Basic properties of the delta function

1D

Definition:

$$\int_{-\infty}^{+\infty} g(x)\delta(x-a)dx = g(a)$$

Normalization and scaling property:

$$\int_{-\infty}^{+\infty} g(x)\delta(\alpha x)dx = \int_{-\infty}^{+\infty} g\left(\frac{u}{\alpha}\right)\delta(u)\frac{du}{|\alpha|} = \frac{g(0)}{|\alpha|} \qquad \Longrightarrow \qquad \delta(\alpha x) = \frac{\delta(x)}{|\alpha|}$$

$$\int_{-\infty}^{+\infty} g(x)\delta(\alpha(x-a))dx = \frac{g(a)}{|\alpha|}$$

Generalized to:

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|} \qquad \text{where } x_i \text{ are simple roots of } f(x)$$

In the integral form:

$$\int_{-\infty}^{+\infty} g(x)\delta(f(x)) = \sum_i \frac{g(x_i)}{|f'(x_i)|}$$

n-D

Generalized in a n-dimensional space (where the roots of $g(\mathbf{r})$ form a continuum and not just a discrete set of points):

$$\int_V g(\mathbf{r})\delta(f(\mathbf{r}))d^n\mathbf{r} = \int_{\partial V} \frac{g(\mathbf{r})}{|\nabla_{\mathbf{r}} f(\mathbf{r})|} d^{n-1}\mathbf{r}$$