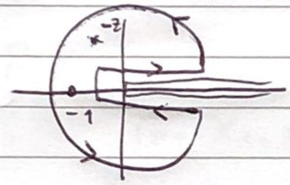
$$\int_0^\infty dx \, \frac{\log x}{(x+1)(x+z)}$$



$$\int_\gamma dw \, \frac{(\log w)^2}{(w+1)(w+z)}$$

$$= 2\pi i \left(\frac{(\log(-1))^2}{z-1} + \frac{(\log(-z))^2}{1-z} \right)$$

$$\log^{(1)}(-z) = \log^{(2)} z + i\pi$$

se $\operatorname{Im} \log^{(1)}$ è comprese tra 0 e $2\pi i$

$\operatorname{Im} \log^{(2)}$ è tra $-\pi i$ e πi

$$= 2\pi i \left(\frac{(i\pi)^2}{z-1} + \frac{(\log z + i\pi)^2}{1-z} \right)$$

$$= 2\pi i \left(\frac{\cancel{-\pi^2}}{z-1} - \frac{(\log z)^2 \cancel{-\pi^2} + 2i\pi \log z}{z-1} \right)$$

$$= -\frac{2\pi i \log z (\log z + 2\pi i)}{z-1}$$

$$\int_\gamma dw \, \frac{(\log w)^2}{(w+1)(w+z)} = \int_0^\infty dx \, \frac{(\log x)^2 - (\log x + 2\pi i)^2}{(x+1)(x+z)}$$

$$= -4\pi i \int_0^\infty \frac{\log x}{(x+1)(x+z)} \, dz + 4\pi^2 \int_0^\infty dx \, \frac{1}{(x+1)(x+z)}$$

$$\int_\gamma dw \, \frac{\log w}{(w+1)(w+z)} = 2\pi i \left(\frac{\log(-1)}{z-1} + \frac{\log(-z)}{1-z} \right)$$

$$= 2\pi i \left(\frac{\cancel{i\pi}}{z-1} + \frac{\log z \pm \cancel{i\pi}}{1-z} \right) = 2\pi i \, \frac{\log z}{1-z}$$

$$\int_\gamma dw \, \frac{\log w}{(w+1)(w+z)} = \int_0^\infty \frac{\log x - (\log x + 2\pi i)}{(x+1)(x+z)} \, dx$$

$$= -2\pi i \int_0^\infty dx \, \frac{1}{(x+1)(x+z)}$$

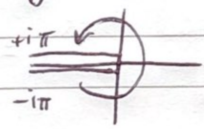
$$\Rightarrow \int_0^\infty dx \, \frac{1}{(x+1)(x+z)} = - \frac{\log z}{1-z}$$

$$\Rightarrow -4\pi i \int_0^\infty \frac{\log x}{(x+1)(x+z)} \, d = \cancel{-4\pi^2 \frac{\log z}{1-z}}$$

$$= 2\pi i \frac{(\log z)^2}{1-z} \cancel{- 4\pi^2 \frac{\log z}{1-z}}$$

$$\Rightarrow \int_0^\infty dx \, \frac{\log x}{(x+1)(x+z)} = -\frac{1}{2} \frac{(\log z)^2}{1-z}$$

$\log z$ con Im part between $-i\pi$ and $i\pi$

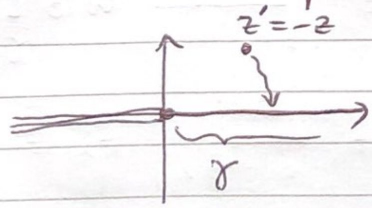


$$\text{Disc}\left[\int_0^\infty dx \, \frac{\log x}{(x+1)(x+z)}\right] \quad \text{per } \begin{matrix} t \in \mathbb{R} \\ t < 0 \end{matrix}$$

$$= -\frac{1}{2} \frac{(\log|t| + i\pi)^2}{1-t} + \frac{1}{2} \frac{(\log|t| - i\pi)^2}{1-t}$$

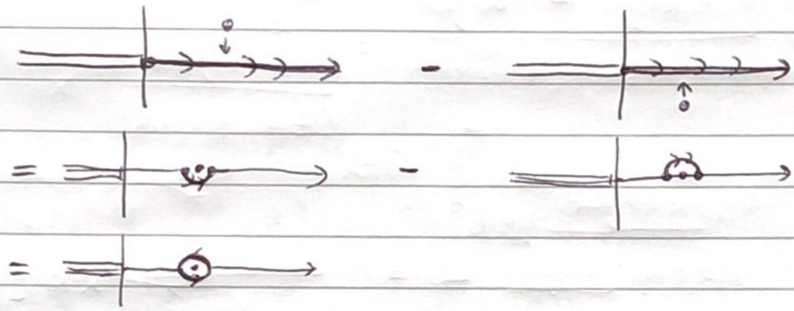
$$= -2\pi i \frac{\log|t|}{1-t}$$

Altro modo per calcolarlo senza fare integrale:



prendi Im $\log \in [-i\pi, i\pi]$ con taglio sull'asse reale negativo e interpreta l'integrale come integrale della funzione olomorfa $\frac{\log w}{(w+1)(w+z')}$ sul cammino in figura

Allora muovendo z' in maniera continua vediamo che:



$$\mathrm{Disc}_{z'=x} = \oint_{\gamma_x} dw \, \frac{\log w}{(w+1)(w-x)} = 2\pi i \, \frac{\log x}{x+1}$$

$x \in \mathbb{R}, \ x>0$

$$\underset{\substack{z'=-z \\ x=-t}}{\iff} \quad \mathrm{Disc}_{z=t} = \underbrace{-2\pi i \, \frac{\log|t|}{1-t}}_{\text{stesso risultato trovato con il calcolo dell'integrale.}}$$

$t \in \mathbb{R}, \ t<0$

($-$ addizionale davanti perché la discontinuità per $z'=-z$ è $-$ la disc per z).

$u(t,x) \qquad x \in [-L, L]$

$$\frac{\partial^2 u}{\partial t^2} = C_1 \frac{\partial^2 u}{\partial x^2}$$

$$u(t,x) = \alpha_0(t) + \sum_{n=1}^{\infty}\left(\alpha_n(t)\cos\left(\frac{n\pi}{L}x\right) + \beta_n(t)\sin\left(\frac{n\pi}{L}x\right)\right)$$

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \ddot{\alpha}_0(t) + \displaystyle\sum_{n=1}^{\infty}\left(\ddot{\alpha}_n(t)\cos\left(\frac{n\pi}{L}x\right) + \ddot{\beta}_n(t)\sin\left(\frac{n\pi}{L}x\right)\right) \\ \dfrac{\partial^2 u}{\partial x^2}(t,x) = \displaystyle\sum_{n=1}\left(-\left(\frac{n\pi}{L}\right)^2\right)\left(\alpha_n(t)\cos\left(\frac{n\pi}{L}x\right) + \beta_n(t)\sin\left(\frac{n\pi}{L}x\right)\right) \end{cases}$$

$$\Rightarrow \quad \ddot{\alpha}_0(t) = 0$$

$$\begin{cases} \ddot{\alpha}_n(t) = -C_1\left(\frac{n\pi}{L}\right)^2 \alpha_n(t) \\ \ddot{\beta}_n(t) = -C_1\left(\frac{n\pi}{L}\right)^2 \beta_n(t) \end{cases} \qquad n \geq 1$$

$$\Rightarrow \quad \alpha_0(t) = A_0 + B_0 t$$

$$\begin{cases} \alpha_n(t) = A_n\cos\left(\sqrt{C_1}\frac{n\pi}{L}t\right) + B_n\sin\left(\sqrt{C_1}\frac{n\pi}{L}t\right) \\ \beta_n(t) = C_n\cos\left(\sqrt{C_1}\frac{n\pi}{L}t\right) + D_n\sin\left(\sqrt{C_1}\frac{n\pi}{L}t\right) \end{cases} \qquad n \geq 1$$

Per determinare $A_0, B_0, A_n, B_n, C_n, D_n$ usiamo le condizione iniziale:

$$u(t=0, x) = 0 \Rightarrow A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right) + C_n\sin\left(\frac{n\pi}{L}x\right) = 0$$

$$\Rightarrow A_0 = A_n = C_n = 0 \quad \forall n \geq 1$$

$\dot{u}(t=0, x) = \alpha x \rightarrow$ per imporre questa condizione dobbiamo sviluppare αx in serie di Fourier

$$\alpha x = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$$

$$a_0 \equiv \frac{1}{2L}\int_{-L}^{L} dx\, \alpha x = \frac{1}{2L}\alpha\left(\frac{x^2}{2}\right)_{-L}^{L} = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} dx\, \alpha x \cos\left(\frac{n\pi}{L}x\right)$$

$$= \frac{1}{L}\int_{-L}^{L} dx\, \alpha x\, \frac{L}{n\pi}\frac{d}{dx}\sin\left(\frac{n\pi}{L}x\right)$$

$$= \frac{1}{n\pi}\left(\alpha x \sin\left(\frac{n\pi}{L}x\right)\right)\Big|_{-L}^{L} - \frac{\alpha}{n\pi}\int_{-L}^{L} dx\, \sin\left(\frac{n\pi}{L}x\right)$$

$$= 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} dx\, \alpha x\, \sin\left(\frac{n\pi}{L}x\right)$$

$$= \frac{1}{L}\int_{-L}^{L} dx\, \alpha x\left(-\frac{L}{n\pi}\frac{d}{dx}\cos\left(\frac{n\pi}{L}x\right)\right)$$

$$= -\frac{1}{n\pi}\left(\alpha x \cos\left(\frac{n\pi}{L}x\right)\right)_{-L}^{L} + \frac{\alpha}{n\pi}\int_{-L}^{+L} dx\, \cos\left(\frac{n\pi}{L}x\right)$$

$$= -\frac{1}{n\pi}\left(\alpha L(-1)^n - \alpha(-L)(-1)^n\right) = \frac{2\alpha L(-1)^{n+1}}{n\pi}$$

$\Rightarrow \dot{u}(t=0, x) = \alpha x$ si riscrive come:

$$B_0 + \sum_{n=1}^{\infty}\left(\sqrt{G}\frac{n\pi}{L}\right)\left(B_n \cos\left(\frac{n\pi}{L}x\right) + D_n \sin\left(\frac{n\pi}{L}x\right)\right)$$

$$= \sum_{n=1}^{\infty}\frac{2\alpha L}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi}{L}x\right)$$

$$\Rightarrow B_0 = B_n = 0 \;\forall n \geq 1, \quad D_n = (-1)^{n+1}\frac{2\alpha L}{n\pi}\cdot\frac{L}{n\pi\sqrt{G}}$$

$$\Rightarrow \boxed{u(t,x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2\alpha L^2}{\sqrt{G}\,\pi^2 n^2} \sin\left(\sqrt{G}\,\frac{n\pi}{L} t\right) \sin\left(\frac{n\pi}{L} x\right).}$$