Chapter 1

Introduction to Fluid Mechanics

1.1 Generalities

As regards *Mechanics*, the states of matter (solid, liquid, gas) are featured by the way they react to extrenal forces. *Fluids*, either liquids or gases, are not capable to resist to shear stresses and to normal, traction stresses, how small they can be. In other words, a fluid deforms continuously (i.e., *flows*) whenever a shear or normal, traction stress is applied on its boundary. Notice that fluids are capable to resist to normal, compressive stresses. A solid, on the contrary, shows a limited, finite deformation under the action of *small* stresses.

This rough distinction does not represent substances that have an intermediate behaviour between solids and fluids, as, e.g., visco-elastic fluids (e.g., paint and jelly), but is adequate for our purposes.

Refer to Chapter 1 of this book for an intriguing introduction to Fluid Mechanics.

1.2 On the continuum assumption

A fluid, or any other substance for that matter, is composed of a large number of molecules in constant motion and undergoing collisions with each other. Matter is therefore discontinuous or discrete at microscopic scales. In principle, it is possible to study the mechanics of a fluid by studying the motion of the molecules themselves, as is done in kinetic theory or statistical mechanics. However, we are generally interested in the gross behaviour of

the fluid, that is, in the average manifestation of the molecular motion. For example, forces are exerted on the boundaries of a container due to the constant bombardment of the molecules; the statistical average of this force per unit area is called pressure, a macroscopic property. So long as we are not interested in the mechanism of the origin of pressure, we can ignore the molecular motion and think of pressure as simply *force per unit area*. It is thus possible to ignore the discrete molecular structure of matter and replace it by a continuous distribution, called a *continuum*. For the continuum or macroscopic approach to be valid, the size of the flow system (characterized, for example, by the size of the body around which flow is taking place) must be much larger than the mean free path of the molecules. For ordinary cases, however, this is not a great restriction, since the mean free path is usually very small. For example, the mean free path for standard atmospheric air is $\sim 5 \times 10^{-8}$m. In special situations, however, the mean free path of the molecules can be quite large and the continuum approach breaks down. In the upper altitudes of the atmosphere, for example, the mean free path of the molecules may be of the order of a meter, a kinetic theory approach is necessary for studying the dynamics of these rarefied gases.

According to **?**, *It is important to appreciate that, once we invoke the continuum hypothesis to obtain continuous fields, such as* $\rho(\boldsymbol{x}, t)$ *and* $U(\boldsymbol{x}, t)$*, we can leave behind all notions of the discrete molecular nature of the fluid, and molecular scales cease to be relevant. We can talk meaningfully of "the density at* $\boldsymbol{x}$*,* t*," even though (in the microscopic view) in all likelihood there is no matter at* $(\boldsymbol{x}, t)$*. Similarly, we can consider differences in properties over distances smaller than molecular scales: indeed we do so when we define gradients,*

$$\frac{\partial \rho}{\partial x_1} \equiv \lim_{h \longrightarrow 0} \left(\frac{1}{h} \left[\rho \left(x_1 + h, x_2, x_3, t \right) - \rho \left(x_1, x_2, x_3, t \right) \right] \right)$$

The Knudsen number (Kn) is a dimensionless number defined as the ratio of the molecular mean free path length[1] λ to a representative physical length scale of the flow. This length scale could be, for example, the radius of a body immersed in a fluid. The Knudsen number helps determine whether the Statistical Mechanics or the Continuum Mechanics perspective of fluid dynamics should be used to model a given flow configuration. If the Knudsen number is near or greater than one, the mean free path of a molecule is comparable to a length scale of the problem, and the continuum assumption of fluid mechanics is no longer a good approximation.

[1] In kinetic theory the mean free path of a particle, such as a molecule, is the average distance the particle travels between collisions with other moving particles.

An expression for the Knudsen number in terms of thermodynamic variables can be provided for ideal gases, according to the kinetic theory. If the velocities of identical gas molecules have a Maxwell distribution, the following relationship applies for the mean free path (see section 1.3):

$$\lambda = \frac{1}{\sqrt{2}\, n\, \sigma} \tag{1.1}$$

where n is the number density of molecules (i.e., the number of molecules per unit volume) while $\sigma = \pi\,(2\,r)^2$ is the effective cross sectional area for spherical particles with radius r. The specific expression for σ is motivated by the representation of a molecule as a hard sphere and by the observation that two identical hard spheres collide whenever their centres get closer than the sum of their radii, i.e., $2\,r$. For an ideal gas the state equation is:

$$p\,V = N\,R_0\,T \tag{1.2}$$

where N is the number of moles contained in the volume V, T is the thermodynamic temperature and $R_0 = 8314$ J/(kmol K) is the universal gas constant. The number density n and the molar density N/V are related as

$$N/V = n/N_a$$

where N_a is the Avogadro's number ($N_a = 6.022 \times 10^{26}$ molecules/kmol). Therefore, we end up with:

$$p = n\,K_B\,T \Longrightarrow n = \frac{p}{K_B\,T} \tag{1.3}$$

where $K_B \equiv R_0/N_a \approx 1.38 \times 10^{-23}$ J/K is the Boltzmann's constant. Substituting n from (1.3) into (1.1) yields:

$$\lambda = \frac{K_B\,T}{\sqrt{2}\,\sigma\,p} \tag{1.4}$$

and, consequently, the Knudsen number is

$$\mathrm{Kn} = \frac{\lambda}{l} = \frac{K_B\,T}{\sqrt{2}\,\sigma\,p\,l} \tag{1.5}$$

which shows that, apart from other considerations, the continuum assumption is better suited to model *low*-temperature, *high*-pressure gases.

Exercise 1.2.1. *As an exercise, calculate the mean free path under usual atmospheric conditions,*

$$p = 100\,kPa,\ T = 293.15\,K$$

assuming a molecular diameter

$$d \approx 0.3 \times 10^{-10}\,m$$

The Knudsen number can be related to the Mach number and the Reynolds number in gas flows. The following results of the kinetic theory of gases are used, to this end:

$$\left.\begin{aligned} \mu &= \frac{1}{2}\,\rho\,\bar{c}\,\lambda \\ \bar{c} &= \sqrt{\frac{8\,K_B\,T}{\pi\,m}} \end{aligned}\right\} \Longrightarrow \lambda = \frac{2\,\mu}{\rho\,\bar{c}} = \frac{\mu}{\rho}\sqrt{\frac{\pi\,m}{2\,K_B\,T}}$$

where $\bar{c}$ denotes the molecular mean velocity from Maxwell's distribution while m denotes the molecular mass, $M = N_a\,m$. The knudsen number is easily expressed as:

$$\mathrm{Kn} = \frac{\mu}{\rho\,l}\sqrt{\frac{\pi\,m}{2\,K_B\,T}} \tag{1.6}$$

The Mach number is defined as:

$$\mathrm{Ma} = \frac{U_\infty}{c_s} \tag{1.7}$$

where, for an ideal gas,

$$c_s = \sqrt{\gamma\,\frac{R_0}{M}\,T}$$

The Reynolds number is defined as

$$\mathrm{Re} = \frac{\rho\,U_\infty\,l}{\mu} \tag{1.8}$$

Thus:

$$\mathrm{Kn} = \frac{\mathrm{Ma}}{\mathrm{Re}}\sqrt{\frac{\pi\,\gamma}{2}} \tag{1.9}$$

which shows that the continuum assumption is better suited to deal with *slow*, subsonic flows. Indeed, *shock fronts* may develop in supersonic (Ma $>$ 1) or hypersonic (Ma $>$ 5) flows: within shock fronts the flow variables experience and abrupt change over a distance of few mean free paths, with little hope to apply the continuum assumption.

Example 1.2.2. *Let us consider for instance the flow over an airliner's wing, with chord length $l = 4m$ (at a given wing section). Let us assume that the aeroplane is flying at $Ma = 0.88$. The kinematic viscosity of air (at $250K$) is $1.132 \times 10^{-5} Pa\ s$. The sound speed through air at $250K$ is given approximately by*

$$c_s = 331.4 + 0.6\ (T - 273.15)\ [m/s]$$

yielding $c_s = 317.28 m/s$ at $T = 250K$. This, in turn, yields $U_\infty = Ma\, c_s = 279.20 m/s$. Thus:

$$\frac{Ma}{Re} = \frac{\nu}{c_s\, l} = 8.9 \times 10^{-9}$$

leading to ($\gamma \approx 1.4$ for air):

$$Kn = \sqrt{\frac{\pi\,\gamma}{2}}\,\frac{Ma}{Re} \approx 1.3 \times 10^{-8} \tag{1.10}$$

This shows that the airflow around an airliner has a very low Knudsen number, making it firmly in the realm of continuum mechanics.

It is commonly believed that Kn $>$ 1 is a suitable criterion for distinguishing *molecular flow* from *continuum flow*. Nevertheless, this inequality has to be considered just as a rule of thumb, as the actual value of the Knudsen number depends on the length scale l.

Let us reconsider the example above, in view of the fact that the chord length is not the only relevant length scale for the considered flow. Another pertinent length scale is the depth δ_v of the boundary layer, the region close to a solid surface where viscous and inertial effects are in approximate balance. Prandtl's equations for incompressible boundary layer flow over a flat plane are a simplification of the full set of Navier-Stokes equations,

owing to the very small thickness of the boundary layer at high Reynolds number:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.11}$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + \frac{1}{\rho}\,\frac{\mathrm{d}p_\infty}{\mathrm{d}y} - \nu\,\frac{\partial^2 u}{\partial y^2} = 0 \tag{1.12}$$

Prandtl's assumption implies:

$$x/\delta_v \gg 1; \qquad y/\delta_v \sim 1$$

From the continuity equation:

$$\frac{u_\infty}{x} \sim \frac{v}{\delta_v}$$

Equating the orders of magnitude of inertial and viscous forces within the b.l. yields:

$$\frac{u_\infty^2}{x} + \frac{u_\infty\,\delta_v}{x}\,\frac{u_\infty}{\delta_v} \sim \nu\,\frac{u_\infty}{\delta_v^2}$$

The following important result is easily derived:

$$\frac{\delta_v}{x} \sim \frac{1}{\sqrt{\mathrm{Re}_x}} \tag{1.13}$$

The Knudsen number for an incompressible fluid is derived as:

$$\mathrm{Kn} = \frac{\lambda}{\delta_v} = \frac{\overline{v}}{\delta_v\,\sqrt{2}\,\mathrm{N}_a\,\sigma}$$

where $\overline{v}$ denotes the molar specific volume of the fluid,

$$\overline{v} = \frac{M}{\rho}$$

with M the molar mass of the liquid. Then, using (1.13),

$$\mathrm{Kn} = \frac{\lambda}{\delta_v} = \frac{\sqrt{\mathrm{Re}_x}\,M/\rho}{x\,\sqrt{2}\,\mathrm{N}_a\,\sigma} \tag{1.14}$$

The continuum assumption holds within the boundary layer as long as $\lambda/\delta_v \ll 1$, which is the case for most of the flows encountered in engineering applications. Nevertheless, as for Example 1.2.2, the Knudsen number based on the boundary-layer thickness turns out to be of the order of 0.02, which may be considered *small*, yet not *negligibly small*.

1.3 Intuitive derivation of λ

Equation (1.1) may be derived following rather intuitive arguments. Let's first consider a stationary molecule, hit by other molecules. We refer to this molecule as *target*. All molecules positioned at time $t = 0$ within a cylindrical region of volume

$$\sigma\,\overline{v}\,t$$

are going to hit the target within the time interval $[0, t]$, where $\overline{v}$ denotes the mean molecular velocity. Thus, there will be

$$n\,\sigma\,\overline{v}\,t \tag{1.15}$$

collisions with the target within the time interval $[0, t]$. On average, the path between two successive collisions can be estimated as

$$\begin{aligned} \lambda &= \frac{\text{distance traveled}}{\text{nr. of collisions}} \\ &= \frac{\overline{v}\,t}{n\,\sigma\,\overline{v}\,t} \\ &= \frac{1}{n\,\sigma} \end{aligned} \tag{1.16}$$

The problem with this derivation is that the target is assumed stationary, while it really moves with mean velocity $\overline{v}$. What the target molecule *sees* is actually the relative velocity of the incoming molecules, not their absolute velocity. Thus, the mean relative velocity $\overline{v}_{rel}$ must be substituted to the mean velocity $\overline{v}$ in equation (1.15). This relative velocity may be estimated as follows:

$$\overline{\boldsymbol{v}}_{rel} = \boldsymbol{v}_1 - \boldsymbol{v}_2$$

Thus:

$$\begin{aligned} \overline{v}_{rel} &= \sqrt{(\boldsymbol{v}_1 - \boldsymbol{v}_2) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_2)} \\ &= \sqrt{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 - 2\,\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \boldsymbol{v}_2 \cdot \boldsymbol{v}_2} \end{aligned}$$

Since $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are random and uncorrelated,

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0$$

yielding

$$\overline{v}_{rel} = \sqrt{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{v}_2}$$

Since the same average velocity would be associated with each molecule, this becomes

$$\overline{v}_{rel} = \overline{v}\,\sqrt{2}$$

yielding, once substituted in (1.16):

$$\lambda = \frac{1}{\sqrt{2}\,n\,\sigma} \tag{1.17}$$

Chapter 2

Hydrostatics

Let's consider the simpler case of incompressible fluids, i.e., fluids with a constant density ρ. By definition, a fluid at rest can not sustain shear- and traction-stresses. Only compressive stresses are *allowed*. Such compressive stress is named *pressure*, p.

Using Cauchy's tetrahedron (figure 2.1) it is easily shown that pressure is a scalar. In other words, at a given point in a fluid at rest, pressure is independent of orientation.

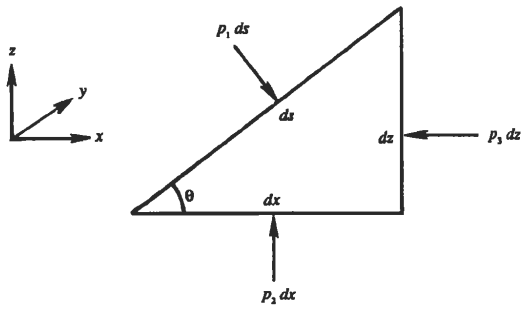

Figure 2.1: Cauchy's tetrahedron.

The force-balance equation for a generic region Ω within a fluid at rest is:

$$-\int_{\partial\Omega} p\,\boldsymbol{n}\,\mathrm{d}S + \int_{\Omega} \rho\,\boldsymbol{g}\,\mathrm{d}V = 0 \tag{2.1}$$

Using Gauss' divergence theorem yields:

$$-\int_{\Omega} \nabla p \, \mathrm{d}V + \int_{\Omega} \rho \boldsymbol{g} \, \mathrm{d}V = 0 \tag{2.2}$$

or, equivalently,

$$\nabla p = \rho \boldsymbol{g} \tag{2.3}$$

Thus, in a fluid at rest, a pressure variation exists only along the vertical direction, with pressure increasing downwards (*Pascal's law*).

2.1 Simple applications

2.1.1 Free surface of a liquid in a rotating, cylindrical tank

Make reference to figure 2.2.

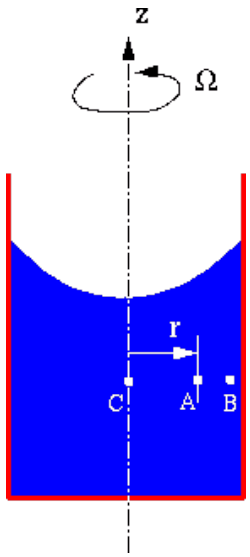

Figure 2.2: Liquid in a rotating tank.

A generic fluid element, of volume $\mathrm{d}V$, is in equilibrium under the action of three forces:

1. Weight:

$$\rho\,\boldsymbol{g}\,\mathrm{d}V$$

2. Inertia (centrifugal) force:

$$\rho\,\Omega^2\,\boldsymbol{r}\,\mathrm{d}V$$

3. Pressure force from surrounding fluid:

$$-\nabla p\,\mathrm{d}V$$

Thus, the equilibrium condition yields:

$$\nabla p = \rho\,\boldsymbol{g} + \rho\,\Omega^2\,\boldsymbol{r}$$

Pressure is uniform on the free surface, as it equals the atmospheric pressure. Thus, ∇p is orthogonal to the free surface. Let's consider the parametric form of the intersection curve between the free surface and a radial plane:

$$\begin{aligned} r &= r \\ z &= z(r) \end{aligned}$$

A tangent vector to the curve is:

$$\boldsymbol{\tau} = \widehat{\boldsymbol{r}} + \frac{\mathrm{d}z}{\mathrm{d}r}\,\widehat{\boldsymbol{k}}$$

Thus:

$$\nabla p \cdot \boldsymbol{\tau} = 0 \Longrightarrow \rho\,\Omega^2\,r - \rho\,g\,\frac{\mathrm{d}z}{\mathrm{d}r} = 0$$

which, in turn, yields:

$$\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{\Omega^2\,r}{g}$$

The free surface is a paraboloid of equation:

$$z(r) - z_0 = \frac{\Omega^2\,r^2}{2\,g}$$

The value of z_0 results from mass conservation. Let's consider a cylindrical tank with circular cross-section and denote by H the height of the liquid in the tank at rest, before rotation is initiated. Mass conservation dictates:

$$\pi\, r_0^2\, H = \int_0^{2\pi} \int_0^{r_0} \int_0^{z(r)} r \,\mathrm{d}z \,\mathrm{d}r \,\mathrm{d}\vartheta$$

Integrating and solving for z_0 yields:

$$z_0 = H - \frac{\Omega^2\, r_0^2}{4\, g}$$

2.1.2 Measurement principle and dynamic behaviour of a U-tube manometer

A U-tube manometer (figure 2.3) is used to measure the pressure of fluids contained within tanks or flowing through pipes. The tube's diameter must be large enough to allow neglecting capillarity effects. Recalling Pascal's principle, we know that the hydrostatic pressure within a homogeneous liquid is uniform on horizontal planes (in gravitational field). Thus:

$$p_A = p_{atm} + \rho_2\, g\, \Delta z - \rho_1\, g\, (z_A - z_1) \tag{2.4}$$

Whenever the *pressurized* fluid is much lighter than the manometric fluid (e.g., air is the pressurized fluid, mercury is the manometric fluid) the formula above may be simplified as

$$p_A = p_{atm} + \rho_2\, g\, \Delta z \tag{2.5}$$

Whenever a U-tube manometer is connected to a positive-pressure source, as a pressurized tank, the manometric liquid column is impulsively accelerated and, due to its inertia and weak friction resistance, oscillates around its equilibrium position. Let's support this intuition by devising a physical, dynamic model for the manometer. Using the mechanical energy conservation theorem yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(K + U) = \dot{W}$$

$$K = \frac{1}{2}\, m\, v^2$$

$$U = (\rho\, a\, z)\; g\, z$$

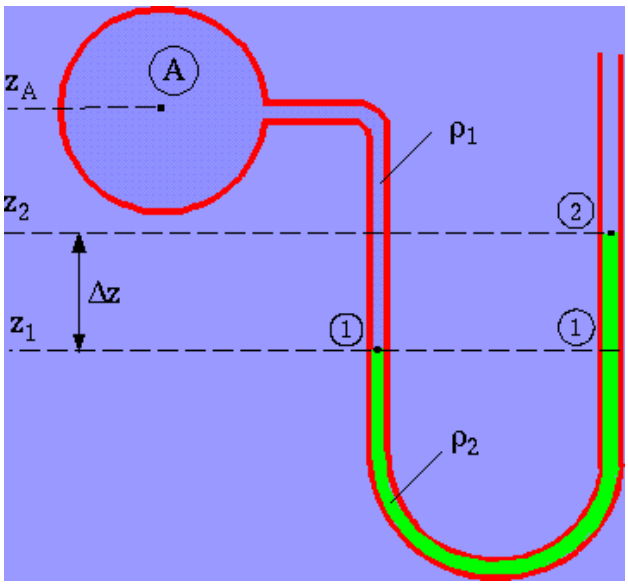

Figure 2.3: Sketch of a U-tube manometer.

$$\dot{W} = (p_A - p_{atm})\, a\, v - \tau_w\, P\, l\, v$$

where K and U denote the kinetic and potential energy of the liquid column, respectively. $\dot{W}$ denotes the work rate of other forces (pressure forces and viscous, wall shear-stress). The mass of the liquid column, m, is given by

$$m = \rho\, a\, l$$

where a is the cross-section's area of the tube and l is the length of the liquid column. z denotes the z-displacement of the free-surface of the liquid column, exposed to atmospheric pressure. The velocity of the liquid column is $v \equiv \dot{z}$. The wall shear-stress τ_w depends linearly on v, under laminar flow conditions,

$$\tau_w \equiv c\, v; \quad c \geq 0$$

Substituting and rearranging yields:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} + 2\,\xi\,\frac{\mathrm{d}z}{\mathrm{d}\tau} + z = f_0 \tag{2.6}$$

where the non-dimensional time τ was introduced, $\tau \equiv \omega_n\, t$, with $\omega_n \equiv \sqrt{2\,g/l}$ *natural* angular frequency of the system. The dissipation coefficient

ξ is defined as

$$\xi \equiv \frac{1}{2} \frac{c\,P}{\rho\,a\,\omega_n}$$

and the pressure-force term is represented by f_0,

$$f_0 \equiv \frac{p_A - p_{atm}}{2\,\rho\,g}$$

Equation (2.6) represents a *second-order dynamical system.* It is well known that, according to the magnitude of the dissipation coefficient ξ, the system may either oscillate, with progressively reduced amplitude, or tend exponentially towards its final equilibrium state.

Chapter 3

Hydrostatic forces on submerged surfaces

3.1 Brief overview on *equivalent* systems of applied vectors

Let us denote by $\{\boldsymbol{f}_h; P_h\}$ a set of applied vectors. Vector $\boldsymbol{f}_h$ is applied on point P_h. In the followings we are going to refer to these vectors as *forces*, as, in practice, this is the most common case in Mechanics.

The *resultant force* vector $\boldsymbol{R}$ is defined as:

$$\boldsymbol{R} \equiv \sum_h \boldsymbol{f}_h \tag{3.1}$$

The *resultant torque* vector w.r.t. a *pole* A is defined as

$$\boldsymbol{M}_A \equiv \sum_h \vec{AP}_h \times \boldsymbol{f}_h \tag{3.2}$$

Two systems of applied vectors are said to be *equivalent* if:

1. They have the same resultant.
2. They have the same resultant torque w.r.t. any pole.

Theorem 3.1.1. *(Torque transposition)*

$$\boldsymbol{M}_B = \vec{BA} \times \boldsymbol{R} + \boldsymbol{M}_A \tag{3.3}$$

Proof.

$$\begin{aligned} \boldsymbol{M}_B &= \sum_h \vec{BP}_h \times \boldsymbol{f}_h \\ &= \sum_h \vec{BA} \times \boldsymbol{f}_h + \sum_h \vec{AP}_h \times \boldsymbol{f}_h \\ &= \vec{BA} \times \boldsymbol{R} + \boldsymbol{M}_A \end{aligned}$$

□

Remark 3.1.2. *The torque transposition theorem yields the consequence that if two systems of applied forces have the same resultant force* $\boldsymbol{R}$ *and the same resultant torque w.r.t. a* specific *pole, then they have the same resultant torque w.r.t any pole and, therefore, are equivalent.*

Two equivalent systems of forces applied to a rigid body cause the same motion. Thus, complex systems of forces can be conveniently reduced to simpler, equivalent ones without affecting the motion of rigid bodies, they are applied onto. In the following we show how a three-dimensional system of forces can be reduced to its resultant force vector and a torque of minimum magnitude.

3.1.1 Central axis

The *invariant trinomial* T is defined as:

$$T \equiv \boldsymbol{M}_A \cdot \boldsymbol{R} \tag{3.4}$$

This definition is well-posed as, using theorem 3.1.1, it can be shown that the choice of the pole is immaterial. This, in turn, implies that $\boldsymbol{M}_A$ depends on the pole A but its projection on $\boldsymbol{R}$ does not.

Let:

$$T = M_A \, R \cos\vartheta$$

where ϑ is the angle between $\boldsymbol{M}_A$ and $\boldsymbol{R}$. Assuming $R \neq 0$, the minimum resultant torque is obtained whenever A is chosen such that $\boldsymbol{M}_A$ and $\boldsymbol{R}$ are either parallel or anti-parallel, yielding:

$$M_{min} = \frac{|T|}{R} \tag{3.5}$$

We aim to find a point C, if any, yielding $\|\boldsymbol{M}_C\|_2 = M_{min}$. To this end, consider the coordinate system sketched in figure 3.1. In this particular coordinate system we have:

$$\boldsymbol{R} = R\,\boldsymbol{k}; \quad \boldsymbol{M}_A = M_y\,\boldsymbol{j} + M_z\,\boldsymbol{k}$$

where the definition of the normal units vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ is provided in figure 3.1. We look for a point $C \equiv (x, y, z)$ such that $\boldsymbol{M}_C$ is parallel or anti-parallel to $\boldsymbol{R}$. According to the transposition theorem:

$$\boldsymbol{M}_C = \boldsymbol{M}_A + \vec{CA} \times \boldsymbol{R} = M_y\,\boldsymbol{j} + M_z\,\boldsymbol{k} + (x\,\boldsymbol{j} - y\,\boldsymbol{i})\; R$$

Then, consider:

$$\boldsymbol{M}_C \parallel \boldsymbol{R} \Longrightarrow \begin{cases} M_y + x\,R &= 0 \\ y\,R &= 0 \end{cases}$$

yielding the position of point C as:

$$C = \left(-\frac{M_y}{R},\, 0,\, z\right)$$

where z is arbitrary. In other words:

$$C = A - \frac{M_y}{R}\,\boldsymbol{i} + z\,\boldsymbol{k} \tag{3.6}$$

This is the equation of the central axis. The resultant torque w.r.t. to any pole C chosen on the central axis is

$$\boldsymbol{M}_C = \frac{T}{R}\,\boldsymbol{k}$$

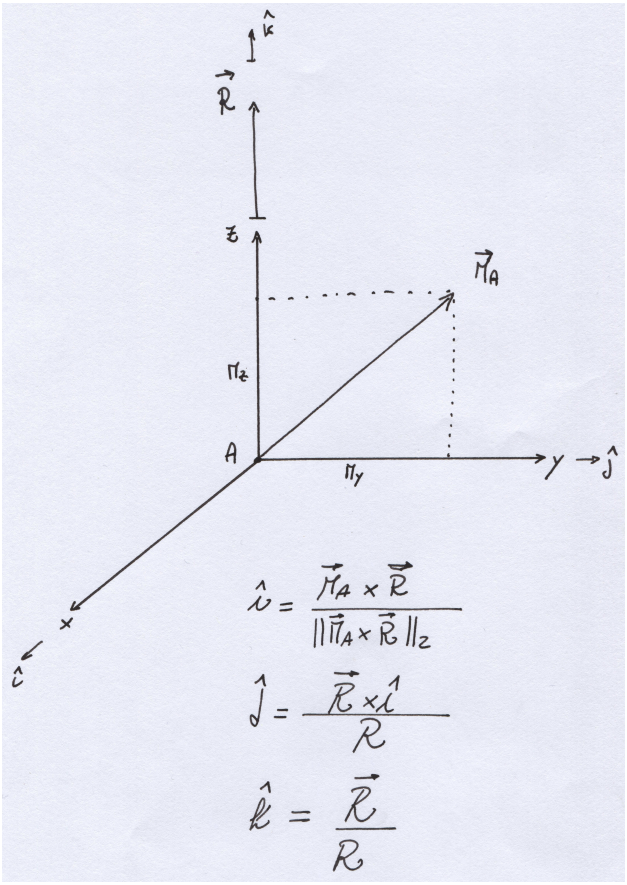

Figure 3.1: Coordinate system used to calculate the location of the central axis.

Remark 3.1.3. *For planar systems, where all force vectors lie in a plane π,*

- *The resultant force lies on π.*

- *The resultant torque is orthogonal to* π.
- $T = 0$ *and* $M_{min} = 0$.
- *The central axis lies on* π.

3.2 Pressure force on immersed surfaces

In an hydrostatic pressure field of an incompressible fluid, the pressure varies linearly with depth:

$$p = p_0 + \rho\, \boldsymbol{g} \cdot (\boldsymbol{x} - \boldsymbol{x}_0)$$

where $\boldsymbol{x}_0$ belongs to the free-surface, where $p \equiv p_0$.

Let us consider a **closed** surface $\partial\Omega$ bounding a solid body Ω of volume V. The resultant pressure force on $\partial\Omega$ is:

$$\boldsymbol{R}_p = \int_{\partial\Omega} (-p\, \boldsymbol{n})\ \mathrm{d}S = -\int_{\Omega} \nabla p\, \mathrm{d}V \tag{3.7}$$

Substituting the hydrostatic pressure field yields:

$$\boldsymbol{R}_p = -\rho\, \boldsymbol{g}\, V \tag{3.8}$$

$\boldsymbol{R}_p$ is the **buoyancy force** in an hydrostatic field, as stated by Archimede's principle.

Let us compute the resultant torque of the pressure forces acting on $\partial\Omega$, w.r.t. a generic pole A[1]

$$\boldsymbol{M}_A = -\int_{\partial\Omega} \boldsymbol{x} \times (p\, \boldsymbol{n})\ \mathrm{d}S = \int_{\Omega} \nabla \times (p\, \boldsymbol{x})\ \mathrm{d}V \tag{3.9}$$

[1]The vector identity used in (3.9) can be easily proved - working in Cartesian coordinates - using Einstein's summation convection and the properties of the permutation tensor.

$$\nabla \times (p\, \boldsymbol{x}) = \nabla p \times \boldsymbol{x} + p\, \underbrace{\nabla \times \boldsymbol{x}}_{\equiv \boldsymbol{0}}$$

$$\begin{aligned} \int_{\Omega} \nabla \times \boldsymbol{f}\, \mathrm{d}V &= \boldsymbol{e}_i \int_{\Omega} \frac{\partial}{x_j} \left(e_{i\,j\,k} f_k\right)\, \mathrm{d}V \\ &= \int_{\partial\Omega} \boldsymbol{e}_i\, e_{i\,j\,k} f_k\, n_j\, \mathrm{d}S = -\int_{\partial\Omega} \boldsymbol{f} \times \boldsymbol{n}\, \mathrm{d}S \end{aligned}$$

Taking $\boldsymbol{f} \equiv p\, \boldsymbol{x}$ makes the deal.

Assuming that $\boldsymbol{g}$ is oriented along the positive z direction, so that $p = p(z)$, it turns out that $\boldsymbol{M}_A$ lies on the (x, y) plane:

$$\boldsymbol{M}_A = (\rho\, g\, V)\ [x_G\, \boldsymbol{j} - y_G\, \boldsymbol{i}]$$

where G denotes the centroid of the immersed volume. Therefore, the system of hydrostatic pressure forces acting on $\partial\Omega$ does not tend to spin the body about a vertical axis.

3.3 Area coordinates for planar triangles

Area coordinates for planar triangles: short intro

3.3.1 Using MATLAB to verify the mapping between Cartesian and area cooridnates

The symbolic toolbox by MATLAB is rather useful to verify that, for a linear triangle,

$$\boldsymbol{x} = L_1\, \boldsymbol{x}_1 + L_2\, \boldsymbol{x}_2 + L_3\, \boldsymbol{x}_3$$

Indeed, try this:

```
>> syms x y x1 x2 x3 y1 y2  y3
>> L1 = det([x y 1;x2 y2 1;x3 y3 1])/det([x1 y1 1;x2 y2 1;x3 y3 1]);
>> L2 = det([x y 1;x3 y3 1;x1 y1 1])/det([x1 y1 1;x2 y2 1;x3 y3 1]);
>> L3 = det([x y 1;x1 y1 1;x2 y2 1])/det([x1 y1 1;x2 y2 1;x3 y3 1]);
>> simplify(L1)

ans =

(x*y2 - x2*y - x*y3 + x3*y + x2*y3 - x3*y2)/(x1*y2 - x2*y1 - x1*y3 + x3*y1 + x2*y3 - x3*y2)

>> simplify(L1*x1+L2*x2+L3*x3)

ans =

x

>> simplify(L1*y1+L2*y2+L3*y3)

ans =

y
```

3.4 Numerical calculation of the resultant force and resultant torque on an immersed, triangulated surface

The surface is represented by a tessellation of triangular facets, sharing common edges. The geometrical characteristics of a triangulated surface are contained in STL files, that have the following general format:

```
solid tetrahedron
facet normal 0 1 0
outer loop
vertex 0 0 0
vertex 1 1 0
vertex 1 0 0
endloop
endfacet
facet normal 0 1 0
outer loop
vertex 1 1 0
vertex 1 1 1
vertex 1 0 0
endloop
endfacet
facet normal 0 1 0
outer loop
vertex 0 0 0
vertex 1 0 0
vertex 1 1 1
endloop
endfacet
facet normal 0 1 0
outer loop
vertex 1 1 1
vertex 1 1 0
vertex 0 0 0
endloop
endfacet
endsolid tetrahedron
```

Thus, an STL file contains information about:

- The coordinates of all vertices of each facet in the tessellation.
- The normal vector to any facet in the tessellation.

Notice that indices are repeated, as they are listed as many times as are the triangles containing them.

There exist several tools to visualize, check and repair STL files, as, e.g.,

- MeshLab

- ViewStl

The resultant pressure-force acting on a triangular facet can be easily computed as follows:

$$\boldsymbol{R}_p = \int_{\triangle} -p\,\boldsymbol{n}\,\mathrm{d}S = -\boldsymbol{n}\int_{\triangle} p\,\mathrm{d}S$$

When p is a **hydrostatic** pressure field:

$$\boldsymbol{R}_p = -\boldsymbol{n}\int_{\triangle} \left[p_0 + \rho\,\boldsymbol{g}\cdot(\boldsymbol{x}-\boldsymbol{x}_0)\right]\,\mathrm{d}S$$

$$\boldsymbol{x}' \equiv \boldsymbol{x} - \boldsymbol{x}_0$$

$$\boldsymbol{R}_p = -\boldsymbol{n}\int_{\triangle} \left[p_0 + \rho\,\boldsymbol{g}\cdot\boldsymbol{x}'\right]\,\mathrm{d}S = -\boldsymbol{n}\,S\,\left[p_0 + \rho\,\boldsymbol{g}\cdot\boldsymbol{x}'_G\right]$$

where $\boldsymbol{x}'_G$ denotes the position of the centroid of the triangular surface (in the translated coordinate system):

$$\boldsymbol{x}'_G \equiv \frac{1}{S}\int_{\triangle} \boldsymbol{x}'\,\mathrm{d}S$$

For a linear triangle, the position of the centroid is obtained as arithmetic average of the positions of the vertices[2]:

$$\boldsymbol{x}_G = \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2 + \boldsymbol{x}_3}{3} \tag{3.10}$$

Using the surface area vector $\boldsymbol{S} \equiv S\,\boldsymbol{n}$ yields:

$$\boldsymbol{R}_p = -\boldsymbol{S}\,p_G \tag{3.11}$$

where p_G is the hydrostatic pressure at the depth of the centroid.[3]

[2]To prove this assert, calculate $\boldsymbol{x}_G$ from the definition of *centroid*, carrying out the quadrature in the transformed space (L_1, L_2).

[3]The mid-point quadrature rule is exact for any linear functions $f(\boldsymbol{x})$:

$$\begin{aligned}
\textstyle\int_\Omega f\,\mathrm{d}\Omega &= \textstyle\int_\Omega f(\boldsymbol{G}) + \nabla f|_{\boldsymbol{G}}\cdot(\boldsymbol{x}-\boldsymbol{G})\,\mathrm{d}\Omega \\
&= f(\boldsymbol{G})\,\Omega + \nabla f|_{\boldsymbol{G}}\cdot\underbrace{\textstyle\int_\Omega (\boldsymbol{x}-\boldsymbol{G})\,\mathrm{d}\Omega}_{=0} \\
&= f(\boldsymbol{G})\,\Omega
\end{aligned}$$

The torque acting on the triangular facet is:

$$\boldsymbol{M}_A = \int_{\triangle} (\boldsymbol{x} - A) \times (-p\,\boldsymbol{n})\,\mathrm{d}S$$

$$\boldsymbol{x}'' \equiv \boldsymbol{x} - A$$

$$\begin{aligned} \boldsymbol{M}_A &= \int_{\triangle} \boldsymbol{x}'' \times (-p\,\boldsymbol{n})\,\mathrm{d}S \\ &= \boldsymbol{n} \times \int_{\triangle} p\,\boldsymbol{x}''\,\mathrm{d}S \end{aligned} \tag{3.12}$$

Let $L_1(\boldsymbol{x}'')$, $L_2(\boldsymbol{x}'')$, $L_3(\boldsymbol{x}'')$ denote the *area-coordinates* of a linear triangle. Both p and $\boldsymbol{x}''$ can be interpolated at any point within the triangle using area coordinates as:

$$p(\boldsymbol{x}'') = \sum_{j=1}^{3} L_j(\boldsymbol{x}'')\,p(\boldsymbol{x}''_j); \quad \boldsymbol{x}'' = \sum_{j=1}^{3} L_j(\boldsymbol{x}'')\,\boldsymbol{x}''_j \tag{3.13}$$

Substituting into (3.12) yields, using Einstein's summation convention:

$$\begin{aligned} \boldsymbol{M}_A &= \boldsymbol{n} \times \int_{\triangle} p_j\,L_j(\boldsymbol{x}'')\,\boldsymbol{x}''_k\,L_k(\boldsymbol{x}'')\,\mathrm{d}S \\ &= \boldsymbol{n} \times \left[p_j\,\boldsymbol{x}''_k \int_{\triangle} L_j(\boldsymbol{x}'')\,L_k(\boldsymbol{x}'')\,\mathrm{d}S \right] \end{aligned} \tag{3.14}$$

The following identity holds for linear triangles:

$$\int_{\triangle} L_1^a\,L_2^b\,L_3^c\,\mathrm{d}S = 2\,S\,\frac{a!\,b!\,c!}{(a+b+c+2)!} \tag{3.15}$$

where S is the triangle's area (it can be computed in several ways, e.g., as half of the modulus of the cross product of two edges). It can be easily recognized that the following simpler expression holds:

$$\int_{\triangle} L_j(\boldsymbol{x}'')\,L_k(\boldsymbol{x}'')\,\mathrm{d}S = (1+\delta_{j\,k})\,\frac{S}{12} \tag{3.16}$$

The resulting expression for $\boldsymbol{M}_A$ is (again, using Einstein's summation convention):

$$\boldsymbol{M}_A = \boldsymbol{S} \times \sum_{j,k=1,2,3} \left[p_j \, \boldsymbol{x}''_k \, (1 + \delta_{j\,k}) \, \frac{S}{12} \right] \tag{3.17}$$

Notice that (3.12) is a general, exact expression for the resultant torque of a pressure field acting on the triangle. As long as the interpolation (3.13) is used, equation (3.17) is only approximate. Being the hydrostatic pressure field linear in the coordinates, the interpolation (3.13) is exact and so is (3.17).

A MATLAB code implementing the algorithm outlined in this section can be found in hydrostatics.m. This figure shows the central axis calculated for a triangulated geometry of a Porsche car. The color pattern corresponds to the absolute pressure distribution on the car, in kPa.

Homework 3.4.1. *In this Homework the Student is requested to:*

1. *Implement his own version of the algorithm for the calculation of the action of a hydrostatic pressure field on an immersed surface.*
2. *Validate the aforementioned implementation calculating the resultant force and resultant torque for a submerged, planar bulkhead, inclined at* $30°$ *w.r.t. the vertical. Assume that:*
   (a) *The bulkhead's dimensions are* $W = 10m$ *(horizontal) and* $H = 5m$ *(sub-vertical).*
   (b) *Fluid's density:* $\rho = 1000kg/m^3$.
   (c) *Free-surface pressure:* $p_0 = 1bar$.
   (d) $g = 9.81m/s^2$.
3. *Assume the bulkhead must be kept in position by a horizontal force, applied along its lower, horizontal edge. Provide the intensity of this force in kN. Consider the configuration shown in figure 3.2.*

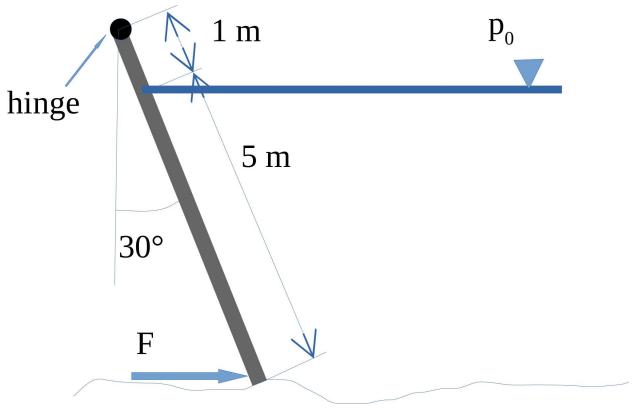

Figure 3.2: Sketch of the bulkhead.

Bibliography