat transport. We will use it below to consider the thermal structure of the lithosphere and of thermal boundary layers more generally. It is an example of a diffusion equation. A similar equation governs the diffusion of chemical species through solids, for example, though without the generation term a. This is why κ is called a diffusivity. As we will see, heat conduction causes temperature differences to spread out and become more uniform, in other words to diffuse. The term *thermal diffusion* is often used interchangeably with heat conduction: they mean the same thing, but in cases where the temperature is changing with time it is useful to emphasise the diffusive nature of the process by using the term thermal diffusion.

7.2 Thermal diffusion time scales

It is a very general feature of thermal diffusion (and other forms of diffusion) that the time it takes for a body to heat up or cool down is related to its size in a particular way. This general property of thermal diffusion is a key to building a simple understanding of thermal convection. It also governs a key part of mantle convection in a particularly simple way. In this section we look at this aspect of thermal diffusion in different ways, in order to provide a clear understanding of it.

7.2.1 Crude estimate of cooling time

Suppose a layer of magma with some high temperature T intrudes between sedimentary rock layers (forming a sill), as illustrated in Figure 7.2. Can we estimate how long it will take to cool? Would it, for example, take hours, or weeks, or centuries? Often it is possible to get some idea of an answer by making very crude approximations. (We have already seen examples of useful rough approximations in Sections 6.8.1 and 6.9.1.)

Suppose the sill thickness is D. At first ($t = 0$) there will be a very steep temperature gradient at the top and bottom of the sill (Figure 7.2), but after some time, t, you might expect the temperature profile to have smoothed out, as sketched. This will be justified more rigorously below. At this stage, a typical temperature difference is T, and a typical length scale over which the temperature varies by this much might be about half the thickness of the sill, $D/2$. Let us try approximating the differentials in Equation (7.1.3) with these large differences (assuming there is no heat generation, so that $a = 0$):

$$\frac{T}{t} = \kappa \frac{T}{(D/2)^2}$$

which yields $t = D^2/4\kappa$. Notice that this is independent of T.

What does this time t mean? According to Equation (7.1.3), T/t is a rough measure of the rate of change of T, so t should be a rough measure of the time it takes for the temperature to change by a significant fraction of T. Suppose D is 10 m and κ is 10^{-6} m^2/s. Then $t \approx 2.5 \times 10^7$ s, which is about 9 months. If this seems to be a surprisingly long time, it illustrates that rocks are not very good conductors of heat. Of course this may only be an approximate result, but it suggests that the cooling time for a 10 m sill is months rather than hours or centuries.

Notice now that the cooling time depends on the square of the thickness D. Thus if D is only 1 m, then $t \approx 3$ days, and if D is 10 cm then $t \approx 40$ minutes. This behaviour is quite characteristic

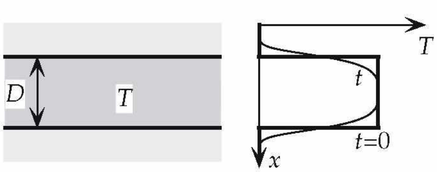

Figure 7.2. Cooling magma layer (or sill).

of diffusion processes: the time scale depends on the square of the length scale, with the diffusivity being the constant of proportionality. In fact this is built into the dimensions of the thermal diffusivity:

$$\kappa \approx \frac{D^2}{t} \tag{7.2.1}$$

This is the fundamental point to understand about time-dependent heat conduction processes. With this simple formula, we can make rough estimates of such things as the thickness of the oceanic lithosphere and the rate at which mantle convection should go. The latter will be done in Chapter 8. You will see in following sections that Equation (7.2.1) always emerges from a more rigorous analysis, with a proportionality constant of the order of 1.

7.2.2 Spatially periodic temperature [*Intermediate*]

In order to keep the mathematics from being unnecessarily complicated, let us approximate the initial temperature variation with depth (x) in Figure 7.2 as

$$T(x, 0) = T_0 \cos px \tag{7.2.2}$$

where $p = 2\pi/(2D)$ is a wavenumber, corresponding to a wavelength of $2D$. You can, if you want, regard this as the first Fourier component of the initial square temperature variation of Figure 7.2. Although this is still a crude approximation to the actual initial temperature distribution, it allows a rigorous solution of Equation (7.1.3) to be derived.

The evolution of the temperature is governed by Equation (7.1.3) with $a = 0$. If the geometry and initial conditions are appropriate, a solution to a partial differential equation such as this can often be found by assuming the solution to be a product of a function of depth, $\chi(x)$, and a function of time, $\Theta(t)$:

$$T(x, t) = \chi(x)\Theta(t) \tag{7.2.3}$$

(This method is called 'separation of variables'.) Substitution of this into Equation (7.1.3) and rearrangement leads to

$$\frac{1}{\Theta}\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \kappa\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2} \equiv -\frac{1}{\tau} \tag{7.2.4}$$

where τ is a constant with dimension time. The first and second parts of this equation must each be equal to a constant because t and x can be varied independently, so the only way the two expressions can remain equal is if they are each equal to the same constant, which I have written with malice aforethought as $-1/\tau$. This equation is now in the form of two ordinary differential equations, each of which can be readily solved. Thus the first and third terms of Equation (7.2.4) can be equated and integrated to yield

$$\Theta = \Theta_0 e^{-t/\tau} \tag{7.2.5}$$

while the second and third terms can be rearranged as

$$\frac{d^2\chi}{dx^2} + \frac{\chi}{\kappa\tau} = 0$$

which has a general solution of the form

$$\chi(x) = a\cos\frac{x}{\sqrt{\kappa\tau}} + b\sin\frac{x}{\sqrt{\kappa\tau}} \tag{7.2.6}$$

We want Equations (7.2.5) and (7.2.6) to combine in Equation (7.2.3) with the constants evaluated so that the solution matches the initial condition, (7.2.2). This requires $b = 0, a\Theta_0 = T_0$ and $p = 1/\sqrt{\kappa\tau}$. The solution is then

$$T(x, t) = T_0 \exp(-t/\tau) \cos px \tag{7.2.7}$$

with

$$\tau = \frac{1}{p^2\kappa} = \frac{D^2}{\pi^2\kappa} \tag{7.2.8}$$

Compare this with the crude estimate in the last section, which yielded a time scale of $D^2/4\kappa$. They differ only by a factor of about 2.5. You can see again the dependence of time scale on the square of the length scale embodied in the dimensionality of κ (Equation (7.2.1)). With this formula, the cooling time of a 10 m sill can be estimated as 4 months.

7.2.3 Why is cooling time proportional to the square of the length scale?

A simple illustration can clarify why there is this general relationship between time scale and length scale in thermal diffusion processes. Compare the two sinusoidal temperature distributions in Figure 7.3, with similar amplitudes, and wavelengths of λ and 2λ. Heat will flow from the hotter part to the colder part. The heat flux at the point where $T = 0$ is only half as great in case (*b*) as in case (*a*), because the temperature gradient is less: the same temperature difference is spread over twice the distance. Thus the rate at which the hot part loses heat to the cool part is only half as great in case (*b*): it is proportional to $1/2\lambda$.

There is another factor to be considered. So far we have accounted for time scale being proportional only to the first power of λ. We must also take account of the fact that in case (*b*) there is twice as much heat to be moved as in case (*a*), because the volume of the hot region is twice as great. Twice as much heat flowing at half the rate will take four times as long. Thus we can conclude that the time scale for a significant reduction in the temperature differences is proportional to λ^2.

7.3 Heat loss through the sea floor

At a midocean rise crest, or spreading centre, two tectonic plates pull apart. It is observed that the zone of rifting is quite narrow, only a few tens of kilometres in width. Beyond the rift zone, each plate is a rigid unit moving away from the spreading centre. If the plates are separating, then of course there must be a replenishing flow of material ascending from below. I will argue in Chapter 12 that at normal midocean rises the upwelling is passive, being simply the flow of mantle material drawn in to replace the material moving away with the plates. In cross-section then, the situation must be like that sketched in Figure 7.4a.

Hot material, at temperature T_{m}, rises close to the surface at the spreading centre. Some of it melts, and the magma rises to the top to form the oceanic crust, but this can be ignored for the

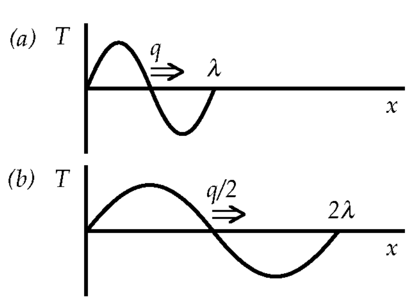

Figure 7.3. Effect of length scale on cooling time.

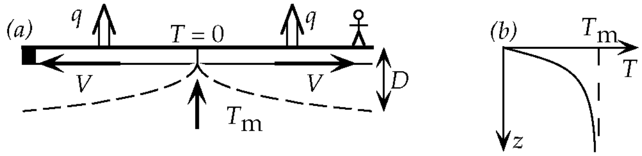

Figure 7.4. Cooling oceanic lithosphere.

moment. As the rising hot material approaches the cool surface, it will begin to cool by conduction, because the surface is maintained at a temperature close to 0 °C. If you imagine standing on one side of the spreading centre (stick figure, Figure 7.4a), you will be carried away with the plate at velocity V. The material under you will continue simply to cool by conduction. Once you are some distance from the spreading centre, its presence and your motion relative to it might be ignored, in which case the only process happening would be the conduction of heat *vertically* to the surface. We will justify this assumption in retrospect. After a time, the temperature profile with depth might look like that sketched in Figure 7.4b. As discussed in Chapter 6, the cold material will behave like an elastic/brittle material, and in this context it will not flow like the deeper (solid) material. In other words, the cooled upper part will behave like a rigid plate, consistent with what is observed.

7.3.1 Rough estimate of heat flux

We can use the results of Section 7.2 to get some idea of the thickness, D, of the cooled zone. From Equation (7.2.1), it should be approximately

$$D = \sqrt{\kappa t}$$

where t is the time for which the cooling has been proceeding. The older parts of the sea floor are over 100 Ma old. Setting t = 100 Ma (1 year $\approx 3.16 \times 10^7$ s) and using $\kappa = 10^{-6}\,\mathrm{m^2/s}$ gives D = 56 km. This says that D should be roughly a few tens to 100 km.

The larger plates are typically 5000–10 000 km across, which is consistent with velocities of 50–100 mm/a (50–100 km/Ma) sustained for about 100 Ma. Thus plate widths are much greater than plate thicknesses. This justifies our assumption that heat conduction is mainly vertical, except within about 100 km of the spreading centre. Putting it another way, the vertical temperature gradients will be much greater than horizontal gradients, except near spreading centres. The plate material will be close to the spreading centre for only about 1 Ma.

We can estimate the order of magnitude of the heat flux through the sea floor to be expected from this conductive cooling process. From Equation (7.1.1) it is $q \sim K\Delta T/D \sim 70\,\mathrm{mW/m^2}$ using $K = 3\,\mathrm{W/m\,°C}$, $\Delta T = 1300\,°\mathrm{C}$ and $D = 56\,\mathrm{km}$. This compares with the average heat flux through the sea floor of about 100 $\mathrm{mW/m^2}$, decreasing to about 50 $\mathrm{mW/m^2}$ through the oldest sea floor.

7.3.2 The cooling halfspace model [*Intermediate*]

The assumptions used above are that the temperature profile at a given location moving with a plate is determined only by heat conduction in the vertical direction and that the conductivity is spatially uniform. This amounts to assuming that the mantle is a uniform infinite halfspace (that half of an infinite space below $z = 0$). With the initial condition that $T(z, 0) = T_\mathrm{m}$, the boundary condition that $T(0, t) = 0$ and the assumption that radioactive heating can be neglected (so $a = 0$) Equation (7.1.3) has the solution

$$T(z, t) = T_\mathrm{m}\mathrm{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right) \qquad (7.3.1)$$

where erf stands for the *error function*:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-\beta^2}\mathrm{d}\beta \qquad (7.3.2)$$

The derivation of this result is outlined in the next section. The error function looks like the temperature profile sketched in Figure 7.4b. It has the value 0.843 at $x = 1$.

The temperature in this solution depends on depth and time only through the combination $[z/2\sqrt{(\kappa t)}]$. Thus, for example, the temperature reaches 84% of its maximum value when $z/2\sqrt{(\kappa t)} = 1$. In other words, the depth, D, to the isotherm $T = 0.84T_\mathrm{m}$ is

$$D = 2\sqrt{\kappa t} \qquad (7.3.3)$$

This is just twice the value resulting from the rough estimate of the last section. Using $\kappa = 10^{-6}\,\mathrm{m^2/s}$, the depth to this isotherm is thus 112 km at 100 Ma. More generally, you can see that the proportionality between length^2 and time has emerged again in this solution. It implies here that the thickness of the lithosphere should be proportional to the square root of its age. Thus D should be 56 km at 25 Ma and 11.2 km at 1 Ma. An impression of this thickening with age is included in Figure 7.4a as the dashed curves.

We can calculate the heat flux through the sea floor from this solution. For this, we need the result

$$\frac{\mathrm{d\,erf}(x)}{\mathrm{d}x} = \mathrm{erf}'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$$

This follows from the fact that erf(x) depends on x only through the upper limit of the integral in Equation (7.3.2), and can be derived using basic calculus methods for differentiating integrals with variable limits. If we identify x with $z/2\sqrt{(\kappa t)}$, we can use the chain rule of differentiation:

$$\frac{\partial T}{\partial z} = T_{\mathrm{m}} \frac{\mathrm{d\,erf}(x)}{\mathrm{d}x} \frac{\partial x}{\partial z} = T_{\mathrm{m}} \frac{2}{\sqrt{\pi}} e^{-x^2} \frac{1}{2\sqrt{\kappa t}}$$

Then the heat flux at $z = 0$ is

$$q_0 = -K \frac{\partial T}{\partial z}\bigg|_{z=0} = -\frac{K T_{\mathrm{m}}}{\sqrt{\pi \kappa t}} \qquad (7.3.4)$$

Thus the heat flux declines with time in proportion to $t^{-1/2}$. The minus in Equation (7.3.4) is because the heat flux is upwards, which is the negative z direction.

We saw in Chapter 4 (Figure 4.7B) that the observations of heat flow through the sea floor follow this behaviour to within the errors of measurement. The values used above yield a heat flux of 39 mW/m^2 for 100 Ma-old sea floor, compared with observed values of 40–50 mW/m^2. This very simple model, which approximates the earth below the sea floor as a uniform halfspace, thus gives a remarkably good description of the observed heat flux through the sea floor.

The physics we have considered here is the same as was considered last century by Lord Kelvin in making his estimate of the age of the earth (Chapter 2). His assumptions were that the earth had started hot and that it had been cooling by conduction to the surface ever since. He asked how long it would take for the near-surface temperature gradient (or the surface heat flux) to fall to the presently observed values. This is explored further in Exercise 4.

7.3.3 The error function solution [*Advanced*]

Since it is a central result in our understanding of oceanic lithosphere, and through that of mantle convection, I will outline the derivation of the error function solution. Another account is given, for example, by Officer [1]. A general form of solution to Equation (7.1.3) (with $a = 0$) is

$$T(z, t; \gamma) = \exp(-\kappa\gamma^2 t)[B(\gamma)\cos\gamma z + C(\gamma)\sin\gamma z] \qquad (7.3.5)$$

where the notation shows explicitly the dependence of T on the wavenumber γ. This form is just a more general version of Equation (7.2.7), and it can be derived in the same way. The coefficients B and C are also assumed to depend on γ because we can use this form to Fourier synthesise the total solution. This can be done by first Fourier analysing the initial condition $T(z > 0, 0) = T_{\mathrm{m}}$, $T(0, 0) = 0$; then the time dependence of each Fourier component will have the above form. Thus the forms of B and C can be derived from the Fourier integrals of the initial condition:

$$B(\gamma) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} T(\zeta, 0)\cos\gamma\zeta\,\mathrm{d}\zeta$$
$$C(\gamma) \equiv \frac{1}{\pi}\int_{-\infty}^{\infty} T(\zeta, 0)\sin\gamma\zeta\,\mathrm{d}\zeta$$

The top boundary condition can be matched by assuming that the solution is antisymmetric about $z = 0 : T(z < 0, 0) = -T(z > 0, 0) = -T_{\mathrm{m}}$. Substitution into these integrals then yields

$$B(\gamma) = 0$$
$$C(\gamma) = \frac{2}{\pi}T_{\mathrm{m}}\int_{0}^{\infty} \sin\gamma\zeta\,\mathrm{d}\zeta$$

The Fourier synthesised solution is then of the form

$$\begin{aligned} T(z, t) &= \int_0^\infty T(z, t; \gamma)\mathrm{d}\gamma \\ &= \int_0^\infty \exp(-\kappa\gamma^2 t)C(\gamma)\sin\gamma z\,\mathrm{d}\gamma \\ &= \frac{2T_{\mathrm{m}}}{\pi}\int_0^\infty\int_0^\infty \exp(-\kappa\gamma^2 t)\sin\gamma\zeta\sin\gamma z\,\mathrm{d}\gamma\,\mathrm{d}\zeta \end{aligned}$$

where the expression for C has been substituted and the order of integration reversed in the third line.

The following two results allow this to be rewritten:

$$\sin A \sin B = [\cos(A - B) - \cos(A + B)]/2$$

$$\int_0^\infty \exp(-\kappa\gamma^2 t)\cos\gamma(\zeta - z)\mathrm{d}\gamma = \frac{1}{2}\sqrt{\frac{\pi}{\kappa t}}\exp[-(\zeta - z)^2/4\kappa t]$$

Then

$$T(z, t) = \frac{T_\mathrm{m}}{2\pi}\sqrt{\frac{\pi}{\kappa t}}\int_0^\infty \left[\mathrm{e}^{-(\zeta-z)^2/4\kappa t} - \mathrm{e}^{-(\zeta+z)^2/4\kappa t}\right]\mathrm{d}\zeta$$

Transform the internal variables in each of these integrals to the following:

$$\beta = \eta(\zeta - z), \qquad \beta' = \eta(\zeta + z)$$

where $\eta = 1/2\sqrt{(\kappa t)}$. Then the integrals have the same form, but over different ranges of β and β', so they can be combined to yield

$$\begin{aligned} T(z, t) &= \frac{T_\mathrm{m}}{\sqrt{\pi}}\int_{-\eta z}^{\eta z} \mathrm{e}^{-\beta^2}\mathrm{d}\beta \\ &= \frac{2T_\mathrm{m}}{\sqrt{\pi}}\int_0^{\eta z} \mathrm{e}^{-\beta^2}\mathrm{d}\beta \\ &= T_\mathrm{m}\operatorname{erf}(\eta z) \end{aligned}$$

which is the solution defined by Equations (7.3.1) and (7.3.2).

7.4 Seafloor subsidence and midocean rises

If the lithosphere cools, it will undergo thermal contraction. As a result, the surface (the sea floor) will subside, and the amount of subsidence can be roughly estimated as follows. If the temperature rises from 0 °C to about 1400 °C through the thickness of the lithosphere, then the average temperature of the lithosphere is about 700 °C. This means that the average temperature *deficit* of the lithosphere relative to the underlying mantle is ΔT = 1400 °C − 700 °C = 700 °C. This will cause the density to increase by the fraction $\Delta\rho/\rho = \alpha\Delta T$, where α is the coefficient of thermal expansion. If the lithosphere thickness is D, then a vertical column of rock of height D through the lithosphere will shorten by this fraction. In other words, the shortening, h, of the top of the column will be given by

$$h/D = \alpha \Delta T$$

This shortening h is not the actual amount by which the surface subsides, since the rock that has subsided away is replaced by water. We have to consider the isostatic balance of the column relative to a similar column at the midocean rise crest. These are illustrated in Figure 7.5. The mass per unit area in the column at the rise crest is $(d + D - h)\rho_{\mathrm{m}}$, while the mass per unit area in the other column is $[d\rho_{\mathrm{w}} + (D - h)\rho_{\mathrm{l}}]$. Equating these, and neglecting second-order terms yields

$$d = h\rho_{\mathrm{m}}/(\rho_{\mathrm{m}} - \rho_{\mathrm{w}}) \tag{7.4.1}$$

Old lithosphere (say 100 Ma) is about 100 km thick and $\alpha \approx 3 \times 10^{-5}/°\mathrm{C}$. Then $h \approx 2.1$ km. Using $\rho_{\mathrm{m}} = 3.3\,\mathrm{Mg/m^3}$ and $\rho_{\mathrm{w}} = 1.0\,\mathrm{Mg/m^3}$, this implies $d \approx 3.0$ km. Old sea floor is indeed observed to be about 3 km deeper than midocean rise crests (Chapter 4, Figures 4.5, 4.6). This result suggests that the greater depth of the old sea floor relative to midocean rise crests may be explained simply by the thermal contraction of the lithosphere. In other words, the existence of the midocean rise topography may be explained by this cooling process.

We can test this idea more rigorously by using the solution to the thermal halfspace model obtained above. Each layer of thickness $\mathrm{d}z$ at depth z will have a temperature deficit of $\Delta T(z, t) = T_{\mathrm{m}} - T(z, t)$. Then the total thermal contraction will be

$$h(t) = \int_0^\infty \alpha \Delta T(z, t)\mathrm{d}z$$

Using the result that

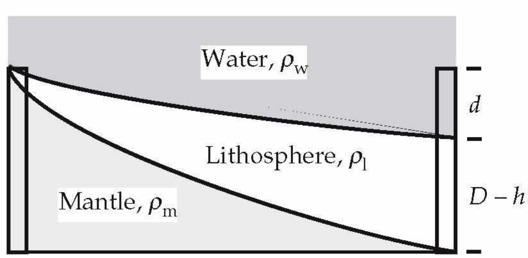

Figure 7.5. Seafloor subsidence by thermal contraction, with isostasy.

$$\int_0^\infty [1 - \mathrm{erf}(x)]\,\mathrm{d}x = \frac{1}{\sqrt{\pi}}$$

and making the appropriate variable transformation in the integral expression for h, this combines with Equation (7.4.1) to yield

$$d = \frac{\rho_\mathrm{m}\alpha T_\mathrm{m}}{(\rho_\mathrm{m} - \rho_\mathrm{w})} 2\sqrt{\frac{\kappa t}{\pi}} \tag{7.4.2}$$

Using the same values as previously, this gives $d = 3.8$ km at $t = 100$ Ma.

Equation (7.4.2) predicts that the seafloor depth should increase in proportion to the square root of its age. We saw in Chapter 4 (Figure 4.6) that the sea floor does follow this behaviour to first order, particularly for ages less than about 50 Ma. At greater ages, many parts of the sea floor are shallower than this by up to about 1 km. The possible reasons for such deviations will be discussed in Chapter 12.

Here I want to emphasise the further success of the cooling halfspace model in accounting for most of the variations of both the heat flux through the sea floor and the depth of the sea floor with age. It is a remarkably simple and powerful result, which suggests that major features of the earth's surface can be explained by a simple process of near-surface thermal conduction. It also suggests that we can usefully think of the midocean rises as standing high by default, because the surrounding sea floor has subsided, rather than the rises having been actively uplifted.

The success of the cooling halfspace model in accounting for seafloor subsidence and heat flux suggests some very important things about the earth and about mantle convection. The midocean rise system is the second-largest topographic feature of the earth after the continents, and we have seen here that it can be accounted for by a simple near-surface process: conductive heat loss to the earth's surface. *Its explanation does not require any process operating deeper than about 100 km*, and in particular it does not require a buoyant convective upwelling under midocean rises, as has often been supposed (see Chapter 3). If the midocean rise topography is not an expression of deep convection, as I am suggesting here, this leads to the question of why there is not some more obvious expression in the earth's topography of deep convection. These questions will be taken up in later Chapters 8, 10, 11 and 12, where they will lead to some important conclusions about the form of mantle convection.

Table 7.1. *Heat-producing isotopes [2].*

Element	Isotope	Half life (Ga)	Power (μW/kg of element)
Uranium	^{238}U	4.468	94.35
	^{235}U	0.7038	4.05
Thorium	^{232}Th	14.01	26.6
Potassium	^{40}K	1.250	0.0035

7.5 Radioactive heating

Radioactivity generates heat, and radioactive heat generated in the earth sustains the earth's thermal regime, as we will see in Chapter 14. There are two aspects that I want to cover here: its effect in modifying continental geotherms and its contribution to the heat budget of the mantle.

The isotopes that make the main heat contributions are ^{40}K, ^{238}U, ^{235}U and ^{232}Th. Each of these has a half life of the order of 1–10 Ga. (If they had shorter half lives, they would not still be present in significant quantities.) Their half lives and current rates of heat production are given in Table 7.1.

Geochemists find that these elements occur in similar proportions relative to each other in the crust and mantle, although their absolute concentrations differ greatly. Thus the mass ratio Th/U is usually 3.5–4 and the ratio K/U is usually $1–2 \times 10^4$. It is sufficient for our purposes to assume the particular values Th/U = 4 g/g and K/U = 10^4 g/g. (The unit g/g may seem to be redundant, but it serves to specify that this is a ratio by weight, rather than by mole or by volume, for example.) With these ratios, the total power production due to all of these isotopes, expressed per kg of uranium in the rock, is 190 μW/(kg of U). Then representative values of the concentration of uranium in different rocks allow us to estimate the total rate of heat production. Such estimates are given in Table 7.2.

These are only representative values, and there is considerable variation, especially in the continental crust. These values probably tend to be on the high side of the distribution. For the upper mantle, Jochum and others [3] have estimated on the basis of measurements of representative rocks that the likely value of the heat production rate is 0.6 pW/kg, with the value unlikely to be as great as 1.5 pW/kg.

You will see in the next section that heat production in the upper continental crust is sufficient to account for about half of typical continental heat fluxes. For example, a heat production rate

Table 7.2. *Radiogenic heat production rates, assuming Th/U = 4 kg/kg, K/U = 10^4 kg/kg. U concentrations from [4, 5].*

Region	Concentration of U	Power (pW/kg)	Density (Mg/m^3)	Power (nW/m^3)
Upper continental crust	5 μg/g	1000	2.6	2600
Oceanic crust	50 ng/g	10	2.9	30
Upper mantle	5 ng/g	1	3.3	3
Chondritic meterorites[a]	20 ng/g	4–6	3.3	12–18

[a]With K/U = 2–6 × 10^4 kg/kg.

of 2.5 μW/m^3 through a depth of 10 km will produce a surface heat flux of 25 mW/m^2, compared with typical continental values of 60–100 mW/m^2. On the other hand, the oceanic crust produces very little heat (30 nW/m^3 through 7 km gives 0.2 mW/m^2).

There is a puzzle about the amount of heat production in the mantle. It is not clear that there is enough radioactivity to account for the heat being lost at the earth's surface. Heat production is small in the upper mantle: 3 nW/m^3 through a depth of 650 km yields 2 mW/m^2. If this heat production rate applied through the whole 3000 km depth of the mantle, the surface heat flux would still be only about 10 mW/m^2. To account for the observed average oceanic heat flux of 100 mW/m^2 requires heat production in the mantle to be closer to that of oceanic crust. Some of the deficit can be accounted for by the slow cooling of the earth's interior, as will be shown in Chapter 14, and some may be explained by a greater heat production in the deeper mantle, either because the deep mantle composition is more 'primitive' (that is, closer to that of chondritic meteorites) or because there is an accumulation of subducted oceanic crust at depth, or both (Chapters 13, 14). Another contribution comes from mantle plumes, but these seem to account for less than 10% of the total (Chapter 11). In any case, there is a significant discrepancy here that has not been entirely accounted for. It is believed that none of the principal heat-producing elements would dissolve in the core in significant quantities, which implies that the discrepancy cannot be made up there. The question is addressed again in Chapters 12 and 14.

7.6 Continents

In the theory of plate tectonics, the continents are part of the lithospheric plates, carried passively as the plates move. Although the assumption that plates are non-deforming is not as good in continental areas as in oceanic areas, it is nevertheless sufficiently true to be a useful approximation. If the continents are part of the lithosphere, then heat transport within them must be by conduction rather than by convection (assuming that heat transport by percolating liquids, such as water or magma, is not important in most places most of the time). This was assumed in Section 7.3 for the oceanic lithosphere. There is, however, an important difference between the continental lithosphere and the oceanic lithosphere, and this is that the continental lithosphere is much older, since we know that most continental crust is much older. Whereas we treated the oceanic lithosphere as a transient (time-dependent) cooling problem, the continental thermal regime is more likely to be near a steady state, as we will now see.

The more stable parts of the continents, the cratons and shields, have not had major tectonic activity for periods ranging from a few hundred million years up to a few billion years. Their heat flux tends to be lower, 40–50 mW/m^2, than younger parts of the continents (Figure 4.8) It is often assumed that they are in thermal steady state, that is the heat input (from below and from radioactivity) balances the heat loss through the surface. Is this reasonable? We saw in Section 7.2 that the time scale for cooling oceanic lithosphere to a depth of 100 km is about 100 Ma. The time scale to cool or equilibrate to a depth of 200 km would then be about 400 Ma. It would thus seem to be reasonable to assume that at least the Archean shields, and perhaps the Proterozoic cratons, had approached equilibrium. I do not want to belabour this point either way. It is instructive to assume that the older continental geotherms are roughly in steady state, but on the other hand most continental areas have had some tectonic activity within the last billion years or so, and little is known about whether the lithosphere might have had a constant thickness during such periods.

The typical heat flux out of continents of about 60 mW/m^2 is due partly to heat generated within the continental crust and partly to heat conducting from the mantle below. The relative proportions of these contributions are not known very accurately, but they seem to be roughly comparable. The heat-producing elements tend to be concentrated in the upper crust, and a common and useful assumption is that their concentration decreases exponentially with depth, with a depth scale of about $h \approx 10$ km. At the surface the heat production rate is of the order of 1 nW/kg, so that the heat production rate per unit volume is $A_0 \approx 2.5\ \mu\text{W/m}^3$. The crust is very heterogeneous, so you should understand that these are merely representative numbers. Thus we might assume that the heat production rate as a function of depth is

$$A = A_0 e^{-z/h} \tag{7.6.1}$$

Let us assume that a piece of continental crust is in thermal steady state, which implies that the lithosphere thickness, the heat production rate and the distribution of heat production with depth have all been constant for a sufficiently long time. Let us also assume that heat is transported only by conduction, which excludes transport by percolating water or magma and transport by deformation of the crust and lithosphere. By steady state I mean simply that the temperature at a given depth is not changing, so that $\partial T/\partial t = 0$. Then Equation (7.1.3), which governs the evolution of temperature by conduction, is

$$\frac{\partial^2 T}{\partial z^2} = -\frac{A}{K} \tag{7.6.2}$$

where K is the conductivity.

We know that the temperature at the earth's surface is about 0 °C and its gradient is constrained by the surface heat flux. From Equation (7.1.1), the surface gradient $T_0' = -q_0/K \approx 20\,°\text{C/km}$, taking $q_0 \approx -60\,\text{mW/m}^2$ and $K = 3\,\text{W/m}\,°\text{C}$. Equation (7.6.2) is a differential equation and these are the boundary conditions:

$$T_0 = 0\,°\text{C}, \qquad T_0' = 20\,°\text{C/km} \tag{7.6.3}$$

Suppose, for the moment, there were no radioactive heat production in the crust: $A = 0$. Then the solution to Equation (7.6.2) with these boundary conditions is

$$T = T_0 + T_0' z \tag{7.6.4}$$

With the values of Equations (7.6.3), the temperature at the base of the crust, about 40 km deep, would be 800 °C, at which temperature the crust would be likely to be melting, depending on its composition. The temperature at 60 km depth would be 1200 °C, at which depth the mantle would almost certainly be melting. Since seismology tells us that the mantle is largely solid, this suggests that at least one of our assumptions becomes invalid at some depth of the order of 60 km.

Now let us return to the assumption that there is radioactive heat generation, and that its variation with depth is given by Equation (7.6.1). Then the solution to Equation (7.6.2) with the same boundary conditions is

$$T = T_0 + \frac{A_0 h^2}{K}\left(1 - e^{-z/h}\right) + \left(T_0' - \frac{A_0 h}{K}\right)z$$
$$= T_0 + T_h\left(1 - e^{-z/h}\right) + T_m' z \tag{7.6.5}$$

The solutions (7.6.4) and (7.6.5) are sketched in Figure 7.6. Equation (7.6.5) approaches an asymptote at depth, line (a), given by

$$T = (T_0 + T_h) + T_m' z \tag{7.6.6}$$

At 40 km depth the term $e^{-z/h}$ is already as small as 0.018, so for greater depths the asymptote is a good approximation. Using this, it is easy to calculate that the temperature at 40 km depth is 550 °C, compared with 800 °C from Equation (7.6.4) without radioactive heating. The depth at which 1200 °C is reached is 96 km, compared with 60 km without radioactive heating. The mantle below the lithosphere is believed to be at a temperature of 1300 °C to 1400 °C. Assuming the latter, the lithosphere could be no more than 113 km thick using the values assumed here. This is relatively thin, and might be appropriate for a relatively young continental province.

It might seem paradoxical that lower temperatures have been calculated when radioactive heating has been included, in Equation (7.6.5), compared with temperatures from Equation (7.6.4) with no radioactive heating. In order to clarify this, I will spend some time explaining some aspects of this solution. The reason for the lower

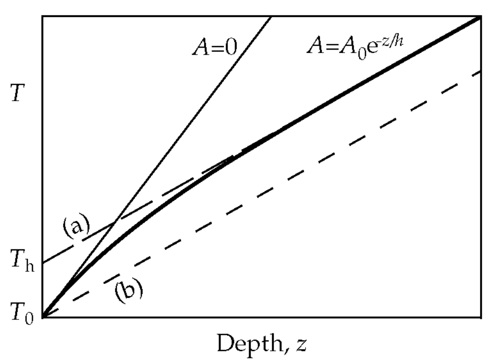

Figure 7.6. Sketch of calculated continental geotherms. Geotherm with no radioactivity ($A = 0$, light solid line), with radioactivity given by Equation (7.6.1) (heavy solid). Line (a) is the asymptote of the heavy curve. Line (b) is the geotherm, with $A = 0$, that would match the temperature at the surface and the temperature gradient below the zone of radioactive heating.

temperatures is that the temperature gradient was fixed at the surface. Physically you can think of it as follows. If the temperature gradient is fixed at the surface, then so is the surface heat flux. If there is no crustal radioactive heat generation, then all of that heat flux must be coming from below the lithosphere. This requires that the temperature gradient must equal the surface value right through the lithosphere (assuming for simplicity that the conductivity is constant). On the other hand, if some of the heat comes from radioactivity near the surface, then below the heat-generating zone the heat flux will be less, so the temperature gradient will also be smaller. This is why the geotherm in this case bends down (Figure 7.6) and reaches lower temperatures at depth than the geotherm with no heat generation (but with the same temperature and temperature gradient at the surface).

If we required instead to keep the same heat flux into the *base* of the lithosphere, then with no heat generation the result would be line (b), with a slope of $T'_m = T'_0 - A_0h/K$. This matches the surface temperature and has the same temperature gradient at the base of the lithosphere. You can see that including heat generation raises the temperature relative to line (b). This comparison is more in accord with simple intuition. It also illustrates what is sometimes called thermal blanketing: the geotherm with heating is hotter by the amount $T_h = A_0h^2/K$. With the values used above, $T_h = 83\,^\circ\mathrm{C}$.

The heat flux through the base from the mantle, using the above values, is $q_m = KT'_m = 35\,\mathrm{mW/m^2}$. The heat flux due to radioactive heating is in this case $q_h = A_0h = 25\,\mathrm{mW/m^2}$. Thus you can see that in this case the total surface heat flux of $60\,\mathrm{mW/m^2}$ is the sum of $35\,\mathrm{mW/m^2}$ from the mantle and $25\,\mathrm{mW/m^2}$ from radioactive heating. If the lithosphere were thicker, the heat flux from the mantle would be less, as would the total surface heat flux.

From the point of view of considering mantle convection, an important question is how much heat escapes from the mantle through the continents, since this heat is not then available to drive mantle convection. This is the amount of heat entering the base of the continental lithosphere, and it is determined essentially by the thickness of the continental lithosphere, as modified by the thermal blanketing effect of radioactivity in the upper crust. Thus the long-term or steady-state conducted heat flux is determined by the temperature gradient $(T_m - T_h)/D$. The mantle temperature T_m does not vary much in comparison with variations in the lithosphere thickness D, and you have just seen an estimate that the thermal blanketing effect of radioactivity is to raise the effective surface temperature by about $80\,^\circ\mathrm{C}$, which also is small in comparison with the total temperature difference. Thus if we take the effective temperature difference across continental lithosphere to be about 1200 °C and a typical range of thickness to be 100–250 km, we get a range of heat flux out of the mantle of 14–36 mW/m^2, with an average of perhaps 20–25 mW/m^2.

Continental crust covers about 40% of the earth's surface, with an area of about 2×10^{14} m^2, so the total heat loss from the mantle through continents is about 4–5 TW, about 10% of the global heat budget. Thus the mantle heat loss through the continents is not a large fraction of the total. In fact it is such a small fraction that we can regard the continents as insulating blankets. The complementary role of oceanic heat loss will be taken up in Chapter 10.

7.7 Heat transport by fluid flow (Advection)

So far we have looked at heat transported by thermal conduction, but heat can also be transported by the motion of mantle material. Suppose, for example, that mantle material with a temperature of $T_h = 1500$ °C replaces normal mantle at $T_m = 1400$ °C. Then the local heat content per unit volume is increased by $\rho C_P \Delta T = \rho C_P(T_h - T_m)$. With $\rho = 3300$ kg/m^3 and $C_P = 1000$ J/kg °C, the increase is 3.3×10^8 J/m^3. Now, referring to Figure 7.7, suppose the hot material is flowing up a vertical pipe of radius $R = 50$ km at a velocity $v = 1$ m/a $\approx 3 \times 10^{-8}$ m/s. Then the volume of hot material that flows past a point on the pipe within unit time is $V = \pi R^2 v \approx 250$ m^3/s. The amount of extra heat that has been carried past this point within unit time is

$$Q = V\rho C_P \Delta T = \pi R^2 v \rho C_P \Delta T \tag{7.7.1}$$

With the above values, $Q \approx 8 \times 10^{10}$ J/s $= 8 \times 10^{10}$ W: this is the heat flow rate. The heat flux is

$$q = Q/\pi R^2 = v\rho C_P \Delta T \tag{7.7.2}$$

Then $q \approx 10$ W/m^2. This heat flux is a much higher value than the conducted heat fluxes we discussed above. The heat flow, Q, is about 0.2% of the global heat budget, despite the small area of the pipe in comparison with the surface area of the earth. This example has been tailored to approximate a mantle plume, and these will be discussed in more detail in Chapter 11.

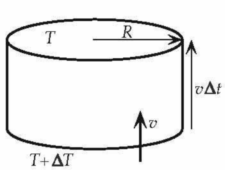

Figure 7.7. Heat carried by fluid flowing with velocity v.

This process of heat transport by mass motion is called *advection*. It usually accounts for most of the heat transport in *convection*. In fact, you will see in Chapter 8 that in a sense convection only occurs when conduction is inadequate to transport heat. You can see from the above example that advection can transport much greater heat fluxes than conduction in some situations.

The distinction between advection and convection is that the term *advection* is used to refer to heat transport by mass motion regardless of the source of the mass motion, whereas in *convection* the motion is due specifically to the internal buoyancies of the material. Thus when you stir your coffee with a spoon, you force fluid motion that transports heat around in the cup by *advection*. On the other hand, if you let the cup sit, the top of the coffee will cool and sink, producing *convection* that will also advect heat.

7.8 Advection and diffusion of heat

7.8.1 General equation for advection and diffusion of heat

The approach just used in Section 7.7 can be used to derive an equation that governs the evolution of temperature in the presence of both conduction (diffusion) and advection. Advection occurs when there is fluid motion and when there are temperature differences within the fluid: if the temperature is homogeneous, then there is no net heat transport. In Section 7.1 we considered heat conduction in one dimension (the x direction). If we now suppose that in addition to the other things happening in Figure 7.1 there is a flow with velocity v in the positive x direction, then we should add two terms to the right-hand side of the heat budget for the little box:

$$Sv\,\mathrm{d}t \cdot \rho C_P T - Sv\,\mathrm{d}t \cdot \rho C_P (T + \Delta T)$$

You can recognise these as the heat advected into the box through the left-hand side, within the time interval $\mathrm{d}t$, minus the heat advected out through the right-hand side. Dividing again by $S \cdot \mathrm{d}x \cdot \mathrm{d}t$ and taking the limit yields the extra term on the right-hand side of Equation (7.1.2)

$$-\rho C_P v \frac{\partial T}{\partial x}$$

Equation (7.1.3) can then be generalised to

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \kappa \frac{\partial^2 T}{\partial x^2} + a \qquad (7.8.1)$$

Here the advection term is placed on the left-hand side, and you can see it depends on there being both a fluid velocity and a temperature gradient.

This equation can readily be generalised to three dimensions just by considering heat transport in the other two coordinate directions of Figure 7.1. The result is

$$\frac{\partial T}{\partial t} + v_i \frac{\partial T}{\partial x_i} = \kappa \frac{\partial^2 T}{\partial x_i \partial x_i} + a \tag{7.8.2}$$

where I have used the summation convention (Box 6.B1). This equation governs the evolution of temperature in the presence of advection, diffusion (conduction) and internal heat generation.

We now have the conceptual and mathematical tools to consider convection. We have looked at viscous fluid flow, including examples driven by buoyancy forces, and at heat transport. Convection involves the combination of these processes. Their general mathematical description is embodied in Equations (6.6.1), (6.6.3) and (7.8.2). Convection will be discussed in Part 3.

7.8.2 An advective-diffusive thermal boundary layer

Here is a relatively simple illustration of the simultaneous occurrence of advection and diffusion. We will see in Chapter 11 that mantle plumes are believed to transport material from the base of the mantle, where mantle material is heated by heat flowing out of the core, which is believed to be hotter. This heat will generate a hot thermal boundary layer at the base of the mantle. The thickness of this boundary layer will depend on the rate at which material flows through it, and also on the form of that flow. If the flow is basically horizontal, like the bottom of a large-scale convection cell, then the boundary layer thickness will depend on the time for which mantle material is adjacent to the hot core. The theory of Section 7.3 will then apply. In this case the bottom thermal boundary layer would be analogous to the top thermal boundary layer (the lithosphere), and its thickness would be proportional to the square root of the time spent at the bottom of the mantle. It is conceivable, however, that the large-scale, plate-related flow penetrates only minimally to the bottom of the mantle (Chapters 10, 12), and that the dominant flow near the bottom is a vertical downwards flow that balances the material flowing upwards in plumes. In this case the relationship between the advection and diffusion of heat would be different from that in Section 7.3. We now look at this possibility.

The situation is sketched in Figure 7.8. Material flows slowly downwards with velocity $v = -V$. Hot, low-viscosity material flows rapidly sideways within a thin layer adjacent to the core, and then into narrow plumes where it rises rapidly upwards. Away from the plumes and above the thin channel at the base, the flow can be approximated as being vertical. We now derive an expression for the temperature in this region, as a function of height, z, above the core.

If we assume that there is a steady state and no heat generation, then Equation (7.8.1) reduces to

$$v\frac{\mathrm{d}T}{\mathrm{d}z} = \kappa\frac{\mathrm{d}^2T}{\mathrm{d}z^2} \tag{7.8.3}$$

Note first of all that if we use a representative temperature scale ΔT and a representative length scale h, then rough approximations to the differentials in this equation yield

$$h = \kappa/v \tag{7.8.4}$$

Thus this ratio contains an implicit length scale, h, which we can also see from the dimensions of κ and v.

Using Equation (7.1.1) for the heat flux, q, Equation (7.8.3) can be rewritten as

$$\frac{1}{q}\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{v}{\kappa} = -\frac{V}{\kappa} \equiv -\frac{1}{h}$$

which defines the length scale $h = \kappa/V$ for the particular problem in Figure 7.8. This equation can be integrated to give

$$q = q_\mathrm{b}\exp\left(-\frac{z}{h}\right)$$

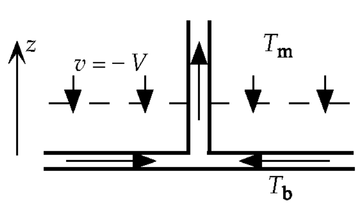

Figure 7.8. Sketch of the flow associated with a mantle plume drawing hot material from the base of the mantle. The temperature at the bottom boundary is T_b, and the temperature of the ambient mantle is T_m. Away from plumes, mantle material is assumed to flow vertically downwards with velocity $v = -V$. A thermal boundary layer (dashed line) forms above the core–mantle boundary.

where q_b is the heat flux into the base. This in turn can be integrated to give

$$T = T_m + \frac{hq_b}{K}\exp\left(-\frac{z}{h}\right) \tag{7.8.5}$$

where K is the conductivity. The temperature at the boundary, $z = 0$, is then

$$T_b = T_m + hq_b/K \tag{7.8.6}$$

In Chapter 11 we will see that the total heat flow carried by plumes is about 3.5 TW, roughly 10% of the heat flowing out of the top of the mantle. This heat is inferred to be flowing out of the core. Since the surface area of the core is only about $\frac{1}{4}$ of the surface area of the earth, the heat flux out of the core is then about 40% of the surface heat flux, or about 30 mW/m^2. In Exercise 12, later, you can deduce that the downward velocity is about $V = 1.3 \times 10^{11}$ m/s (0.4 mm/a). Then taking the density at the base of the mantle to be 5600 kg/m^3 and other quantities from Table 7.3, below, we obtain $h = 115$ km and $T_b - T_m = 385\,°C$.

The physics described by this solution is that mantle material slowly flows down towards the hot interface with the core and heat conducts upwards against this flow. Thus upwards thermal diffusion is competing against downwards advection of heat. In the steady state, the temperature declines exponentially towards the ambient mantle temperature as a function of height above the interface. This thermal boundary layer has a characteristic thickness of the order of 100 km and a temperature increase across it of about 400 °C, according to the numerical values we have used.

7.9 Thermal properties of materials and adiabatic gradients

7.9.1 Thermal properties and depth dependence

We have already encountered most of the important thermal properties of materials that we will be needing in this book. It is useful to summarise them here, with some typical values. This is done in Table 7.3. However there is one aspect that we have not yet encountered, and that is the variation with depth of some of these properties, and of the temperature in the convecting mantle. Although we will not be much concerned with these depth variations, because the effects are secondary to the main points I want to demonstrate, they are nevertheless significant and worth noting.

Table 7.3. *Representative thermal properties of the mantle [6].*

Quantity	Symbol	Value $P = 0$	Value CMB	Units
Specific heat at constant pressure	C_P	900	1200	J/kg °C
Thermal conductivity	K	3	9	W/m °C
Thermal diffusivity	κ	10^{-6}	1.5×10^{-6}	m^2/s
Thermal expansion coefficient	α	3×10^{-5}	0.9×10^{-5}	/°C
Grüneisen parameter	γ	1.0–1.5	0.9	

Some estimates of values at the base of the mantle (CMB or core–mantle boundary) are included in Table 7.3. These are modified from Stacey's [6] values, mainly by using higher values of the thermal expansion coefficient.

7.9.2 Thermodynamic Grüneisen parameter

The thermodynamic relationships governing the depth dependence of the temperature can be expressed most concisely in terms of a parameter known as the thermodynamic Grüneisen parameter, γ. It is related to the thermal expansion coefficient, and this relationship is most directly evident if we define γ as [6]

$$\gamma = \frac{1}{\rho C_V}\left(\frac{\partial P}{\partial T}\right)_V \tag{7.9.1}$$

where C_V is the specific heat at constant volume. This definition shows that γ is a measure of the rate at which pressure increases as heat is input while volume is held constant. For comparison, the thermal expansion coefficient is

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P \tag{7.9.2}$$

Thus α is a measure of the rate at which volume increases as heat is input while pressure is held constant, and α and γ are complementary measures of the effect of heating. Another way to think of γ is that it measures the pressure required to prevent thermal expansion.

Two other useful expressions for γ can be derived with the help of thermodynamic identities. The latter are complicated, and can be found in standard thermodynamics texts. A concise summary is provided by Stacey [6]. I will just quote the results here. The first form is

$$\gamma = \frac{\alpha K_S}{\rho C_P} = \frac{\alpha v_b^2}{C_P} \tag{7.9.3}$$

where $K_S = \rho(\partial P/\partial \rho)_S$ is the adiabatic bulk modulus, subscript S indicates constant entropy, and v_b is the bulk sound speed (Section 5.1.4). The second form is

$$\gamma = -\frac{V}{T}\left(\frac{\partial T}{\partial V}\right)_S = \frac{\rho}{T}\left(\frac{\partial T}{\partial \rho}\right)_S \tag{7.9.4}$$

where V is specific volume (that is, volume per unit mass: $V = 1/\rho$).

7.9.3 Adiabatic temperature gradient

As mantle material rises and sinks in the course of mantle convection, thermal diffusion is so inefficient at large scales that through most of the mantle it can be neglected. At the same time, there are large changes of pressure accompanying the vertical motion. A process of compression with no heat exchange with surroundings is called *adiabatic compression*. If it happens slowly, so that it is reversible, it is characterised by having constant entropy. A parcel of mantle that sinks slowly through the mantle experiences such adiabatic compression. During adiabatic compression, although there is no heat exchange with surroundings, the increasing pressure does work on the material as it compresses, and this increases the internal energy of the material, which is expressed as a rise in temperature. We will now estimate this adiabatic increase in temperature with depth in the mantle.

The Grüneisen parameter provides a convenient way to make this estimate. The Grüneisen parameter in the mantle can be estimated most reliably from Equation (7.9.3), since K_S, ρ and v_b are known from seismology (Section 5.1.4). C_P does not vary much with pressure. The thermal expansion coefficient is the least well constrained, and it is likely to decrease substantially under pressure [7], as indicated in Table 7.3. This is counteracted by the increase of v_b with depth (Figure 5.3). The result is that γ does not vary greatly with depth, being about 1–1.5 in the peridotite and transition zones and decreasing to slightly less than 1 at the bottom of the mantle.

If γ does not vary greatly through the mantle, then the assumption that it is constant will be a reasonable approximation. In this case, Equation (7.9.4) can be integrated to yield

$$\frac{T_l}{T_u} = \left(\frac{\rho_l}{\rho_u}\right)^\gamma \tag{7.9.5}$$

where the subscripts l and u refer to lower mantle and upper mantle, respectively. With $\rho_l = 5500\,\mathrm{kg/m^3}$, $\rho_u = 3300\,\mathrm{kg/m^3}$, and $\gamma = 1.0$–1.5, this yields $T_l/T_u = 1.7$–2.15. However, about 800 kg/m^3 of the density increase through the mantle is due to phase transformations, through which Equation (7.9.4) does not apply. If we take instead $\rho_l = 4700\,\mathrm{kg/m^3}$, then $T_l/T_u = 1.4$–1.7. With $T_u = 1300\,°\mathrm{C}$, this indicates that the adiabatic increase of temperature through the mantle is about 500–900 °C, and $T_l =$ 1800–2200 °C.

A schematic temperature profile through the earth is shown in Figure 7.9. A more quantitative version is not given here, both because we are not concerned with details, and because the uncertainties are so large that greater detail is hardly justified. For example, various estimates put the temperature jump across the lower thermal boundary layer of the mantle at anything between 500 °C and 1500 °C, with some estimates even higher [6, 7], so that $T_b = 2300$–$3700\,°\mathrm{C}$. However, it is hard to reconcile these higher values with the dynamics of plumes (Chapter 11), even taking account of the likelihood of a layer of denser material at the base of the mantle (Chapter 5). Stacey [6] estimates the adiabatic temperature increase through the core to be about 1500 °C, so that the temperature at the centre of the earth might be $T_c = 3800$–$5000\,°\mathrm{C}$.

7.9.4 The super-adiabatic approximation in convection

Although the adiabatic increase of temperature through the mantle is quite large, it is not of great concern to us in this book. This is because convection will only occur if the actual temperature gradient exceeds the adiabatic gradient, as I will explain in a moment. We can therefore focus on this *super-adiabatic* gradient. An effective way to do this is to subtract the adiabatic gradient out of the mantle temperature profile for convection calculations, or in other words to neglect this effect of pressure.

To see that convection requires a super-adiabatic gradient, suppose that the interior of the mantle has an adiabatic gradient, as sketched in Figure 7.9. You might suppose at first that since the deeper mantle is hotter than the shallow mantle, it will be buoyant and therefore drive convection. However, if a small portion of this deep mantle rises vertically, it will decompress adiabatically as it rises and its temperature will follow the adiabatic profile. Thus it

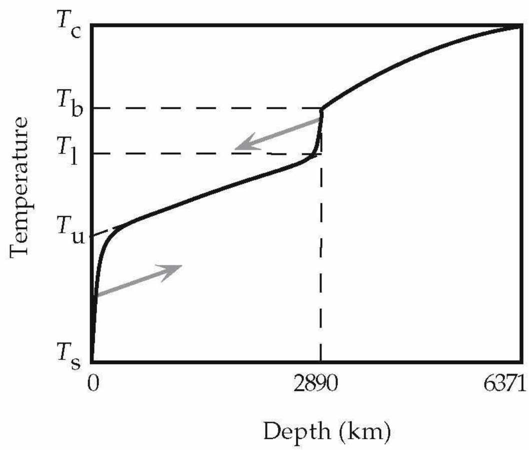

Figure 7.9. Schematic temperature profile through the earth. Thermal boundary layers are assumed at the top of the mantle (the lithosphere) and the bottom of the mantle. Numerical values of the temperatures are quite uncertain (see text). The grey arrows show adiabatic compression and decompression paths of material from the thermal boundary layers.

will remain at the same temperature as its surroundings and no thermal buoyancy will be generated.

In order to have buoyancy that will drive convection, a deep portion of the mantle must start hotter than its surroundings, as would for example material from the lower thermal boundary layer. It may then follow an adiabatic decompression path that is sub-parallel to the mantle adiabat, and consequently remain hotter and buoyant as it rises. Such a path is illustrated in Figure 7.9. An analogous path is also shown for descending, cool, negatively buoyant lithospheric material. Of course these portions of the mantle may exchange heat with the surrounding mantle by thermal diffusion, in which case their paths will tend to converge towards the mantle adiabat, but their initial buoyancy will be approximately preserved within a larger volume of material.

7.10 References

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7.11 Exercises

1. Use Equation (7.2.1) to estimate the time it would take for a sill of thickness 100 m to cool substantially.
2. During the ice age, glaciers kept the surface of Canada cooler than at present. The glaciers had melted by about 10 000 years ago. To about what depth in the crust would the subsequent warming of the surface have penetrated?
3. [*Intermediate*] Complete the derivation of Equation (7.2.7). Either integrate the equations or show that the forms used are solutions of the relevant equations. Apply the initial condition to evaluate the constants of integration.
4. Use Lord Kelvin's argument to estimate the age of the earth from the fact that the rate of temperature increase with depth in mines and bore holes is about 20 °C/km and assuming the upper mantle temperature to be 1400 °C. Comment on the relationship between your answer and the age of oceanic lithosphere.
5. [*Advanced*] Derive the general solution (7.3.5), using the same approach as in Exercise 3.
6. Using values in Table 7.2, calculate the thicknesses of layers composed of (i) upper continental crust, (ii) oceanic crust, and (iii) chondritic meteorites required to produce the average heat flux of 80 mW/m^2 observed at the earth's surface. What constraints does this impose on the composition of the continental crust and the mantle?
7. Calculate, by integration from the surface to great depth, the total rate of heat production per unit surface area implied by Equation (7.6.1).
8. Derive the solution (7.6.4) for temperature versus depth from Equations (7.6.2) and (7.6.3).