Fourier-Galerkin solution of the heat equation

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\begin{cases}
\dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2} & x \in (0,L) \quad (1a) \\
u(0,t) = u(L,t) = 0 & (1b) \\
u(x,0) = f(x) & x \in (0,L) \quad (1c)
\end{cases}
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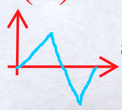

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We seek an approximate solution in the "trial space" S_{2N}, the set of all sine functions of degree $\le N$:

$$S_N \equiv \left\{ \sin\left(\frac{2\pi k x}{L}\right),\ 1 \le k \le N,\ k \in \mathbb{Z} \right\}$$

Thus:

$$\hat{u}(x,t) = \sum_{k=1}^{N} c_k(t) \sin(2\pi k x/L) \qquad (2)$$

Notice that $\hat{u}(x,t)$ automatically satisfies the boundary conditions (1b). In the Fourier-Galerkin method, the space of test functions is coincident with the space of trial functions.

At this point it is worth recalling that S_{2N} is an orthonormal space w.r.t. the inner product (of $L_2(0,L)$):

$$\langle f, g \rangle \equiv \frac{2}{L} \int_0^L f(x)\, g(x)\, dx$$

The residual of (1a) with the approximation (2) is:

$$R(\hat{u}) = \frac{\partial \hat{u}}{\partial t} - \alpha \frac{\partial^2 \hat{u}}{\partial x^2} = \sum_{k=1}^{N} \left(\dot{c}_k + \frac{4\pi^2 k^2}{L^2} \alpha\, c_k \right) \sin\left(\frac{2\pi k x}{L}\right)$$

Then, following the method of weighted residuals:

(3) $0 = \langle R(\hat{u}), \sin\left(\frac{2\pi m x}{L}\right)\rangle = \dot{c}_m + \frac{4\pi^2 m^2}{L^2} c_m, \quad 1 \le m \le N-1$

Thus, we are left with a set of independent IVPs, that are easily solved:

$$c_m(t) = c_m(0)\, e^{-\frac{4\pi^2 m^2 \alpha t}{L^2}}$$

The coefficients $\{c_m(0)\}_{m=-N}^{N}$ are found using (1c):

$$\hat{f}(x) = \sum_{k=1}^{N-1} b_k \sin\left(\frac{2\pi k x}{L}\right)$$

$$\Rightarrow b_k = \langle f(x), \sin\left(\frac{2\pi k x}{L}\right)\rangle$$

Then, by (1c) and linear independence of the considered sine functions, it is found that

$$c_m(0) = b_m$$

Eventually:

$$\hat{u}(x,t) = \sum_{k=1}^{N-1} b_k\, e^{-\frac{4\pi^2 k^2 \alpha t}{L^2}}$$

Now, let's solve (3) by the explicit Euler method:

$$c_m^{n+1} - c_m^{n} = -\frac{4\pi^2 m^2 \alpha}{L^2} \Delta t\, c_m^{n}$$

The amplification factor $\tau_m \equiv \left| \frac{C_m^{n+1}}{C_m^n} \right|$ is:

$$\tau_m = \left| 1 - \frac{4\pi^2 m^2 \alpha \Delta t}{L^2} \right|,$$

Stability of the method requires:

$$\tau_m \leq 1 \;\; \forall\, m : |m| \leq N$$

$$\Rightarrow \frac{4\pi^2 m^2 \alpha}{L^2} \Delta t \leq 2, \quad 1 \leq m \leq N-1$$

The most restrictive condition corresponds to $m = \pm N$:

$$\frac{4\pi^2 N^2}{L^2} \alpha \Delta t \leq 2$$

Define the grid-spacing Δx as:

$$\Delta x \equiv \frac{L}{2N} \quad (4)$$

to get

$$\left(\frac{\alpha \Delta t}{\Delta x^2} \right) \leq \frac{2}{\pi^2} \simeq 0{,}203 \quad (5)$$

According to Nyquist's sampling theorem, Δx, defined as in eq. (4), corresponds to the largest grid spacing that can be used to "resolve" (i.e, identify) a Fourier mode of order N.

The stability limit (5) is about 2.5 times more stringent than the corresponding one, when the central difference scheme is used to approximate the second-order spatial derivative.

Remark: the $k=0$ mode is identically zero and must be disregarded.

Pseudo-spectral approximation of the heat equation

In the following, only the major differences w.r.t. the Fourier-Galerkin approximation are emphasized.

- Instead of using the Method of Weighted Residuals, a collocation approach is pursued to figure out the "best" approximation $\hat{u}$:

$$x_j = \frac{j}{2N} L \quad , \quad j = 0 \ldots 2N-1$$

$$R(\hat{u}(x_j, t)) = 0 \quad \forall_j$$

(6) $\Rightarrow$
$$\sum_{-N}^{N-1}{}_{k} \left(\dot{c}_k(t) + \frac{4\pi^2 k^2 \alpha}{L^2} c_k \right) \sin\left(\frac{2\pi m j}{2N}\right) = 0 \quad \forall_j$$

Recall that:

$$\langle f, g \rangle_{2N} \equiv \frac{2}{2N} \sum_{0}^{2N-1}{}_{j} f(x_j)\, g(x_j)$$

and

$$\left\langle \sin\left(\frac{2\pi m j}{2N}\right) \sin\left(\frac{2\pi k j}{2N}\right) \right\rangle_{2N} = \delta_{km}$$

Thus, taking $\sum_{-N}^{N-1}{}_{j}$ of (6) yields:

$$\dot{c}_k(t) + \frac{4\pi^2 k^2 \alpha}{L^2} c_k(t) = 0 \quad \forall k$$

Thus, the same stability limit as for the Fourier-Galerkin method is derived.