Boundary-fitted mesh

We aim to lay down a mesh, that:

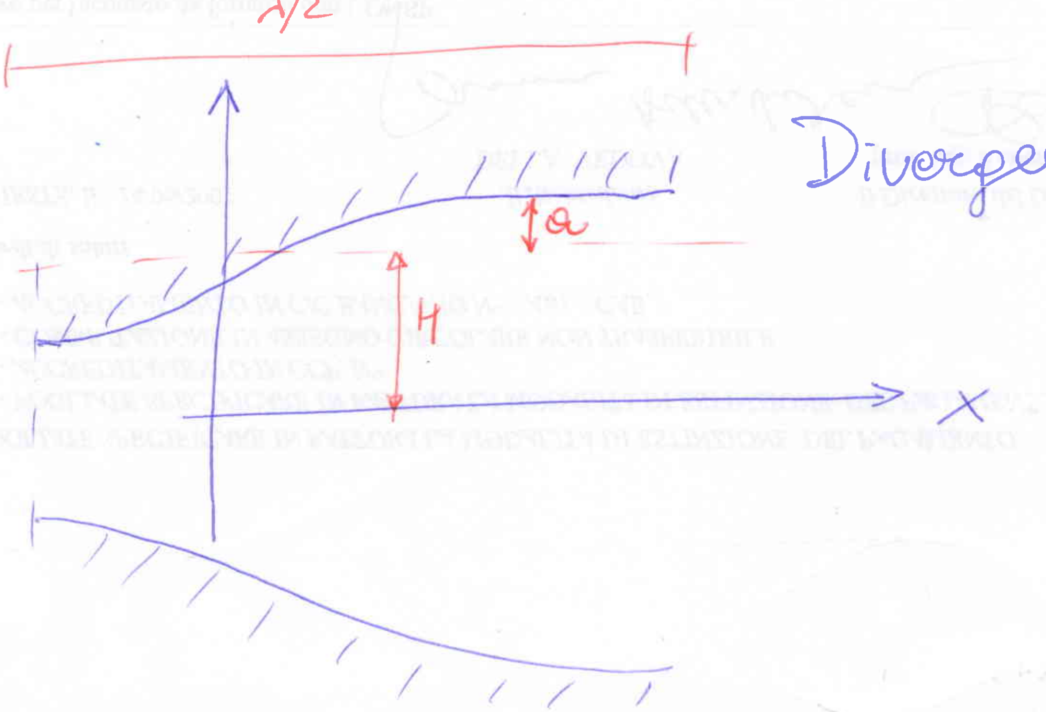
- is "structured"
- matches the boundaries of the computational domain.
- is mapped onto a Cartesian mesh in a "computational space".

In the following we consider meshes in 2D. The extension to 3D space is immediate.

We consider just two methods for generating boundary-fitted meshes:

- analytical mesh-generation
- elliptic, numerical mesh generation.

Analytical mesh generation



Divergent nozzle

$$g(x) = H + a \sin\left(\frac{2\pi x}{\lambda}\right)$$

$$x = -\frac{\lambda}{4} + \frac{\lambda}{2}\xi \qquad \xi, \eta \in [0,1]$$

$$y = g(x)\left[2\eta - 1\right]$$

In general, more accurate results are obtained on orthogonal meshes. In this case:

$$\frac{\nabla\xi \cdot \nabla\eta}{\|\nabla\xi\| \, \|\nabla\eta\|} \longrightarrow \text{quantification of non-orthogonality}$$

$$\varphi = \left(x + \frac{\lambda}{4}\right)\frac{2}{\lambda}$$

$$\eta = 1 + \frac{1}{2}\left[\frac{y}{g(x)}\right]$$

$$\nabla \varphi = \frac{2}{\lambda}\hat{\imath}$$

$$\nabla \eta = -\frac{a y \pi \cos\left(\frac{2\pi x}{\lambda}\right)}{\lambda\left[H + a \sin\left(\frac{2\pi x}{\lambda}\right)\right]^2}\hat{\imath}$$

$$+ \frac{1}{2H + 2a \sin\left(\frac{2\pi x}{\lambda}\right)}\hat{\jmath}$$

Try to plot $\cos^{-1}\left(\frac{\nabla\varphi \cdot \nabla\eta}{\|\nabla\varphi\| \|\nabla\eta\|}\right)\frac{180}{\pi}$ with MATLAB (I used the Symbolic Toolbox). The angle btw the mesh lines lies in the range $[\sim 70° \div \sim 110°]$.

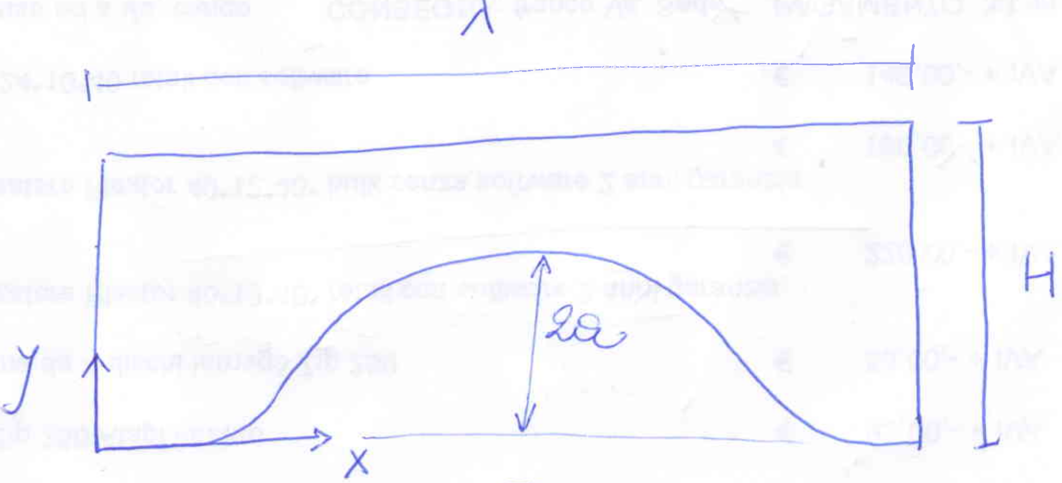
Elliptic mesh generation

Consider:

$$\Delta \xi = 0$$

$$\Delta \zeta = 0 \qquad (x, y) \in \Omega$$

Assign (ξ, ζ) on boundary $\partial\Omega$, e.g.:



Bottom: $y_b = a\left[1 - \cos\left(\frac{2\pi x}{\lambda}\right)\right]$

$$ds = \sqrt{1 + (y_b')^2}\, dx$$

$$s = \int_0^x \sqrt{1 + (y_b')^2}\, dx$$

$$s_{max} = \int_0^\lambda \sqrt{1 + (y_b')^2}\, dx \leftarrow \text{can be calculated numerically.}$$

$$\left.\begin{array}{l} \xi = \dfrac{\Delta}{\Delta_{max}} \\ \eta = 0 \end{array}\right\} \text{ on bottom boundary}$$

$$\left.\begin{array}{l} \xi = \dfrac{x}{\lambda} \\ \eta = 1 \end{array}\right\} \text{ on top boundary}$$

$$\left.\begin{array}{l} \xi = 0 \\ \eta = \dfrac{y}{H} \end{array}\right\} \text{ on left boundary}$$

$$\left.\begin{array}{l} \xi = 1 \\ \eta = \dfrac{y}{H} \end{array}\right\} \text{ on right boundary}$$

We aim to carry out the calculations in the (ξ,η) plane. Thus, consider:

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1}$$

We are able to approximate $\frac{\partial x}{\partial \xi}$, $\frac{\partial x}{\partial \eta}$, $\frac{\partial y}{\partial \xi}$, $\frac{\partial y}{\partial \eta}$ by finite-differences in the (ξ,η) plane.

Inverting the $\frac{\partial \vec{x}}{\partial \vec{\xi}}$ Jacobian matrix we end up with:

$$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix} = \frac{1}{D} \begin{bmatrix} \frac{\partial y}{\partial \eta} & -\frac{\partial x}{\partial \eta} \\ -\frac{\partial y}{\partial \xi} & \frac{\partial x}{\partial \xi} \end{bmatrix}$$

$$D = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial \xi}{\partial x}\right)\frac{\partial \eta}{\partial x}$$

$$= \frac{1}{D}\frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}\left[\frac{1}{D}\frac{\partial y}{\partial \eta}\right] - \frac{1}{D}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\left[\frac{1}{D}\frac{\partial y}{\partial \eta}\right]$$

$$\frac{\partial^2 \xi}{\partial y^2} = -\frac{1}{D}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\left[-\frac{1}{D}\frac{\partial x}{\partial \eta}\right] + \frac{1}{D}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\left[-\frac{1}{D}\frac{\partial x}{\partial \eta}\right]$$

$$= \frac{1}{D}\frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\left[\frac{1}{D}\frac{\partial x}{\partial \eta}\right] - \frac{1}{D}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\left[\frac{1}{D}\frac{\partial x}{\partial \eta}\right]$$

$$\Delta \xi = 0 \iff \frac{\partial y}{\partial \eta}\frac{\partial}{\partial \xi}\left[\frac{1}{D}\frac{\partial y}{\partial \eta}\right] - \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \eta}\left[\frac{1}{D}\frac{\partial y}{\partial \eta}\right]$$

$$+ \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\left[\frac{1}{D}\frac{\partial x}{\partial \eta}\right] - \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta}\left[\frac{1}{D}\frac{\partial x}{\partial \eta}\right] = 0$$

$$\frac{\partial^2 \zeta}{\partial x^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \zeta}\left(\frac{\partial \zeta}{\partial x}\right)\frac{\partial \zeta}{\partial x}$$

$$= \frac{1}{D}\frac{\partial y}{\partial \zeta}\frac{\partial}{\partial \xi}\left(-\frac{1}{D}\frac{\partial y}{\partial \xi}\right) - \frac{1}{D}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \zeta}\left(-\frac{1}{D}\frac{\partial y}{\partial \xi}\right)$$

$$= \frac{1}{D}\frac{\partial y}{\partial \xi}\frac{\partial}{\partial \zeta}\left(\frac{1}{D}\frac{\partial y}{\partial \xi}\right) - \frac{1}{D}\frac{\partial y}{\partial \zeta}\frac{\partial}{\partial \xi}\left(\frac{1}{D}\frac{\partial y}{\partial \xi}\right)$$

$$\frac{\partial^2 \zeta}{\partial y^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \zeta}\left(\frac{\partial \zeta}{\partial y}\right)\frac{\partial \zeta}{\partial y}$$

$$= -\frac{1}{D}\frac{\partial x}{\partial \zeta}\frac{\partial}{\partial \xi}\left(\frac{1}{D}\frac{\partial x}{\partial \xi}\right) + \frac{1}{D}\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \zeta}\left(\frac{1}{D}\frac{\partial x}{\partial \xi}\right)$$

$$\Delta \zeta = 0 \iff \frac{\partial y}{\partial \xi}\frac{\partial}{\partial \zeta}\left(\frac{1}{D}\frac{\partial y}{\partial \xi}\right) - \frac{\partial y}{\partial \zeta}\frac{\partial}{\partial \xi}\left(\frac{1}{D}\frac{\partial y}{\partial \xi}\right)$$

$$+ \frac{\partial x}{\partial \xi}\frac{\partial}{\partial \zeta}\left(\frac{1}{D}\frac{\partial x}{\partial \xi}\right) - \frac{\partial x}{\partial \zeta}\frac{\partial}{\partial \xi}\left(\frac{1}{D}\frac{\partial x}{\partial \xi}\right) = 0$$

Boundary conditions in the form of

$$x = x(\xi) \quad \text{or} \quad x = x(\eta)$$

$$y = y(\xi) \quad \text{or} \quad y = y(\eta)$$

can be derived by inverting the equations relating (ξ, η) to (x, y) on each boundary.