FEM solution of the wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(0,t) = \cos(\omega t) \equiv u_0(t) \\ \dfrac{\partial u}{\partial x}(1,t) = 0 \end{cases}$$

$u \longrightarrow \hat{u}$ chosen in suitable, finite-dimensional space ("trial functions")

$$\frac{\partial^2 \hat{u}}{\partial t^2} + c^2 \frac{\partial^2 \hat{u}}{\partial x^2} = R(\hat{u})$$

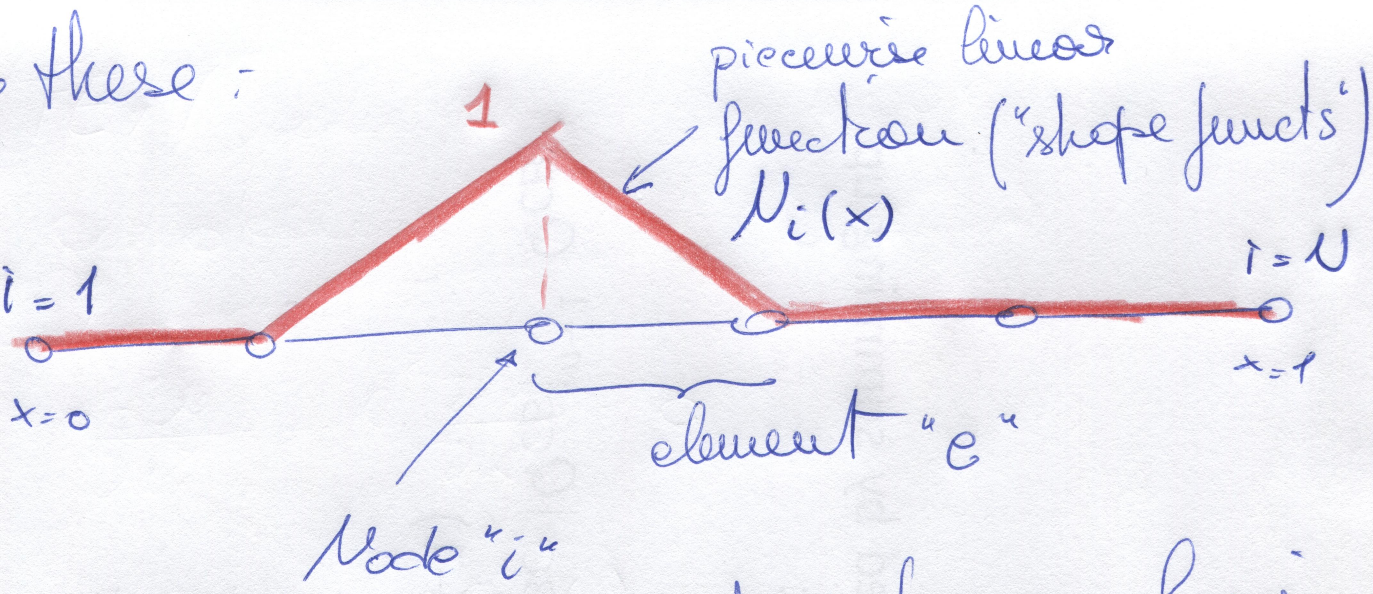
Method of weighted residuals:

$$\int_0^1 v \frac{\partial^2 \hat{u}}{\partial t^2} dx + \int_0^1 v c^2 \frac{\partial^2 \hat{u}}{\partial x^2} dx = 0$$

$\forall$ v in a suitable, finite-dimensional space of "test" functions.

Galerkin method: test- and trial-space are coincident.

Try these:



The set of all shape functions forms a basis of both the test- and trial-function spaces.

The resulting approximation is of interpolating type:

$$\hat{u}(x) = \sum_{i=1}^{N} \hat{u}_i(t) N_i(x)$$

$$\hat{u}(x_i) = \hat{u}_i \quad \text{as} \quad N_i(x_j) = \delta_{ij}$$

Then:

$$\int_0^1 N_i \left(\frac{\partial^2 \hat{u}_j}{\partial t^2}(t) \right) N_j \, dx + c^2 \int_0^1 N_i \frac{\partial^2 N_j}{\partial x^2} \, dx \, \hat{u}_j = 0$$

Integrate by parts to reduce the regularity requirements on $\{N_i\}_{i=1}^N$:

$$\int_0^1 N_i \frac{\partial^2 N_j}{\partial x^2} dx = \int_0^1 \frac{\partial}{\partial x}\left(N_i \frac{\partial N_j}{\partial x}\right) dx$$

$$- \int_0^1 \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx =$$

$$= N_i \frac{\partial N_j}{\partial x}\Big|_0^1 - \int_0^1 \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx$$

Thus:

$$0 = \left(\int_0^1 N_i N_j dx\right) \ddot{\hat{u}}_j + c^2 \left(N_i \frac{\partial \hat{u}}{\partial x}\Big|_0^1\right)$$

$$- c^2 \left(\int_0^1 \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx\right) \hat{u}_j \quad \forall i = 1 \dots N$$

Define

Mass matrix $M_{ij} \equiv \int_0^1 N_i N_j \, dx$

Elementwise mass matrix $M_{ij}^e \equiv \int_{\Omega^e} N_i^e N_j^e \, dx$

Stiffness matrix $K_{ij} \equiv \int_0^1 \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} dx$

Elementwise stiffness matrix $K_{ij}^e \equiv \int_{\Omega^e} \frac{\partial N_i^e}{\partial x} \frac{\partial N_j^e}{\partial x} dx$

$$0 = M_{ij} \ddot{\hat{u}}_j - c^2 K_{ij} \hat{u}_j + b_i$$

$$b_i \equiv c^2 \left[N_i \frac{\partial \hat{u}}{\partial x} \right]_0^1$$

At $x = 0$, $\hat{u}$ is known. Therefore one doesn't need to "weight" the left boundary: choose $N_i(x=0) = 0 \ \forall i$, and $\hat{u}$ is chosen such that $\hat{u}(x=0,t) = u_0(t)$.

On right boundary $\frac{\partial \hat{u}}{\partial x} = 0$. Thus

$$b_i = 0 \quad \forall\, i$$

$\ddot{\hat{u}}$: approximate by FD:

$$[M]\hat{u}^{n+1} = c^2[K]\hat{u}^{n} + 2[M]\hat{u}^{n} - [M]\hat{u}^{n-1}$$

$$= \left(2[M] + c^2[K]\right)\hat{u}^{n} - [M]\hat{u}^{n-1}$$

Dirichlet b. cond:

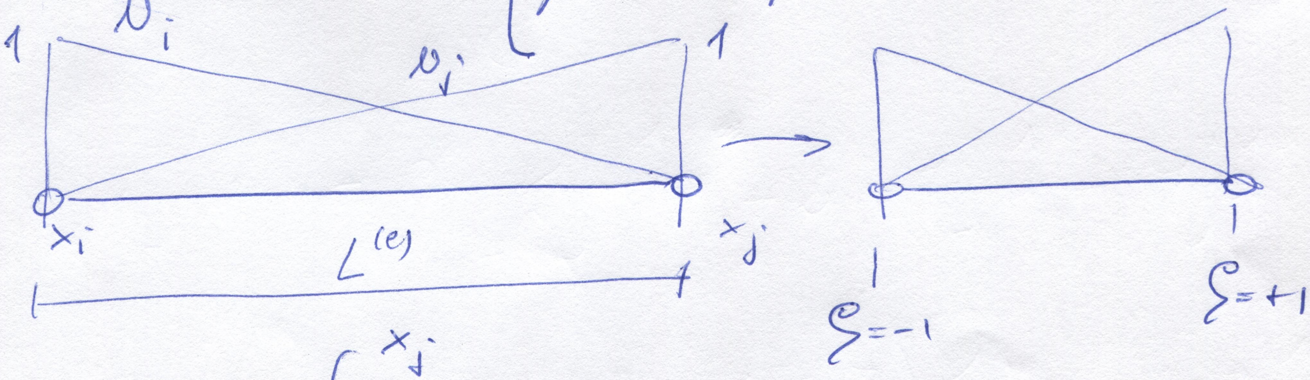
$$[M] \longrightarrow [\tilde{M}] \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \left[\; M_{2:N,2:N} \;\right] & \\ 0 & & & \end{pmatrix}$$

On right-hand side:

$$\{r\} \equiv \left(2[M] + c^2[K]\right)\hat{u}^{n} - [M]\hat{u}^{n-1}$$

$$\{r\} \longmapsto \{\tilde{r}\} = \begin{Bmatrix} \hat{u}_0(t^{n+1}) \\ r_2 - M_{21}u_0(t^{n+1}) \\ \vdots \\ r_N - M_{N1}u_0(t^{n+1}) \end{Bmatrix}$$

$$\int_{\Omega_e} N_i N_j \, dx = \frac{L_e}{2} \begin{cases} 0 & \text{if either node "i" or node "j" is } \underline{\text{not}} \text{ incident to element "e"} \\ 2/3 & i=j \text{ incident to element } e \\ 1/3 & i\neq j \text{ both incident to element } e \end{cases}$$



$$N_i = \int_{x_i}^{x_j} N_i N_j \, dx$$

$$\xi = 2\frac{x-x_i}{L^e} - 1; \quad x = (\xi+1)\frac{L_e}{2} + x_i$$

$$= \int_{-1}^{+1} N_i(\xi) N_j(\xi) \frac{L_e}{2} d\xi$$

$$= \frac{L_e}{2} \cdot \begin{cases} \int_{-1}^{+1} \frac{(1-\xi)^2}{4} d\xi = 2/3 \\ \int_{-1}^{+1} \frac{(1+\xi)^2}{4} d\xi = 2/3 \\ \int_{-1}^{+1} \frac{(1-\xi)(1+\xi)}{4} d\xi = 1/3 \end{cases}$$

$$\int_{\Omega^e} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\,dx = L_e \, \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}$$

$$= L_e \cdot \begin{cases} 1/L_e^2 & i=j \text{ incident} \\ -1/L_e^2 & i \neq j \text{ both incident} \\ 0 & \text{both } i \text{ and } j \text{ NOT incident} \end{cases}$$

Thus:

$$[M^e] = \begin{bmatrix} L_e/3 & L_e/6 \\ L_e/6 & L_e/3 \end{bmatrix}$$

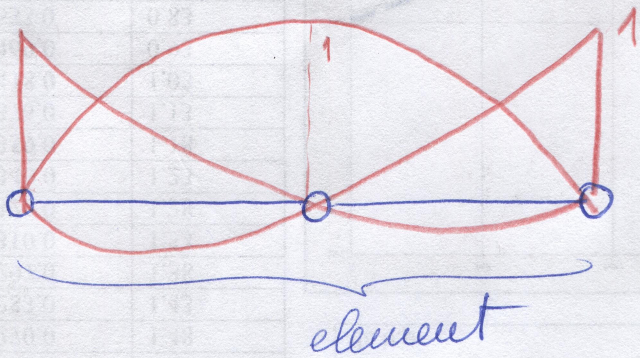
$$[K^e] = \begin{bmatrix} 1/L_e & -1/L_e \\ -1/L_e & 1/L_e \end{bmatrix}$$

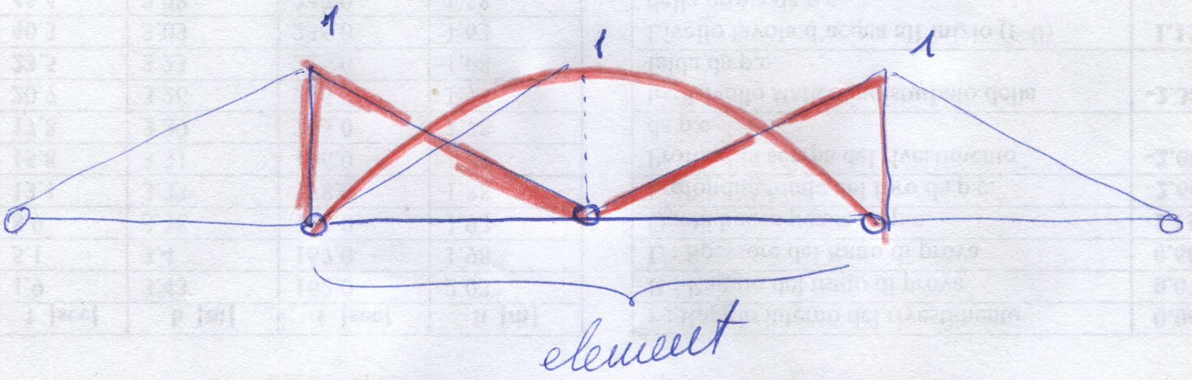
Homework

- Implement solution of wave eq. by the proposed FEM method.

- Try also this higher-order sets of shape functions:

Lagrange family of II order





Compare accuracy.