The error induced by the finite-difference approximation to the first derivative ~~[illegible]~~ is quantified by k'/k or by the "modified phase speed" $c' = ck'/k$. Compact f.d. schemes or symmetric stencils result in a purely-real modif. phase speed, which in terms implies only "dispersion" but not dissipation. For completeness, let's consider the simplest upwind-biased f.d. scheme, i.e. the 1st-order upwind scheme

$$\left.\frac{\delta f}{\delta x}\right|_j \simeq \frac{f_j - f_{j-1}}{h}$$

$$f = e^{ikx} \Rightarrow \left.\frac{\delta f}{\delta x}\right|_j = i k' e^{ikx}$$

$$k' = i\,\frac{\sin(kh/2)}{h/2}\, e^{-ikh/2}$$

$$\frac{k'}{k} = i\,\frac{\sin(w/2)}{w/2}\, e^{-iw/2}$$

~~The solution of the semi-discrete transport equation becomes;~~

$$\tilde{f}(x,t) = \sigma_0\, e^{ik(x - c k'/k\, t)}$$

$$= \sigma_0\, e^{ikx}\, e^{i c t k (a+ib)}$$

$$= \sigma_0\, e^{ikx}\, e^{i(ac)kt}\, e^{-bckt}$$

$$= \sigma_0\, e^{ik(x-c't)}\, e^{-\tilde{\alpha} k^2 t}$$

$$\tilde{\alpha} = bc/k$$

$$\frac{k'}{k} = \frac{\sin \omega/2}{\omega/2}\left(\sin(\omega/2) + i\cos(\omega/2)\right)$$

$$= \frac{\sin^2(\omega/2)}{\omega/2} + i\,\frac{\sin(\omega)}{\omega}$$

$$\tilde{f}(x,t) = \sigma_0\, e^{ik(x-c't)}\, e^{-\tilde{\alpha} k^2 t} \qquad (3)$$

$$\frac{c'}{c} = R(\omega)$$

$$k^2 \tilde{\alpha} = I c k$$

The real part of the modified wavenumber induces a dispersion error, while the imaginary part induces dissipation: the waveform is progressively damped as time passes by.

In order to provide a physical interpretation for I, let's consider the solution to the transport-diffusion equation

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = \alpha \frac{\partial^2 f}{\partial x^2}$$

$$x \in \mathbb{R}$$
$$t > 0$$
$$f(x,0) = \sigma_0 e^{ikx}$$

$$\Rightarrow \quad \dot{\sigma} e^{ikx} + ick\sigma e^{ikx} = -\alpha k^2 \sigma e^{ikx}$$

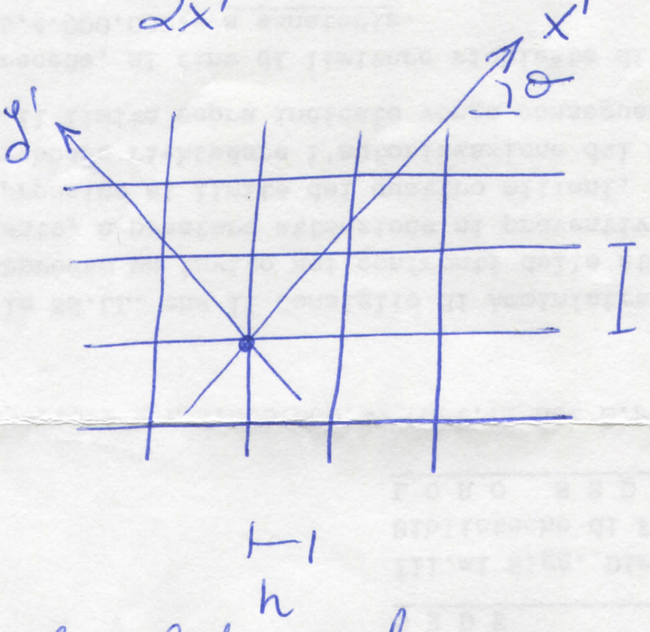
$$\Rightarrow \quad \dot{\sigma} + (ick - \alpha k^2)\sigma = 0$$

$$\Rightarrow \quad \sigma(t) = \sigma_0 e^{ik(x-ct)} e^{-\alpha k^2 t} \qquad (4)$$

Comparing (3) and (4) it turns out that the imaginary part of $\frac{k'}{k}$ corresponds to a "numerical diffusion" coefficient.

Anisotropy of a f.d. scheme deals with the representation of a Fourier mode, oriented along an off-grid direction. As for the transport equation, consider the following problem.

$$5) \quad \frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x'} = 0$$



The general solution of (5) is:

$$f = \sigma_0 e^{ik(x'-ct)}$$

When eq (5) is to be solved on a 2D, uniform Cartesian mesh ~~not oriented~~ not aligned with x', a coordinate mapping must be applied:

$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

The corresponding PDE reads:

$$\frac{\partial f}{\partial t} + c\left[\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right] = 0$$

with initial condition.

$$f(x,y,0) = \sigma_0 e^{iK(x\cos\theta + y\sin\theta)}$$

$$= \sigma_0 e^{i[(K\cos\theta)x + (K\sin\theta)y]}$$

Looking for a f_γ^d solution in the form

$$\tilde{f}(x,y,t) = \sigma(t)\, e^{i[(K\cos\theta)x + (K\sin\theta)y]}$$

yields

$$\dot{\sigma} + \Big[c\, i\, K'(K\cos\theta)\cos\theta + c\, i\, K'(K\sin\theta)\sin\theta\Big]\sigma = 0$$

$$\Rightarrow \cancel{\tilde{f}(x,y,t) = \sigma_0 e^{i}}$$

or

$$\dot{\sigma} + i c K'_{eff}\, \sigma = 0$$

$$\Rightarrow \sigma(t) = \sigma_0 e^{-icK'_{eff} t}$$

$$\Rightarrow \tilde{\rho}(x,y,t) = \rho_0 e^{ik[x' - c_{f.d.} t]}$$

$$C_{f.d.} = c \frac{k'_{eff}}{k}$$

$$\frac{C_{f.d.}}{c} = \frac{k'(k\cos\theta)\cos\theta + k'(k\sin\theta)\sin\theta}{k}$$

$$= \frac{\omega'(\omega\cos\theta)\cos\theta + \omega'(\omega\sin\theta)\sin\theta}{\omega}$$

Considering for instance the second-order central finite difference.

$$\omega'(\omega) = \sin(\omega)$$

$$\frac{C_{f.d.}}{c} = \left[\sin(\omega\cos\theta) \cdot \cos\theta + \sin(\omega\sin\theta)\sin\theta \right] \cdot \frac{1}{\omega} = \frac{\sin(\omega\cos\theta)\cos\theta}{\omega} + \frac{\sin(\omega\sin\theta)\sin\theta}{\omega}$$

It turns out that compact difference schemes have superior properties as regards anisotropy. For all considered schemes, the best resolution is obtained for $\theta = 45°$.

Resolving efficiency of second-order differencing scheme.

$$\frac{\delta^2 f}{\delta x^2} = -k_{eff}^2 \, e^{ikx}$$

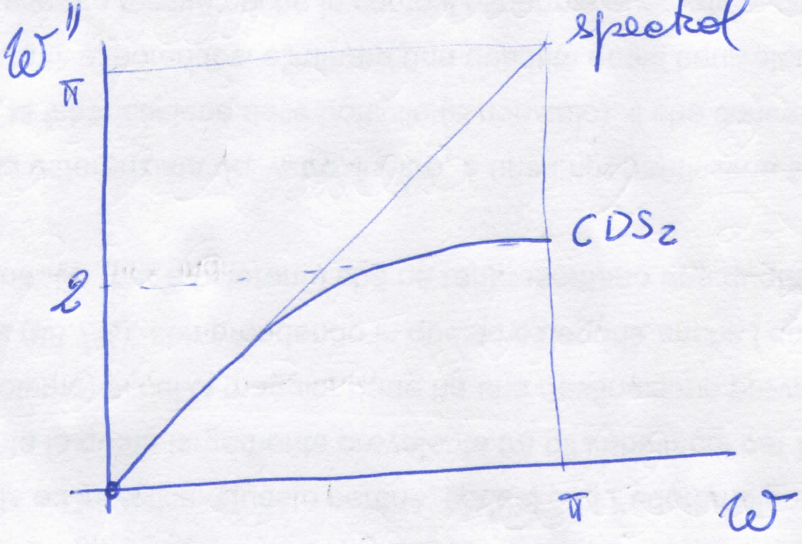
For instance for the CDS:

$$\left.\frac{\delta^2 e^{ikx}}{\delta x^2}\right|_j = \frac{e^{ikh} - 2 + e^{-ikh}}{h^2} \, e^{ikx_j}$$

$$= \frac{2\cos(kh) - 2}{h^2} \, e^{ikx_j}$$

$$K_{eff}^2 = \frac{2 - 2\cos(kh)}{h^2}$$

$$\left((kh)''\right)^2 \equiv \omega''^2 = 2\left(1 - \cos(kh)\right)$$



The modified wavenumber for compact difference schemes of type (D2) results in:

$$(\omega'')^2 = \frac{2a(1-\cos\omega) + (b/2)(1-\cos(2\omega)) + (2c/9)(1-\cos 3\omega)}{1 + 2\alpha\cos\omega + 2\beta\cos 2\omega}$$

NON-PERIODIC BOUNDARIES

Non-symmetric formulations for the first- and second derivatives must be derived near boundary nodes.

As for the first derivative, consider for the boundary node $i=1$:

$$f_1' + \alpha f_2' = \frac{1}{h}\left(a f_1 + b f_2 + c f_3 + d f_4\right)$$

$$\left.\begin{aligned} a &= -\frac{3+\alpha+2d}{2} \\ b &= 2+3d \\ c &= -\frac{1-\alpha+6d}{2} \end{aligned}\right\} \text{2}^{\text{nd}}\text{-order accurate}$$

$$\left.\begin{aligned} a &= -\frac{11+2\alpha}{6} \\ b &= \frac{6-\alpha}{2} \\ c &= \frac{2\alpha-3}{2} \\ d &= \frac{2-\alpha}{6} \end{aligned}\right\} \text{3}^{\text{rd}}\text{-order accurate}$$

$$\left.\begin{aligned} &\alpha = 3 \quad a = -17/6 \\ &b = 3/2 \quad c = 3/2 \\ &d = -1/6 \end{aligned}\right\} \text{4}^{\text{th}}\text{-order accurate.}$$