Thermal network

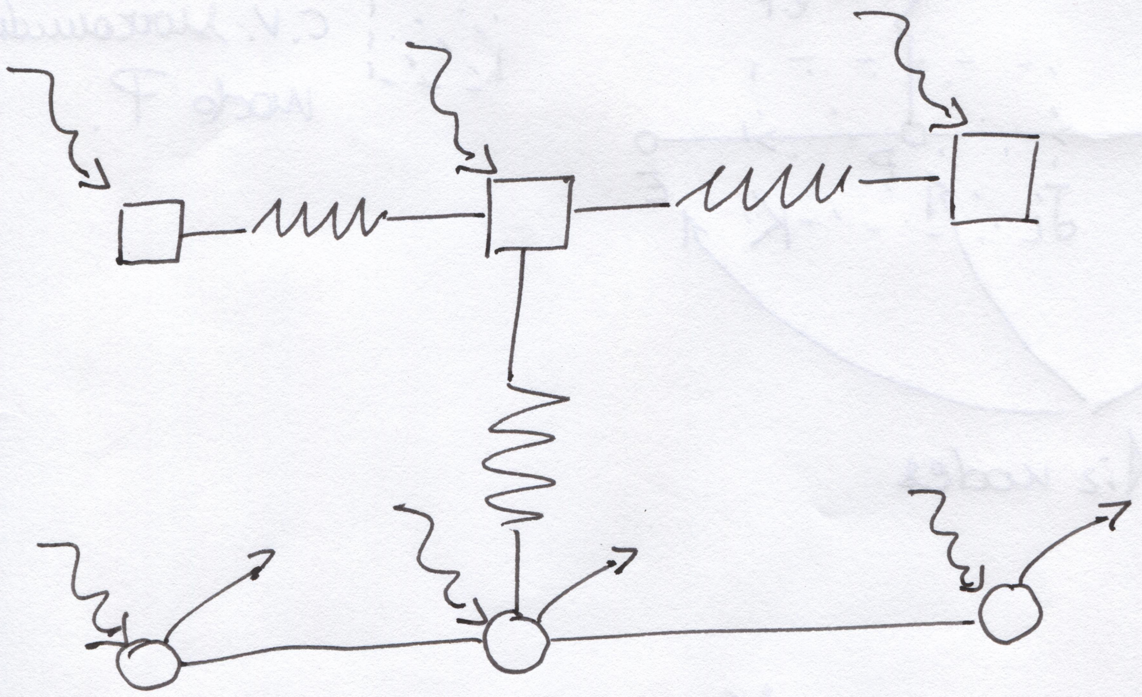
We have to deal with either "solid" nodes or "air" nodes.

Solid nodes represent "blocks" of solid material with nearly uniform temperature. E.g., the end turns or parts of them.

Air nodes represent parts of air channels with nearly uniform temperature.

The Finite Volume approach is the most natural approach to model this problem.

A representative sketch of a thermal network:



thermal resistance (conduction, convection, radiation)

mass flow out of node

solid node

air node

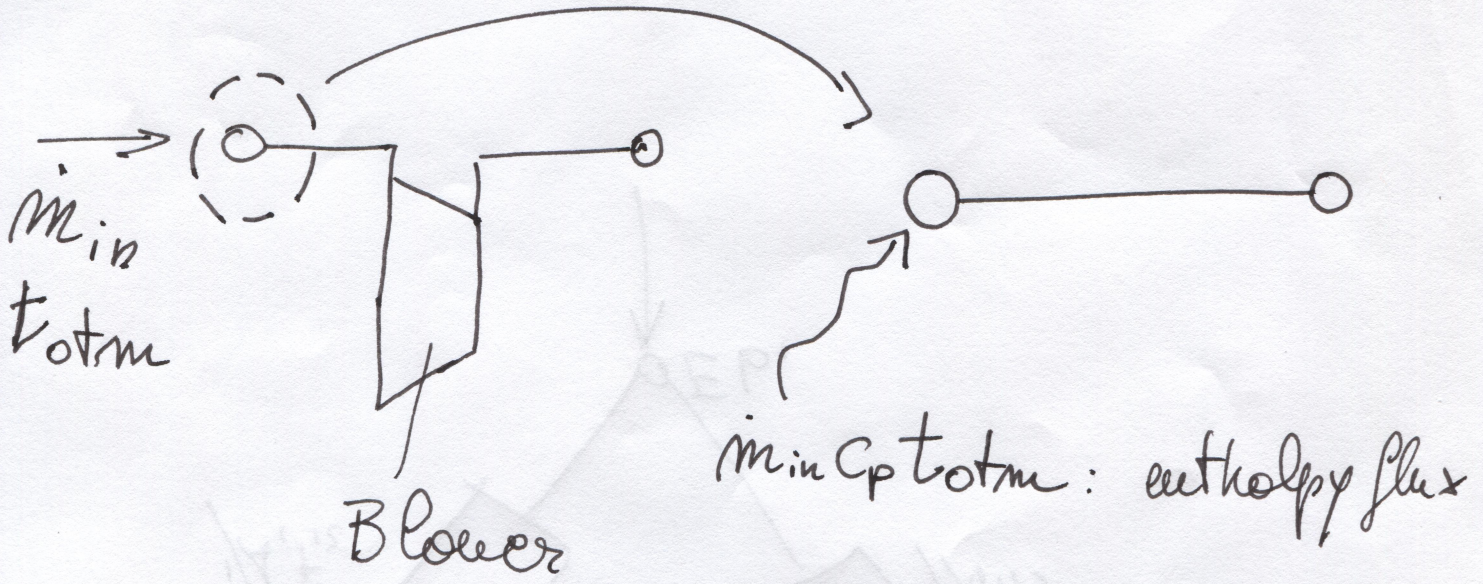
air channel

heat input into node (with sign)

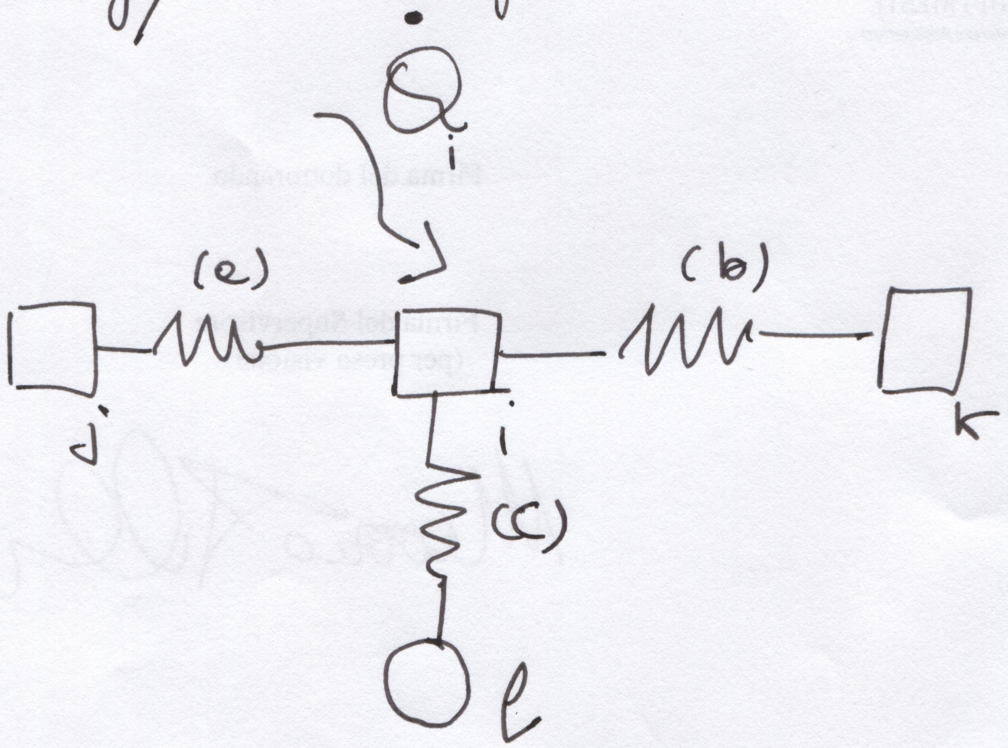
Remark

Heat input into node includes also enthalpy fluxes into the node.

Consider for instance an entry node into the pneumatic network:

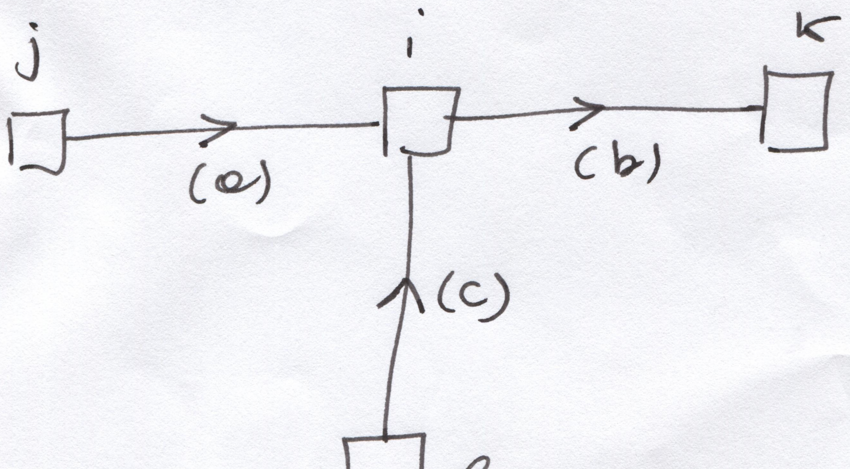


Energy balance for solid node:



$$0 = \dot{Q}_i + \frac{T_j - T_i}{R_t^{(a)}} + \frac{T_k - T_i}{R_t^{(b)}} + \frac{T_l - T_i}{R_t^{(c)}}$$

Network representation as D-graph:



$\dot{q}^{(a)}$: heat flowing along thermal resistance $R_t^{(a)}$. $\dot{q}^{(a)} > 0$ if heat flows from first to second node of the branch. $\dot{q}^{(a)} < 0$ otherwise

Energy balance for solid node "i" reads:

$$\sum_{a=1}^{n} \underbrace{\dot{q}^{(a)} C_{ai}}_{\text{heat flowing into node "i" from thermal resistance } R_t^{(a)}} + Q_i = 0$$

Also: $\dot{q}^{(a)} = -\sum_{k=1}^{N} \dfrac{C_{ak} T_k}{R_t^{(a)}}$

Thus, the energy balance for solid node "i" reads:

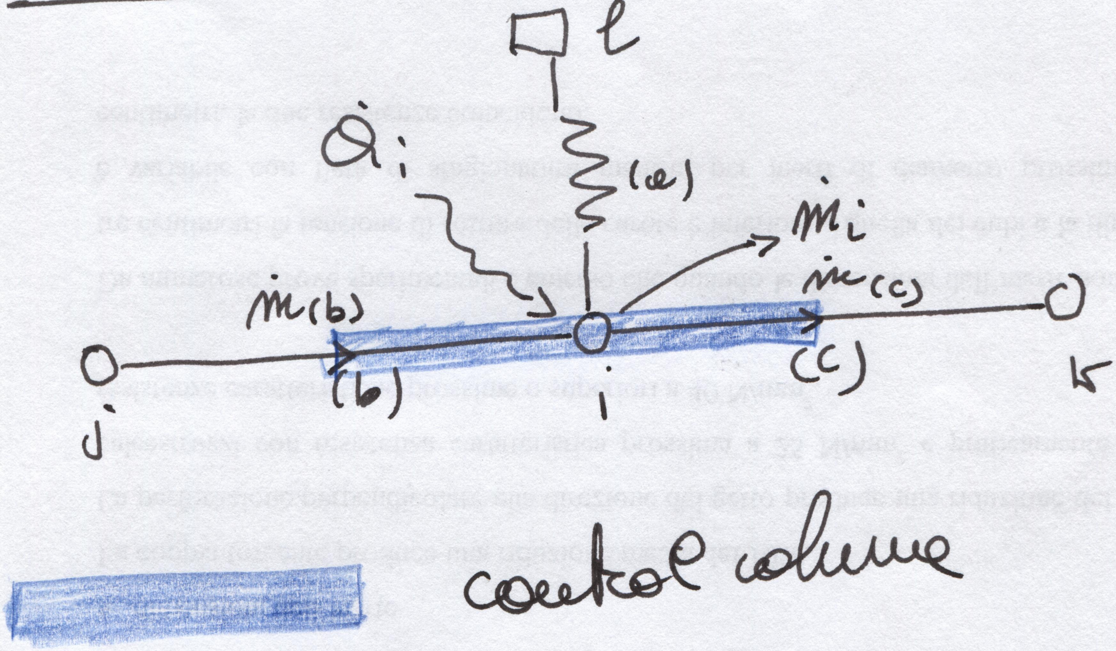
$$Q_i - \sum_{a=1}^{n} C_{ai} \sum_{k=1}^{N} \frac{C_{ak} T_k}{R_t^{(a)}} = 0$$

In matrix form:

$$[C]^T \lceil K_t \rfloor [C] \{T\} = \{\dot{Q}\} \quad (B1)$$

where $\lceil K_t \rfloor = \lceil 1/R_t \rfloor$ matrix of thermal conductances.

Energy balance for air nodes



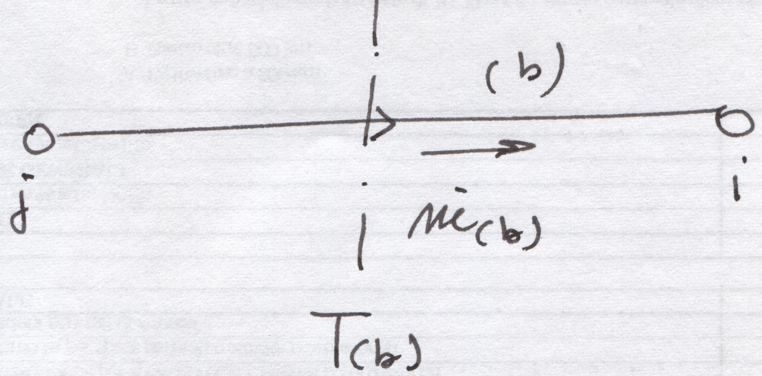
$$\dot{Q}_i + \dot{m}_{(b)} c_p T_{(b)} - \dot{m}_{(c)} c_p T_{(c)}$$

$$- \dot{m}_i c_p T_i + \frac{T_e - T_i}{R_t^{(a)}} = 0$$

$T_{(b)}$: air temperature at control-volume face located at middle section of branche "b".

$T_{(c)}$: similar to $T_{(c)}$, but for branche "c".

Several possible approaches to approximate $T_{(b)}$ and $T_{(c)}$. We choose the <u>upwind</u> strategy:



$$T_{(b)} \equiv T_j$$ where j is the <u>upstream</u> node.

This corresponds to <u>neglect the axial diffusion.</u>

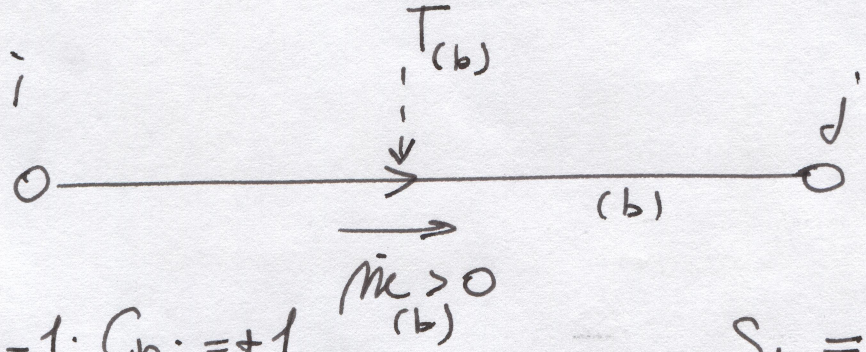
Automatic calculation of $T_{(b)}$:

$$T_{(b)} = \sum_j^N \max\left(-C_{bj} S_b, 0\right) T_j$$

where

$$S_b \equiv \cancel{\max\left(\dot{m}_{(b)}, 0\right)} \ \mathrm{sign}\left(\dot{m}_{(b)}\right).$$

Let's check this formula:



$C_{bi} = -1$; $C_{bj} = +1$. $\qquad S_b = +1$

$$\max\left(-C_{bi} S_b, 0\right) = \max\left(-(-1)\cdot(+1), 0\right) = 1$$

$$\max\left(-C_{bj} S_b, 0\right) = \max\left(-(+1)\cdot(+1), 0\right) = 0$$

$$T_{(b)} = 1 \cdot T_i + 0 \cdot T_j = T_i$$

<u>OKAY!</u>

The net enthalpy flux <u>into</u> node "i" from all incident air channels can be computed automatically as:

$$\sum_{b}^{n} \dot{m}_{(b)} c_p T_{(b)} C_{bi} =$$

$$= \sum_{b}^{n} \dot{m}_{(b)} c_p C_{bi} \sum_{j}^{N} \max\left(-C_{bj} S_{b,0}\right) T_j$$

In matrix form:

$$[C]^T \ulcorner \dot{m}c_\lrcorner \ [\tilde{C}] \{T\}$$

where:

$\ulcorner \dot{m}c_\lrcorner \in \mathbb{R}^{n \times n}$ diagonal matrix

$$\dot{m}c_{rr} = \dot{m}_{(r)} c_p$$

$$[\tilde{C}] = \max\left(-\ulcorner S_\lrcorner [C], [\phi]\right)$$

with

$\ulcorner S_\lrcorner \in \mathbb{R}^{n \times n}$ diagonal matrix

$$S_{rr} \equiv \text{sign}\left(\dot{m}_{(r)}\right) \Rightarrow \lceil S \rfloor = \text{sign} \lceil \dot{M} C \rfloor$$

$[\phi]$ is the identically-zero, $M \times N$ matrix.

The matrix form for the energy balance of an air node is thus:

$$\{\dot{Q}\} + [C]^T \lceil \dot{M} C \rfloor [\breve{C}] \{T\} - \lceil \dot{m} c \rfloor \{T\} - [C]^T \lceil K_t \rfloor [C] \{T\} = 0 \qquad (B2)$$

where

$$\lceil mc \rfloor \in \mathbb{R}^{N \times N}$$

$$mc_{ii} = \begin{cases} \dot{m}_i c_p & \text{if } \dot{m}_i > 0 \\ 0 & \text{if } \dot{m}_i \leq 0 \end{cases}$$

Thus $\dot{m}c_{ii} \neq 0$ only where air is withdrawn from the node "i".

Comparing (B1) and (B2) it turns out that (B2) is more general and holds both for solid and for air nodes, provided:

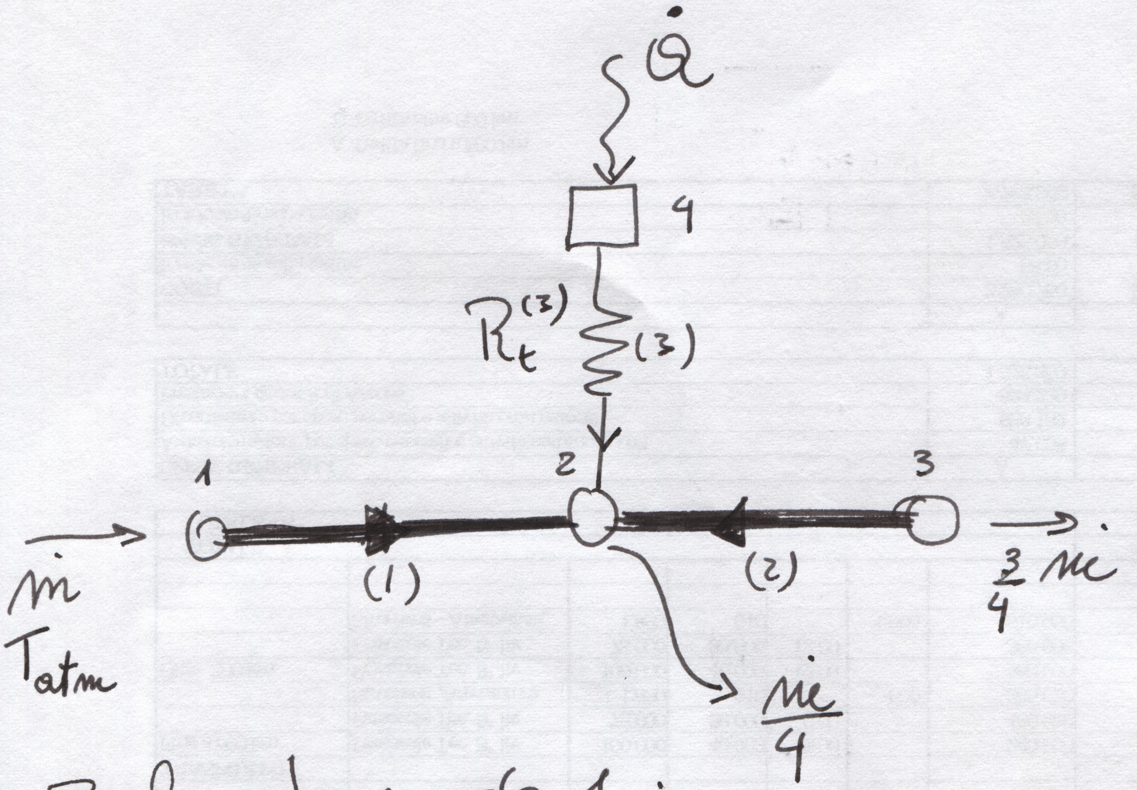
a) $\dot{m}_{(r)}$ is set equal to zero for a branch "r" representing a thermal resistance.

b) K_t is set equal to zero for branches corresponding to air channels.

c) $\dot{Q}_i$ encomposses both heat fluxes into node "i" by conduction/convection/radiation and enthalpy fluxes into node "i".

d) mc set to zero for solid nodes.

Example



Balance for node 1:

$$\dot{m} c_p T_{atm} - \dot{m} c_p T_1 = 0 \Rightarrow T_1 = T_{atm}$$

Balance for node 2:

$$\dot{m} c_p T_1 - \frac{3}{4}\dot{m} c_p T_2 - \frac{\dot{m}}{4} c_p T_2 + \frac{T_4 - T_2}{R_t^{(3)}} = 0$$

Balance for node 3:

$$\frac{3}{4}\dot{m} c_p T_2 - \frac{3}{4}\dot{m} c_p T_3 = 0 \Rightarrow T_3 = T_2$$

Balance for node 4:

$$\dot{Q} + \frac{T_2 - T_4}{R_t^{(3)}} = 0$$

We need to solve a 2x2 system for T_2 and T_4:

$$\begin{bmatrix} -\dot{m}c_p - \frac{1}{R_t^{(3)}} & \frac{1}{R_t^{(3)}} \\ \frac{1}{R_t^{(3)}} & -\frac{1}{R_t^{(3)}} \end{bmatrix} \begin{bmatrix} T_2 \\ T_4 \end{bmatrix} = \begin{bmatrix} -\dot{m}c_p T_{otm} \\ -\dot{Q} \end{bmatrix}$$

$$\Rightarrow T_2 = T_{otm} + \frac{\dot{Q}}{\dot{m}c_p}$$

$$T_4 = T_{otm} + \dot{Q}\left[R_t^{(3)} + \frac{1}{\dot{m}c_p}\right]$$

Let's solve the same problem using (B2):

$$\{\dot{Q}\} = \begin{Bmatrix} \dot{m}c_p T_{otm} \\ 0 \\ 0 \\ \dot{Q} \end{Bmatrix}$$

$$[C] = \begin{bmatrix} -1 & +1 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & +1 & 0 & -1 \end{bmatrix}$$

$$\ulcorner \dot{m}C \lrcorner = \begin{bmatrix} \dot{m}c_p & 0 & 0 \\ 0 & -\frac{3}{4}\dot{m}c_p & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ulcorner S \lrcorner = \begin{bmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ulcorner S \lrcorner [C] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 0 & \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\tilde{C}] = \max\left(- \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\ulcorner \dot{m}C \lrcorner = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & +\frac{\dot{m}c_p}{4} & 0 & 0 \\ 0 & 0 & \frac{3}{4}\dot{m}c_p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[K_{tj}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/R_t^{(3)} \end{bmatrix}$$

Thus:

$$\begin{Bmatrix} \dot{m} c_p T_{other} \\ 0 \\ 0 \\ \dot{Q} \end{Bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{m} c_p & 0 & 0 \\ 0 & -\frac{3}{4}\dot{m} c_p & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\dot{m}}{4} c_p & 0 & 0 \\ 0 & 0 & \frac{3}{4}\dot{m} c_p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix}$$

$$- \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/R_t^{(3)} \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

This system yields exactly the same solution as we have already obtained!

I checked this out with MATLAB!