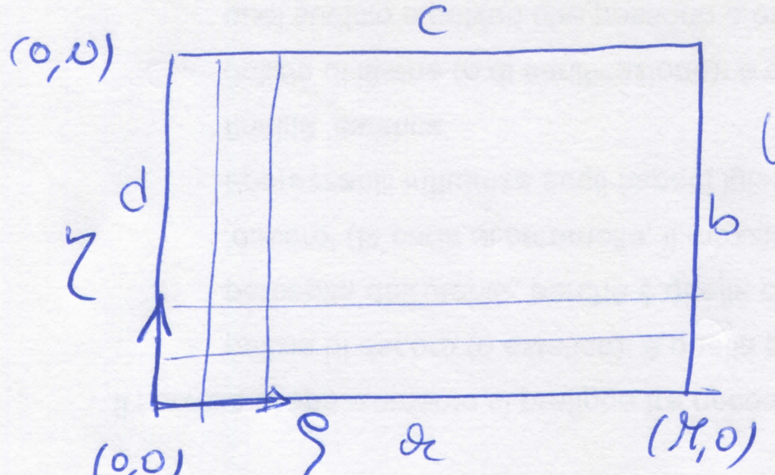
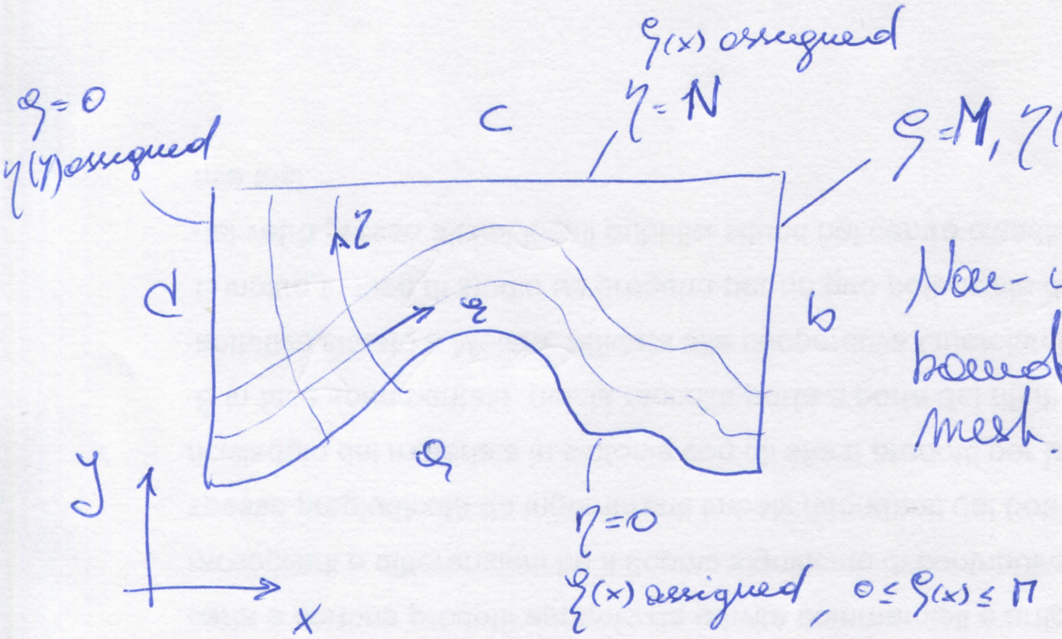
Boundary-fitted mesh

A Laplacian mesh-generator



Uniform (1x1) mesh



Non-uniform, boundary-fitted mesh

Consider: $\Delta \xi = 0$ with ξ assigned on boundaries

$\Delta \eta = 0$ with η assigned on boundaries

ξ and η do not take values out of $[0,M]$ and $[0,N]$, respectively, due to the properties of Δ.

Now recast in terms of $\frac{\partial \vec{x}}{\partial \vec{\xi}}$:

$$\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{\left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}\right)} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

$$= \frac{1}{D} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial}{\partial \xi}\left[\frac{\partial \xi}{\partial x}\right]\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left[\frac{\partial \xi}{\partial x}\right]\frac{\partial \eta}{\partial x}$$

$$= \frac{\partial}{\partial \xi}\left[\frac{1}{D}\frac{\partial y}{\partial \eta}\right]\frac{1}{D}\frac{\partial y}{\partial \eta} + \frac{\partial}{\partial \eta}\left[\frac{1}{D}\frac{\partial y}{\partial \eta}\right]\left(-\frac{1}{D}\frac{\partial y}{\partial \xi}\right)$$

$$\frac{\partial^2 \xi}{\partial y^2} = \frac{\partial}{\partial \xi}\left[\frac{\partial \xi}{\partial y}\right]\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\left[\frac{\partial \xi}{\partial y}\right]\frac{\partial \eta}{\partial y} =$$

$$= \frac{\partial}{\partial \xi}\left[-\frac{1}{D}\frac{\partial x}{\partial \eta}\right]\left(-\frac{1}{D}\frac{\partial x}{\partial \eta}\right)$$

$$+ \frac{\partial}{\partial \eta}\left[-\frac{1}{D}\frac{\partial x}{\partial \eta}\right]\left(\frac{1}{D}\frac{\partial x}{\partial \eta}\right)$$

$$\Delta \xi = 0 \overset{(D\neq 0)}{\Longleftrightarrow} \frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}\left(\frac{1}{D}\frac{\partial y}{\partial \eta}\right) - \frac{\partial y}{\partial \xi}\left(\frac{1}{D}\frac{\partial y}{\partial \eta}\right)$$

$$+ \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\left(\frac{1}{D}\frac{\partial x}{\partial \eta}\right) - \frac{\partial x}{\partial \eta}\left(\frac{1}{D}\frac{\partial x}{\partial \eta}\right) = 0$$

An analogous equation is derived for $\Delta \eta = 0$.
They can be solved by finite-differences.

Homework