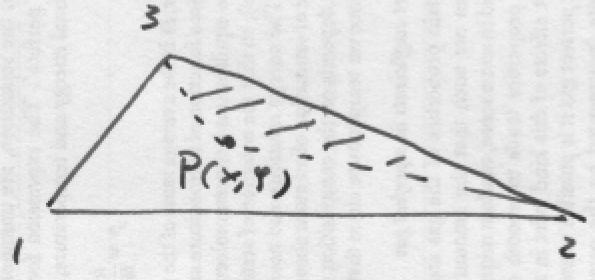
Area coordinates for planar triangles

- Widely used as shape functions in FEM
- 

$$L_1 = \frac{\text{Area}(\triangle P\,2\,3)}{\text{Area}(\triangle 1\,2\,3)}$$

$$L_2 = \frac{\text{Area}(\triangle P\,3\,1)}{\text{Area}(\triangle 1\,2\,3)}$$

$$L_3 = \frac{\text{Area}(\triangle P\,1\,2)}{\text{Area}(\triangle 1\,2\,3)}$$

$$L_3 = 1 - L_1 - L_2 \quad !$$

$$L_1 = \frac{\begin{vmatrix} x & y & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}{2A_\Delta} \qquad L_2 = \frac{\begin{vmatrix} x & y & 1 \\ x_3 & y_3 & 1 \\ x_1 & y_1 & 1 \end{vmatrix}}{2A_\Delta}$$

$$L_3 = \frac{\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}}{2A_\Delta} \qquad A_\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

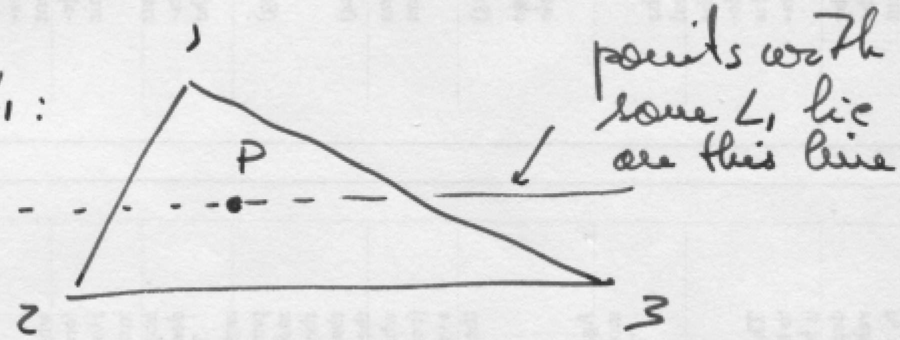
• Mapping of triangular element onto standard triangle

$$(L_1, L_2) \longleftrightarrow (x, y)$$

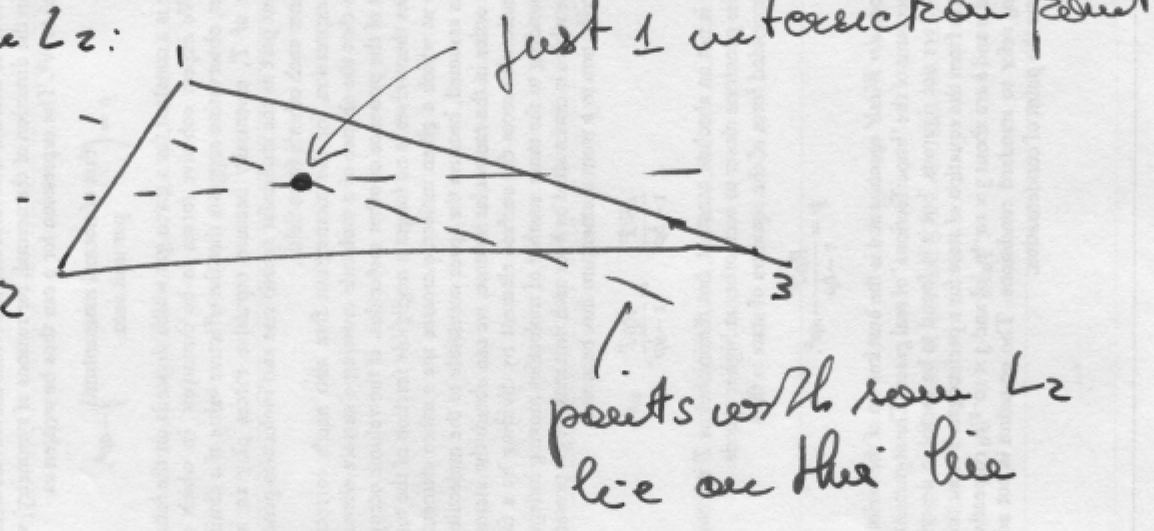
This is a one-to-one relation (mapping). Bilinear.

• Why is 1-to-1?

Assign L_1:

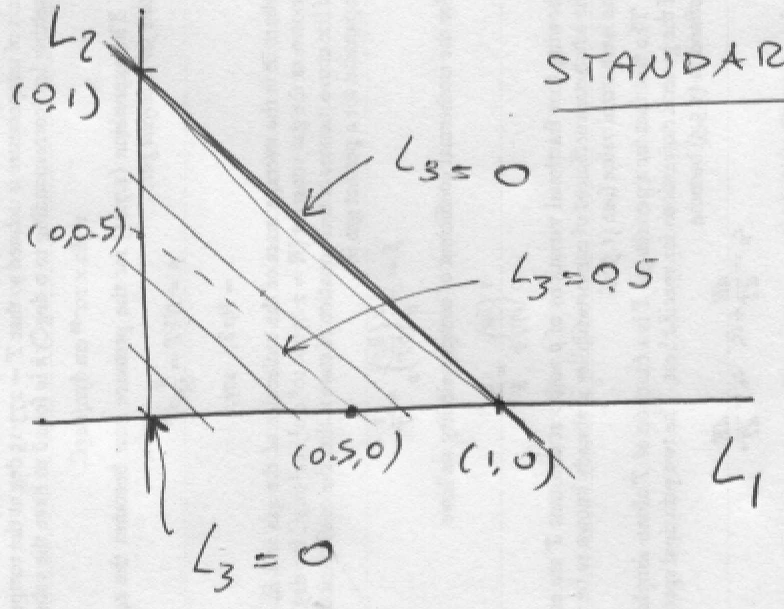


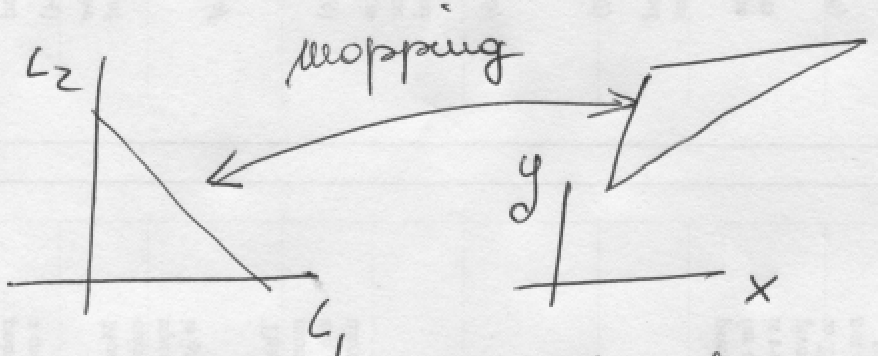
Assign L_2:



• $L_1, L_2, L_3 \in [0,1]$

STANDARD TRIANGLE





• Notice that, by construction:

$$L_i(\vec{x}_j) = \delta_{ij}$$

Thus, $\{L_i\}_{i=1}^{3}$ provide a set of interpolating shape functions.

• It can be shown (e.g., by inverting L_1 and L_2 w.r.t. x and y) that the coordinate mapping can be rewritten as:

$$\boxed{\vec{X} = L_1 \vec{X_1} + L_2 \vec{X_2} + L_3 \vec{X_3}}$$

- Given any function $f : \mathbb{R}^2 \longmapsto \mathbb{R}$, an interpolatory approximation to $f|_\Delta$ is

$$\boxed{f|_\Delta = \sum_{1}^{3} L_j(\vec{x}) f_j}$$

Homework

- $f(x,y) = e^{-(x^2+y^2)} \cos(2x) \sin(3y)$ defined on $\Omega = [0,1] \times [0,1]$.
- Build several triangular meshes on Ω (e.g., using "delaunay" function of MATLAB) with progressively smaller diameter
- Implement an algorithm that:

- approximates f on the triangles using a bilinear interpolation
- calculates:

1) $\int_\Omega f(x,y)\,dx\,dy$ $\left(\simeq 0.21187196\right)$

2) $\int_\Omega \frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\,dx\,dy$ $\left(\simeq -6.580038 \times 10^{-4}\right)$

Carry out quadrature in the transformed space !

Provide error estimates with decreasing mesh diameter (specify error norm).