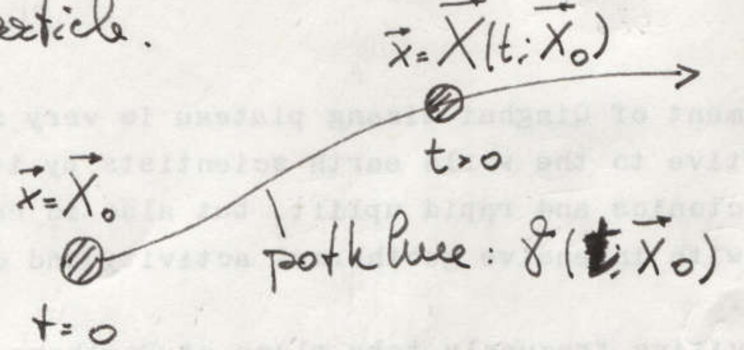
Acoustic propagation equation

- Substantial derivative: time-rate-of-change of a quantity following the motion of a fluid particle.



φ: generic quantity (e.g. density)

$$\varphi=\varphi(\vec{x},t)$$

Following the motion of a fluid particle

$$\varphi\big|_{\gamma(t;\vec{x}_0)}=\varphi(\vec{X}(t;\vec{x}_0),t)$$

$$\frac{d\varphi\big|_{\gamma(t;\vec{x}_0)}}{dt}=\frac{\partial\varphi}{\partial x_k}\bigg|_{(\vec{X}(t;\vec{x}_0),t)}\frac{dX_k}{dt}+\frac{\partial\varphi}{\partial t}$$

$$\frac{dX_k}{dt}=u_k(\vec{X}(t;\vec{x}_0),t)$$

$$\Rightarrow\ \frac{d\varphi\big|_{\gamma(t;\vec{x}_0)}}{dt}=\frac{\partial\varphi}{\partial x_k}u_k+\frac{\partial\varphi}{\partial t}$$

Notation: $$\frac{D\varphi}{Dt}=\frac{d\varphi\big|_{\gamma(t;\vec{x}_0)}}{dt}$$

• Physical significance of $\nabla\cdot\vec{u}$

$\frac{D}{Dt}\int_{\Omega_m} 1\,dV$: time-rate-of-change of the volume of a material particle

Reynolds' transport theorem (Leibniz theorem applied to material volume):

$$\frac{D}{Dt}\int_{\Omega_m} 1\,dV = \int_{\Omega_m}\frac{\partial 1}{\partial t}\,dV + \int_{\partial\Omega_m} 1\,\vec{u}\cdot\hat{n}\,dS$$

$$= \int_{\Omega_m}\nabla\cdot\vec{u}\,dV$$

$$\Rightarrow \boxed{\nabla\cdot\vec{u} = \frac{1}{\Omega_m}\frac{D\Omega_m}{Dt}}$$

$\nabla\cdot\vec{u} > 0$ denotes expansion of fluid particles

$\nabla\cdot\vec{u} < 0$ denotes compression of fluid particles

- A simple derivation of the mass-conservation equation:

$$\frac{D\,\delta m}{Dt} = 0$$ where δm is the mass of a fluid particle

$$\delta m = \rho\,\delta V$$

$$0 = \frac{D\,\delta m}{Dt} = \rho\frac{D\,\delta V}{Dt} + \delta V\frac{D\rho}{Dt}$$

$$= \rho\,\delta V\frac{1}{\delta V}\frac{D\,\delta V}{Dt} + \rho\,\delta V\frac{1}{\rho}\frac{D\rho}{Dt}$$

$$= \delta m\left[\nabla\cdot\vec{u} + \frac{1}{\rho}\frac{D\rho}{Dt}\right]$$

$$\Longrightarrow \boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{u} = 0} \quad (1)$$

Alternative derivation: integral form. M: mass of a material volume. Then: DM/Dt = 0. Then, use Leibnitz's theorem to get the integral, conservation form of the mass-conservation equation.

Then, use Gauss' theorem to derive the differential, conservation form.

- A simple derivation of Euler's equations

Consider an "inviscid flow", where viscous effects are negligible (e.g., flows in regions far from solid bodies, where shear stresses are relatively weak). Then:

$$\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}$$

Newton's law of motion yields:

$$\frac{D}{Dt}(\delta m\,\vec{u}) = \nabla \cdot \underline{\underline{\sigma}}\,\delta V$$

$$\Rightarrow \delta m \frac{D\vec{u}}{Dt} = \nabla \cdot \underline{\underline{\sigma}}\,\delta V \quad \text{as } \delta m = \text{const}$$

$$\Rightarrow \boxed{\rho \frac{D\vec{u}}{Dt} = -\nabla p} \quad (2)$$

• Propagation of weak acoustic (i.e., "pressure") disturbances

Consider a quiescent, ideal gas. In this equilibrium state, $p = p_0$ and $\rho = \rho_0$ throughout the "flow" field. Consider a weak perturbation propagating through the field, i.e. a perturbation giving rise to a very small "condensation" δ

$$\delta \equiv \frac{\rho - \rho_0}{\rho_0} << 1$$

Assume that viscous effects are negligible, so that eq (1) and (2) can be used to model the flow. Since gases are poor heat conduction media, the flow can be assumed as adiabatic.

Thus, the flow is inviscid and adiabatic or in other words, isentropic.

For isentropic transformations

$$de = T ds - p dv = - p dv$$

For an ideal gas:

$$e = c_v T \qquad pv = \frac{R_0}{M} T \equiv R T$$

Thus:

$$C_v \, dT = -\frac{RT}{v} \, dv$$

$$\Rightarrow \frac{dT}{T} = -\frac{R}{C_v}\frac{dv}{v}$$

$$\Rightarrow T v^{R/C_v} = \text{const}$$

Mayer's identity: $C_p - C_v = R$

Definition of $\gamma = C_p/C_v$

$$\Rightarrow C_p = \frac{\gamma R}{\gamma - 1} \qquad C_v = \frac{R}{\gamma - 1}$$

$$\Rightarrow \boxed{T v^{\gamma - 1} = \text{const}}$$

Using the equation of state:

$$\boxed{\begin{gathered} P v^{\gamma} = \text{const} \\ \frac{P}{\rho^{\gamma}} = \text{const} \\ P T^{\frac{\gamma}{1-\gamma}} = \text{const} \end{gathered}}$$

Let's reconsider eq. (1):

$$\frac{1}{\rho_0(1+\delta)}\rho_0\frac{D\delta}{Dt} \simeq \frac{\rho_0}{\rho_0}\frac{D\delta}{Dt}$$

$$= \frac{D\delta}{Dt}$$

Thus, the mass conservation equation yields:

$$\frac{D\delta}{Dt} + \nabla\cdot\vec{u} = 0$$

We make the additional assumption that the velocity perturbation $\vec{u}$, i.e. the velocity of the fluid particles interested by the pressure disturbance propagation, is very small compared to the velocity of the disturbance itself, i.e.,

$$\frac{u}{c} = Ma \ll 1$$

where c denotes the rock water. Then

Consider f(x-ct); df/dt = -cf'; df/dx = f'

$$\frac{D\delta}{Dt} = \underset{\wr}{\frac{\partial\delta}{\partial t}} + \underset{\wr}{u\frac{\partial\delta}{\partial x}} \simeq \frac{\partial\delta}{\partial t}$$

$$\frac{\delta c}{L} \gg \frac{u\delta}{L}$$

Thus, the mass conservation equation reduces to:

$$\boxed{\frac{\partial\delta}{\partial t} + \nabla\cdot\vec{u} = 0} \quad (3a)$$

Let's reconsider (2):

$$\rho \frac{D\vec{u}}{Dt} \sim \rho_0 \frac{\partial \vec{u}}{\partial t}$$ (linear for convective terms / as to force)

$$\Rightarrow \boxed{\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p} \quad (3b)$$

From the state equation for isentropic transformation:

$$\frac{P}{\rho^\gamma} = \text{const} \Rightarrow \frac{dP}{P} - \gamma \frac{d\rho}{\rho} = 0$$

$dP \sim P'$ acoustic pressure

$$P' = P - P_0$$

$$\frac{dP}{P} \sim \frac{P'}{P_0}$$

$$\frac{d\rho}{\rho} \sim \delta$$

$$\Rightarrow \boxed{\frac{P'}{P_0} \simeq \gamma \delta} \quad (3c)$$

Let's rewrite (3a), (3b), (3c) for convenience:

$$\boxed{\begin{aligned} &\frac{\partial \delta}{\partial t} + \nabla \cdot \vec{u} = 0 && (4a) \\ &\rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p' && (4b) \\ &\delta = \frac{1}{\gamma} \frac{P'}{P_0} && (4c) \end{aligned}}$$

From (4a) and (4c):

$$\frac{1}{\gamma p_0}\frac{\partial p'}{\partial t} = -\nabla\cdot\vec{u} \quad (5a)$$

$$\Rightarrow \frac{1}{\gamma p_0}\frac{\partial^2 p'}{\partial t^2} = -\nabla\cdot\frac{\partial \vec{u}}{\partial t} \quad (5b)$$

From (4b):

$$\frac{\rho_0}{p_0}\,\nabla\cdot\frac{\partial \vec{u}}{\partial t} = -\nabla^2 p' \quad (5c)$$

Substituting (5b) in (5c) yields:

$$-\frac{\rho_0}{\gamma p_0}\frac{\partial^2 p'}{\partial t^2} + \nabla^2 p' = 0$$

$$\Rightarrow \boxed{\frac{\partial^2 p'}{\partial t^2} - \frac{\gamma p_0}{\rho_0}\nabla^2 p' = 0}$$

$$c^2 \equiv \frac{\gamma p_0}{\rho_0} = \gamma R T_0 \quad \text{speed of sound}$$

$$\boxed{\frac{\partial^2 p'}{\partial t^2} - c^2\nabla^2 p' = 0} \quad \text{wave equation}$$

Meaning of c^2 is very clear in 1D where the general solution in "open field" is

$$p' = f(x-ct) + g(x+ct)$$

c is indeed the propagation velocity of the pressure disturbance.

In air:

$$\gamma \simeq 1.4 \qquad R \simeq 287 \frac{J}{kg\,K}$$

$$T_0 \simeq 293\,K$$

$$c = \sqrt{\gamma R T_0} \simeq 343\ m/s$$

In Helium:

$$R = 2077\ J/kg\,K$$

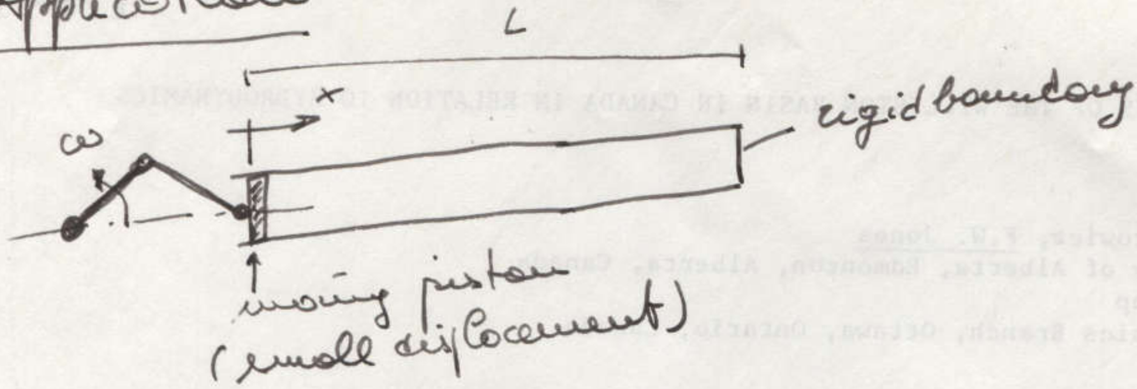
$$c \simeq 923\ m/s$$

In CO_2:

$$R = 188.9\ J/kg\,K$$

$$c = 278\ m/s$$

Application



6a) $\frac{\partial^2 p'}{\partial t^2} - c^2 \frac{\partial^2 p'}{\partial x^2} = 0$

6b) $u'(0,t) = U \cos \omega t$

6c) $u'(L,t) = 0$

6d) $p'(x,0) = 0, \; u'(x,0) = 0$

From (4b):

$$\rho_0 \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x}$$

From 6b):

$$\rho_0 \frac{\partial u'}{\partial t} = -\rho_0 \omega U \sin \omega t = -\frac{\partial p'}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial p'}{\partial x} = \rho_0 U \omega \sin \omega t} \quad \text{at } x = 0$$

From 6c):

$$\boxed{\frac{\partial p'}{\partial x} = 0} \quad \text{at } x = L$$

Thus, we are left with:

$$\frac{\partial^2 p'}{\partial t^2} - c^2 \frac{\partial^2 p'}{\partial x^2} = 0 \qquad x \in (0,L)$$

$$\frac{\partial p'}{\partial x} = \rho_0 U \omega \sin \omega t \qquad \text{at } x = 0$$

$$\frac{\partial p'}{\partial x} = 0 \qquad \text{at } x = L$$

$$P'(x,0)=0, \quad u'(x,0)=0$$

From $u'(x,0)=0$ and (5a):

$$\frac{\partial S}{\partial t}=0$$

From (5c):

$$\frac{\partial}{\partial t}\left(\frac{1}{\gamma}\frac{P'}{P_0}\right)=0 \Rightarrow \frac{\partial P'}{\partial t}(x,0)=0$$

A very simple F.D. solution:

$$\frac{P_j^{n+1}-2P_j^{n}+P_j^{n-1}}{(\Delta t)^2}-c^2\,\frac{P_{j+1}^{n}-2P_j^{n}+P_{j-1}^{n}}{h^2}=0$$

B.C. at $x=0$

$$\frac{P_1^{n+1}-P_0^{n+1}}{h}=\rho_0 U\omega\sin\omega t \Rightarrow P_0^{n+1}=P_1^{n+1}-\rho_0 U h\omega\sin\omega t$$

B.C. at $x=L$

$$P_{M+1}^{n+1}=P_M^{n+1}$$

I.C. 1:

$$P_j^0=0 \quad \forall j$$

I.C. 2:

$$P_j^1=P_j^0 \quad \forall j$$ Why?

Homework:

HOMEWORK

1) Make problem dimensionless

2) Solve analytically (neglect transient)

3) Solve numerically

4) Derive stability limit (e.g., by von Neumann analysis).

5) Derive the modified PDE for the proposed scheme. Comment results

6) Try an implicit scheme (i.e. $\frac{\delta^2 P}{\delta x^2}$ computed at t_{n+1}

or $\left.\frac{\delta^2 P}{\delta x^2}\right| \simeq (1-\alpha)\frac{\delta^2 P^n}{\delta x^2} + \alpha \frac{\delta^2 P^{n+1}}{\delta x^2}$)

Derive stab. limit.

7) Verify your algorithms using either analytical solutions or MMS.

Von Neumann Stability analysis

$$P_j^{n+1} = -P_j^{n-1} + 2(1-\alpha^2)P_j^n + \alpha^2(P_{j+1}^n + P_{j-1}^n)$$

$$\alpha = \frac{c\Delta t}{h}$$

Look for a harmonic solution

$$P_j^n = \theta^n e^{i\tilde{k}x_j} = \theta^n e^{2\pi i j\frac{k}{L}}$$

$$\tilde{k} = \frac{2\pi k}{L}$$

Introduce the "modified" wavenumber

$$\hat{k} = \tilde{k}h$$

$$P_j^n = \theta^n e^{i\hat{k}j}$$

Then:

$$\theta^2 = 2(1-\alpha^2)\theta - 1 + 2\alpha^2 \cos(\hat{k})\,\theta$$

$\rightarrow$ $\frac{1}{2}[\cdot]$ $\cos\cos\left(\frac{\hat{k}}{2}+\frac{\hat{k}}{2}\right)$ and expand...

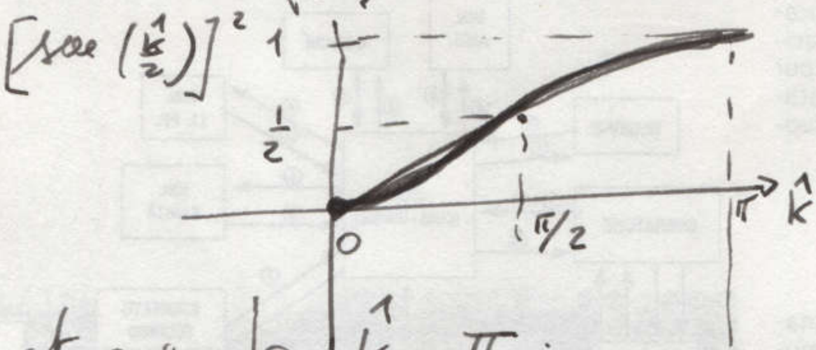
Let $\beta = 1 - 2\alpha^2 \sin^2\left(\frac{\hat{k}}{2}\right)$ to get:

$$\boxed{\theta_{1,2} = \beta \pm \sqrt{\beta^2 - 1}}$$

For $|\beta| > 1$ at least one of the $|\theta_{1,2}|$ is bigger than 1. Thus, we look for $|\beta| \le 1$ for stability.

$$-1 \le 1-2\alpha^2 \sin^2\left(\frac{\hat{k}}{2}\right) \le 1$$

$$-1 \le 1-2\alpha^2 \sin^2\left(\frac{\hat{k}}{2}\right) \Rightarrow 1-\alpha^2 \sin^2\left(\frac{\hat{k}}{2}\right) \ge 0$$



Worst case for $\hat{k} = \pi$:

$$1-\alpha^2 \ge 0 \Rightarrow \boxed{\alpha \le 1}$$

$$1-2\alpha^2 \sin^2\left(\frac{\hat{k}}{2}\right) \le 1 \Rightarrow -2\alpha^2 \sin^2\left(\frac{\hat{k}}{2}\right) \le 0 \quad \text{True } \forall \hat{k}$$

Thus, the stability condition derived from V. N. analysis is $\boxed{\alpha \le 1}$

Analytical solution of the wave equation in 1D

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

Solve by

- d'Alembert's method
- Fourier Transform
- Separation of variables

Let's try to solve (1) by d'Alembert's method.

$$\xi = x - ct \qquad \eta = x + ct$$

$$u_{,x} = u_{,\xi}\,\xi_{,x} + u_{,\eta}\,\eta_{,x}$$

$$= u_{,\xi} + u_{,\eta}$$

$$u_{,t} = u_{,\xi}(-c) + u_{,\eta}\,c$$

$$= c\,(u_{,\eta} - u_{,\xi})$$

$$u_{,xx} = u_{,\xi\xi} + 2u_{,\xi\eta} + u_{,\eta\eta}$$

$$u_{,tt} = c\left((-c)\,u_{,\xi\eta} - (-c)\,u_{,\xi\xi} + c\,u_{,\eta\eta} - c\,u_{,\eta\xi}\right)$$

$$= c^2\left(u_{,\xi\xi} - 2u_{,\xi\eta} + u_{,\eta\eta}\right)$$

Thus eq (1) becomes

$$c^2\left(u_{,\xi\xi} - 2u_{,\xi\eta} + u_{,\eta\eta}\right) = \left(u_{,\xi\xi} + 2u_{,\xi\eta} + u_{,\eta\eta}\right)c^2$$

$$\Rightarrow \boxed{u_{,\xi\eta} = 0}$$

See what happens solving along $\xi = \text{const}$ or $\eta = \text{const}$ lines: you find that $u = \text{const}$ along these lines. They are characteristic lines for eq (1).

However

$$\frac{\partial}{\partial \xi}\frac{\partial u}{\partial \eta} = 0 \Rightarrow \frac{\partial u}{\partial \eta} = f(\eta)$$

$$\Rightarrow \boxed{u = F(\eta) + G(\xi)}$$

This is a general solution for eq (1) and can be written more conveniently as

$$\boxed{u(x,t) = F(x+ct) + G(x-ct)}$$

To get F and G we need the initial conditions.

$$u_{,tt} - c^2 u_{,xx} = 0 \qquad t \geq 0$$

$$u(x,0) = f(x)$$

$$u_{,t}(x,0) = g(x)$$

We need to express F and G in terms of f and g.

Thus:

$$u(x,0) = F(x) + G(x) = f(x) \qquad (2a)$$

$$u_{,t}(x,0) = F_{,\xi}\,\xi_{,t} + G_{,\eta}\,\eta_{,t}$$

$$= F_{,\xi}\big|_{t=0}\, c - G_{,\eta}\big|_{t=0}\, c$$

$$= c\left(F'(x) - G'(x)\right) = g(x) \qquad (2b)$$

Differentiate (2a):

$$F'(x) + G'(x) = f'(x) \qquad (2c)$$

Solve (2b), (2c) in terms of F' and G'.

$$F'(x) = \frac{1}{2}\left[f'(x) + \frac{1}{c}\,g(x)\right]$$

$$G'(x) = \frac{1}{2}\left[f'(x) - \frac{1}{c}\,g(x)\right]$$

Hence:

$$F(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(y)\,dy + C_1$$

$$G(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(y)\,dy + C_2$$

Notice that, from (2a):

$$f(x) = F(x) + G(x) = f(x) + C_1 + C_2$$

yielding

$$C = C_2 = -C_1$$

and

$$F(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(y)\,dy - C$$

$$G(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(y)\,dy + C$$

The ~~general~~ solution of the IVP is thus:

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy$$