$$(2.13) \qquad \delta W = \int d^3\bar{x}\, \delta\rho(\bar{x})\, \phi(\bar{x})$$

Potenziale additivo $\quad D^2(\delta\phi) = 4\pi G\, \delta\rho(\bar{x})$

$$(2.14) \qquad \delta W = \frac{1}{4\pi G} \int d^3x\, \phi(\bar{x})\, D^2(\delta\phi)$$

(*) [Ricordiamo (56 bis): $\bar{D}\cdot(\lambda\bar{u}) = \lambda\bar{D}\cdot\bar{u} + \bar{u}\cdot\bar{D}\lambda$]

se $\lambda \to \phi$, $\bar{u} \to \bar{D}(\delta\phi)$ $\to$ $\bar{D}\cdot(\phi\bar{D}(\delta\phi)) = \phi\underbrace{\bar{D}\cdot\bar{D}(\delta\phi)}_{D^2(\delta\phi)} + \bar{D}(\delta\phi)\cdot\bar{D}\phi$

da cui
$$\phi D^2(\delta\phi) = \bar{D}\cdot[\phi\bar{D}(\delta\phi)] - \bar{D}(\delta\phi)\cdot\bar{D}\phi$$

da questa $\to$ (2.15) usando il teorema della divergenza.

Inoltre
$$\delta[\bar{D}\phi\cdot\bar{D}\phi] = 2\bar{D}\phi\cdot\delta\bar{D}\phi$$

ma $\delta\bar{D}\phi = \bar{D}(\phi+\delta\phi) - \bar{D}\phi = \bar{D}\phi + \bar{D}(\delta\phi) - \bar{D}\phi = \bar{D}(\delta\phi)$

da cui $\bar{D}\phi\cdot\bar{D}(\delta\phi) = \frac{1}{2}\delta[|\bar{D}\phi|^2]$

e
$$\delta W = -\frac{1}{4\pi G}\int d^3\bar{x} \cdot \frac{1}{2}\delta[|\bar{D}\phi|^2]$$

Ma la variazione della somma è la somma delle variazioni, per cui

$$(2.16) \qquad \delta W = -\frac{1}{8\pi G}\delta\int d^3\bar{x}\, |\bar{D}\phi|^2$$

Uso di nuovo la (*) con $\lambda \to \phi$ e $\bar{u} \to \bar{D}\phi$:

$$\bar{D}\cdot(\phi\bar{D}\phi) = \phi\bar{D}\cdot\bar{D}\phi + \underbrace{\bar{D}\phi\cdot\bar{D}\phi}_{|\bar{D}\phi|^2} \to |\bar{D}\phi|^2 = \bar{D}\cdot(\phi\bar{D}\phi) - \phi D^2\phi$$

$$W = -\frac{1}{8\pi G}\int_{Vol} d^3\bar{x}\, |\bar{D}\phi|^2 = -\frac{1}{8\pi G}\Big[\underbrace{\int_{Vol} d^3\bar{x}\, \bar{D}\cdot(\phi\bar{D}\phi)}_{\int_{sup} d\bar{S}\cdot\phi\bar{D}\phi} - \int_{Vol} d^3\bar{x}\, \phi \underbrace{D^2\phi}_{4\pi G\rho(\bar{x})}\Big]$$

$$\underbrace{\int_{sup} d\bar{S}\cdot\phi\,\bar{D}\phi}_{\to 0 \text{ se } r\to\infty}, \quad d\bar{S} \propto r^2,\ \phi \sim \frac{1}{r},\ \bar{D}\phi \sim \frac{1}{r^2}$$

da cui

$$W = \frac{1}{2}\int d^3\bar{x}\, \rho(\bar{x})\phi(\bar{x}) \qquad (2.18)$$

Ricorda: integrazione per parti:

$$\int f' g \, ds = fg - \int f g' \, ds$$

con

$$g \to GM(s) \qquad f \to \frac{1}{s}$$

$$\int -\frac{1}{s^2} GM(s)\, ds = \frac{GM(s)}{s} - \int \frac{1}{s} \cdot G \frac{dM}{\cancel{ds}} \cancel{ds}$$

$$-\int_r^\infty \frac{GM(s)}{s^2}\, ds = \underbrace{\frac{GM(s)}{s}\Big|_r^\infty}_{-\frac{GM(r)}{r}} - \int_r^\infty \frac{G}{s}\, dM(s)$$

$$-\frac{GM(r)}{r} = -\frac{G}{r}\int_0^r dM(s)$$

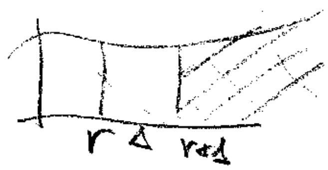
cioè

$$\Phi(r) = -\frac{G}{r}\int_0^r dM - G\int_r^\infty \frac{dM}{s} = \qquad (2.28)$$

$$= -\int_r^\infty \frac{GM(s)}{s^2}\, ds \quad //$$

$\bar{g} = -\bar{\nabla}\Phi$ lungo la direzione radiale: $g_r = -\frac{d\Phi}{dr}$

$$g_r = \frac{d}{dr}\int_r^\infty \frac{GM(s)}{s^2}\, ds$$



cioè

$$g_r = \lim_{\Delta \to 0} \frac{1}{\Delta}\left[\int_{r+\Delta}^\infty - \int_r^\infty\right] = -\frac{1}{\Delta}\int_r^{r+\Delta} \frac{GM(s)}{s^2}\, ds =$$

$$= -\frac{1}{\Delta} \cdot \Delta \cdot \frac{GM(r)}{r^2} \to g_r = -\frac{GM(r)}{r^2} \quad //$$

cvd

$$\boxed{M(r,\alpha,\beta) = 4\pi\rho_0 a^3 \int_0^{r/a} \frac{s^{2-\alpha}}{(1+s)^{\beta-\alpha}}\, ds}$$

Jaffe: $\beta=4,\ \alpha=2$

$$\int \frac{dx}{(1+x)^2} = -\frac{1}{1+x}$$

$$M_J(r) = 4\pi\rho_0 a^3 \int_0^{r/a} \frac{ds}{(1+s)^2} = 4\pi\rho_0 a^3\left[-\frac{1}{1+r/a}+1\right] = 4\pi\rho_0 a^3\,\frac{\cancel{-1}+\cancel{1}+r/a}{1+r/a}$$

$$\boxed{M_J(r) = 4\pi\rho_0 a^3 \cdot \frac{r/a}{1+r/a}}$$

Se $r\to\infty$, $M_J(r)\to 4\pi\rho_0 a^3$ finita

Herquist: $\beta=4,\ \alpha=1$

$$\int \frac{x}{(1+x)^3}dx = -\frac{1}{1+x}+\frac{1}{2(1+x)^2}$$

$$M_H(r) = 4\pi\rho_0 a^3 \int_0^{r/a} \frac{s}{(1+s)^3}ds = 4\pi\rho_0 a^3\left[\underbrace{-\frac{1}{1+r/a}+\frac{1}{2(1+r/a)^2}+1-\frac{1}{2}}\right]$$

$$\frac{1}{2}-\frac{1}{1+r/a}+\frac{1}{2(1+r/a)^2} = \frac{\cancel{1}+(\frac{r}{a})^2+\cancel{2\frac{r}{a}}-\cancel{2}-\cancel{2\frac{r}{a}}+\cancel{1}}{2(1+r/a)^2}$$

$$\boxed{M_H(r) = 4\pi\rho_0 a^3 \cdot \frac{(r/a)^2}{2(1+r/a)^2}}$$

Se $r\to\infty$, $\boxed{M_H\to 2\pi\rho_0 a^3}$

NFW: $\beta=3,\ \alpha=1$

$$\int \frac{x}{(1+x)^2}dx = \ln(1+x)+\frac{1}{1+x}$$

$$M_{NFW}(r) = 4\pi\rho_0 a^3 \int_0^{r/a} \frac{s}{(1+s)^2} = 4\pi\rho_0 a^3\left[\ln\left(1+\frac{r}{a}\right)+\underbrace{\frac{1}{1+r/a}-1}_{\frac{\cancel{1}-\cancel{1}-r/a}{1+r/a}}\right]$$

$$\boxed{M_{NFW}(r) = 4\pi\rho_0 a^3\left[\ln\left(1+\frac{r}{a}\right)-\frac{r/a}{1+r/a}\right]}$$

(logaritmicamente)

$M_{NFW}\to\infty$ se $r\to\infty$!

Non va bene a $r\gg a$

70 ter

$$\boxed{v_c(r) = \sqrt{\frac{GM(r)}{r}}}$$

$$v_c^2(r)_J = \frac{G}{\cancel{r}} \cdot 4\pi\rho_0 a^{\cancel{3}\,2} \frac{\cancel{r/a}}{1+r/a} = 4\pi G\rho_0 a^2 \cdot \frac{1}{1+r/a}$$

$$\boxed{v_c(r)_J = \sqrt{4\pi G\rho_0 a^2} \cdot \frac{1}{(1+r/a)^{1/2}}}$$

$$v_c^2(r)_H = \frac{G}{\cancel{r}} 4\pi\rho_0 a^{\cancel{3}\,2} \cdot \frac{(r/a)^{\cancel{2}\,1}}{2(1+r/a)^2} = 4\pi G\rho_0 a^2 \cdot \frac{r/a}{2(1+r/a)^2}$$

$$\boxed{v_c(r)_H = \sqrt{4\pi G\rho_0 a^2} \cdot \frac{\sqrt{r/a}}{\sqrt{2}(1+r/a)}}$$

$$v_c^2(r)_{NFW} = \frac{G}{r} \cdot 4\pi\rho_0 a^3 \cdot \left[\ln(1+\frac{r}{a}) - \frac{r/a}{1+r/a}\right] = 4\pi G\rho_0 a^2 \left[\frac{\ln(1+r/a)}{r/a} - \frac{1}{1+r/a}\right]$$

$$\boxed{v_c(r)_{NFW} = \sqrt{4\pi G\rho_0 a^2} \cdot \left[\frac{\ln(1+r/a)}{r/a} - \frac{1}{1+r/a}\right]^{1/2}}$$

$$\boxed{\Phi_{(r)} = -G\int_r^\infty \frac{M(s)}{s^2} ds}$$

$$\int \frac{dx}{x(1+x)} = -\ln\left|\frac{1+x}{x}\right|$$

$$\Phi_J(r) = -G\int_r^\infty 4\pi\rho_0 a^3 \frac{s/a}{1+s/a} \cdot \frac{1}{s^2} \frac{ds}{a} \cdot a = -4\pi G\rho_0 a^2 \int_{r/a}^\infty \frac{1}{x(1+x)} dx$$

$$= -4\pi G\rho_0 a^2 \cdot \ln\left[1 + \frac{a}{r}\right]$$

$$\int \frac{dx}{(1+x)^2} = -\frac{1}{1+x}$$

$$\Phi_H(r) = -4\pi G\rho_0 \int_r^\infty a^3 \cdot \frac{1}{s^2} \cdot \frac{(s/a)^2}{2(1+s/a)^2} \frac{ds}{a} \cdot a = -4\pi G\rho_0 a^2 \underbrace{\int_{r/a}^\infty \frac{dx}{2(1+x)^2}}_{\frac{1}{2} \cdot \frac{1}{1+r/a}}$$

$$\boxed{\Phi_H(r) = -4\pi G\rho_0 a^2 \cdot \frac{1}{2(1+r/a)}}$$

$$\Phi_{NFW}(r) = -4\pi G\rho_0 \int_r^\infty a^3 \cdot \frac{1}{s^2} \cdot \left[\ln(1+\frac{s}{a}) - \frac{s/a}{1+s/a}\right] \frac{ds}{a} \cdot a =$$

$$= -4\pi G\rho_0 a^2 \left[\int_{r/a}^\infty \frac{\ln(1+x)}{x^2} dx - \int_{r/a}^\infty \frac{dx}{x(1+x)}\right]$$

$$\int \frac{\ln(1+x)}{x^2} dx = \ln x - \left(\frac{1}{x}+1\right)\ln(1+x)$$

$$\Phi_{NFW}(r) = -4\pi G\rho_0 a^2 \left[\cancel{\ln x} - \underbrace{\frac{1+x}{x}\ln(1+x) + \ln(1+x)}_{\ln(1+x)\left[1-\frac{1+x}{x}\right] = \ln(1+x)\,\frac{\cancel{x}-1-\cancel{x}}{x} = -\frac{\ln(1+x)}{x}} - \cancel{\ln x} \right]_{r/a}^{\infty}$$

70 queker

$$\boxed{\Phi_{NFW}(r) = -4\pi G\rho_0 a^2 \cdot \frac{\ln(1+r/a)}{r/a}}$$