Oscillatory instability for the heat equation

Consider the simple IVP:

$$\begin{cases} \dot{y} = -\alpha y \\ y(0) = y_0 \end{cases}$$

Explicit-Euler scheme yields:

$$\frac{y^{n+1}}{y^n} = 1 - \alpha \Delta t$$

The method becomes unstable for $\alpha \Delta t > 2$.

Nevertheless, consider what happens for $\alpha \Delta t > 1$:

$$\alpha \Delta t > 1 \Rightarrow \frac{y^{n+1}}{y^n} < 0$$

Thus, while the exact solution shows an exponential decay from y_0 towards 0, for $\alpha \Delta t > 1$ the numerical solution, though progressively decaying, develops an unphysical behavior: it changes sign from one time-step to the next.

The explicit Euler / central difference approximation of the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \alpha \dfrac{\partial^2 u}{\partial x^2} & x \in (0, L) \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

can be cast as:

$$\vec{u}^{\,n} \equiv (u_1 \text{———} u_N)^T$$

$$\vec{u}^{\,n+1} = \frac{\alpha \Delta t}{(\Delta x)^2} [A] \vec{u}^{\,n} + \vec{u}^{\,n} =$$

$$[A] = Tr(N, 1, -2, 1)$$

The eigenvalues:

$$\lambda_k = -2 + 2\cos\left(\frac{k\pi}{N+1}\right), \quad k = 1 \ldots N$$

The largest_magnitude eigenvalue is:

$$\lambda_N = -2 + 2 \cdot \cos\left(\frac{N\pi}{N+1}\right) \simeq -4 \text{ (for large } N)$$

The method is stable when:

$$\left|1 + \frac{\alpha \Delta t}{\Delta x^2} \lambda_k\right| \leq 1 \Rightarrow |\lambda_k| \frac{\alpha \Delta t}{\Delta x^2} \leq 2 \quad \forall k$$

and develops oscillatory instability in time whenever

$$1 + \frac{\alpha \Delta t}{(\Delta x)^2}\lambda_k < 0 \quad \text{for some } k.$$

The most restrictive condition holds for $\lambda_N \simeq -4$, yielding

$$1 - 4\frac{\alpha \Delta t}{(\Delta x)^2} < 0 \Rightarrow \frac{\alpha \Delta t}{(\Delta x)^2} > \frac{1}{4}$$

Notice that the corresponding eigenvector is:

$$v_j^{(N)} = (-1)^j \sin\left(\frac{N\pi j}{N+1}\right), \quad j = 1 \dots N$$

This eigenvector changes sign at each grid point:

$$v_{j+1}^{(N)} = (-1)^j \sin\left(\frac{N\pi j}{N+1}\right)\cos\left(\frac{N\pi}{N+1}\right) + (-1)^j \cos\left(\frac{N\pi j}{N+1}\right)\sin\left(\frac{N\pi}{N+1}\right)$$

$$\simeq -(-1)^j \sin\left(\frac{N\pi j}{N+1}\right) \quad (N \text{ large})$$

Thus, the component $\bar{v}^{(N)}$ of the numerical solution oscillates both in space and in time while it decays towards zero

(provided $\frac{\kappa \Delta t}{(\Delta x)^2} < \frac{1}{2}$).