Semiclassical theory of conduction in metals - The Boltzmann equation

M. Peressi, University of Trieste
Condensed Matter Physics I
Dec. 2018
(reference: Ashcroft & Mermin, Solid State Physics, Ch 13 & 16)

1   Introduction - Electronic transport - generalities

Semiclassical model of electron dynamics applied in cases of :
- static electric fields
- uniform and static magnetic fields

From now on we will treat a **more general case** (presence of $\mathbf{E}$, $\mathbf{H}$, ∇T, functions of $\mathbf{r}$ and t; μ dependent on $\mathbf{r}$) but always considering:

- Independent electrons
- semiclassical motion between collisions
- no interband transition (conservation of band index n)
- no spin change (conservation of spin).

2   Sources of electronic scattering (collisions)

Ch 16 §I

- Perfect periodic crystal $\Longrightarrow$ NO COLLISIONS

- INDEPENDENT ELECTRON PICTURE (here!); main source of collisions:

    1. point defects, impurities, vacancies
    2. thermal effects $\Longrightarrow$ vibrations (small!) of electrons around their equilibrium positions (amplitude of vibrations depending on T: important scattering source in DC and main responsible of the T dependence of conductivity around R.T.; as $T \to 0$K, defects dominate)

- BEYOND INDEPENDENT ELECTRON APPROXIMATION:
  $e^- - e^-$ scattering ($e^- - e^-$ interactions) that are however:
  $<<$ T-dependent lattice vibrations effects at hight T;
  $<<$ impurity effects at low T

3   Non-equilibrium distribution function

A&M Ch. 13 §Introduction p 244 + Ch. 16 §IV p 319 + CH. 16 §I p 315

3.1 Generalities

We consider a **non equilibrium distribution function $g(\mathbf{r}, \mathbf{k}, t)$ in phase space** (in general, due to *applied fields* or *temperature gradients*). It gives the **occupancy** of state $\mathbf{k}$ at position $\mathbf{r}$ and time t.
External forces act to drive the distribution function **away** from equilibrium.
The limiting case at equilibrium, with NO applied fields, NO T gradients), corresponds to:

$$g^0(\mathbf{k}) = f(\mathcal{E}(\mathbf{k})) = \frac{1}{e^{(\mathcal{E}(\mathbf{k})-\mu)/k_B T} + 1} \tag{1}$$

AIM: derive a **closed** expression for $g(\mathbf{r}, \mathbf{k}, t)$ (when possible!) using:
- assumption of semiclassical equations of motion between collisions
- simple treatment of collisions

GENERALIZATION (the simplest possible!) OF THE RELAXATION TIME:
We continue to assume that an e^- experiences a collision in a time interval dt with probability dt/τ, but τ is in general $\tau(\mathbf{r},\mathbf{k})$.
Rationale: even in the independent electron approximation, collisions are not totally random and uncorrelated. They depend on the distribution of all the e^- (for instance, on the occupation of the other states $\mathbf{k}$).

3.2 Differential equation for $g(\mathbf{r}, \mathbf{k}, t)$

We construct in general $g(\mathbf{r}, \mathbf{k}, t)$ at time t from its value at time $t' = t - dt$
a) first assuming NO collision during the infinitesimal time interval dt
b) consider $\mathbf{r}$ and $\mathbf{k}$ evolving according to the semiclassical equations of motion:

$$\dot{\mathbf{r}} = \mathbf{v}(\mathbf{k}) \tag{2}$$

$$\hbar\dot{\mathbf{k}} = -e(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{H}) = \mathbf{F}(\mathbf{r}, \mathbf{k}). \tag{3}$$

We consider explicitly the time evolution of $\mathbf{r}$ and $\mathbf{k}$ from t' to t (linear expansion in dt):

$$t' = t - dt \qquad \longrightarrow t \tag{4}$$

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}(\mathbf{k})dt \qquad \longrightarrow \mathbf{r} \tag{5}$$

$$\mathbf{k}' = \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt \qquad \longrightarrow \mathbf{k} \tag{6}$$

The number of e^- that occupy the volume of phase space $\Delta\mathbf{r}\Delta\mathbf{k}$ centered in $\mathbf{r}$ and $\mathbf{k}$ at the time t is:

$$\frac{\Delta\mathbf{r}\Delta\mathbf{k}}{8\pi^3}g(\mathbf{r}, \mathbf{k}, t). \tag{7}$$

(we do not consider spin degeneracy, since we focus on a precise spin channel and we remain in that one). Analogously, the number of e^- occupying the volume of phase space

$\Delta \mathbf{r}'\Delta \mathbf{k}'$ centered in $\mathbf{r}$' and $\mathbf{k}$' at the time t' is:

$$\frac{\Delta \mathbf{r}'\Delta \mathbf{k}'}{8\pi^3}g(\mathbf{r}',\mathbf{k}',t') = \frac{\Delta \mathbf{r}'\Delta \mathbf{k}'}{8\pi^3}g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right). \tag{8}$$

If there are **no collisions (no scattering)**, the trajectory of the e^- in phase space is such that all the e^- that are in the volume of phase space $\Delta \mathbf{r}'\Delta \mathbf{k}'$ centered in $\mathbf{r}$' and $\mathbf{k}$' at the time t', are also in the volume of phase space $\Delta \mathbf{r}\Delta \mathbf{k}$ centered in $\mathbf{r}$ and $\mathbf{k}$ at the time t, therefore:

$$\frac{\Delta \mathbf{r}\Delta \mathbf{k}}{8\pi^3}g(\mathbf{r},\mathbf{k},t) = \frac{\Delta \mathbf{r}'\Delta \mathbf{k}'}{8\pi^3}g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right). \tag{9}$$

For the Liouville theorem:

$$\frac{\Delta \mathbf{r}\Delta \mathbf{k}}{8\pi^3} = \frac{\Delta \mathbf{r}'\Delta \mathbf{k}'}{8\pi^3} \tag{10}$$

therefore in absence of scattering:

$$g(\mathbf{r},\mathbf{k},t) - g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right) = 0. \tag{11}$$

Finally, we consider the expansion of $g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right)$ in terms of $g(\mathbf{r},\mathbf{k},t)$ up to the linear term in dt and we get:

$$g\left(\mathbf{r} - \mathbf{v}(\mathbf{k})dt, \mathbf{k} - \frac{\mathbf{F}}{\hbar}dt, t - dt\right) = g(\mathbf{r},\mathbf{k},t) - \frac{\partial}{\partial \mathbf{r}}g \cdot \mathbf{v}(\mathbf{k})dt - -\frac{\partial}{\partial \mathbf{k}}g \cdot \frac{\mathbf{F}}{\hbar}dt - -\frac{\partial}{\partial t}g \; dt \tag{12}$$

therefore in absence of scattering we get:

$$\frac{\partial}{\partial \mathbf{r}}g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial \mathbf{k}}g \cdot \frac{\mathbf{F}}{\hbar} + \frac{\partial}{\partial t}g = 0 \tag{13}$$

Introducing collisions, i.e., generalizing **in presence of scattering**, the right hand side of the previous equation is no longer zero, but it is the contribution due to scattered electrons:

$$\underbrace{\frac{\partial}{\partial \mathbf{r}}g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial \mathbf{k}}g \cdot \frac{\mathbf{F}}{\hbar} + \frac{\partial}{\partial t}g}_{} = \underbrace{\left(\frac{\partial g}{\partial t}\right)_{coll}}_{} \tag{14}$$

where the left side is the *DRIFT term* and the right side is the *COLLISION term*.

This is the BOLTZMANN EQUATION.

If we specify the forces and the collision term, we have an initial value problem to determine $g(\mathbf{r},\mathbf{k},t)$. The difficult part now is related to the explicit knowledge of the collision term, that in general could be very complicate.
The simplest possible approximation is the *relaxation time approximation* to replace the right hand side term with something much simpler, but this will come later on.

3.3 Change of $g(\mathbf{r}, \mathbf{k}, t)$ due to collisions

A&M CH. 16 §IV

We assume that the collisions are very well **localized in space and time**, so that those occurring to electrons in $\mathbf{r}$ at time t are totally determined by the properties of the system in the neighborhoods of $\mathbf{r}$ and close to time $t \implies$ for simplicity, we drop the explicit dependence of $g(\mathbf{r}, \mathbf{k}, t)$ on $\mathbf{r}$ and t in what follows.

We consider therefore collisions that change istantaneously the crystal momentum from a volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ to a volume $\Delta\mathbf{k}$' centered in $\mathbf{k}$' and viceversa. We distinguish scattering events that **increase** the occupancy of electronic states at $\mathbf{k}$ and those that **decrease** it:

$$\left(\frac{\partial g}{\partial t}\right)^{IN}_{coll} > 0, \qquad \left(\frac{\partial g}{\partial t}\right)^{OUT}_{coll} < 0. \tag{15}$$

We do not need to specify which is the scattering mechanism; it will be described in any case by some *scattering matrix* $W_{\mathbf{kk'}}$ for one electron suffering a scattering from a state $\mathbf{k}$ to $\mathbf{k}$'.

OUT: The e^- that are scattered OUT from the volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ will go somewhere else in phase space, in volumes $\Delta\mathbf{k}$' centered in all the possible $\mathbf{k}$', provided that the states are available (not already occupied, because the Pauli principle must be satisfied). Therefore, the contribution of such scattering events to the variation of $g(\mathbf{r}, \mathbf{k}, t)$ is (in the continuum limit):

$$\left(\frac{\partial g}{\partial t}\right)^{OUT}_{coll} = -g(\mathbf{k}) \int \frac{d\mathbf{k}'}{8\pi^3} W_{\mathbf{kk'}} \left(1 - g(\mathbf{k}')\right) \tag{16}$$

where we have also introduced a weighting factor $g(\mathbf{k})$ which is the occupancy of the states in the volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ and the sign "$-$" since it is a reduction in the occupancy.

IN: The e^- that are scattered IN the volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ come from somewhere else in phase space, from volumes $\Delta\mathbf{k}$' centered in all the possible $\mathbf{k}$', provided that those states are filled. Therefore, the contribution of such scattering events to the variation of $g(\mathbf{r}, \mathbf{k}, t)$ is (in the continuum limit):

$$\left(\frac{\partial g}{\partial t}\right)^{IN}_{coll} = (1 - g(\mathbf{k})) \int \frac{d\mathbf{k}'}{8\pi^3} W_{\mathbf{k'k}} g(\mathbf{k}') \tag{17}$$

where we have also introduced a weighting factor $(1-g(\mathbf{k}))$ which is the availability of the states in the volume $\Delta\mathbf{k}$ centered in $\mathbf{k}$ (they should not be already occupied, because the Pauli principle must be satisfied).

Keep in mind that these scattering events are supposed to be LOCALIZED in space and time (depending only on $\mathbf{r}$ and t, not on $\mathbf{r}$' and t'). The **total balance** of the scattering events gives:

$$\left(\frac{\partial g}{\partial t}\right)_{coll} = -\int \frac{\Delta \mathbf{k}'}{8\pi^3} \left\{ W_{\mathbf{k}\mathbf{k}'} g(\mathbf{k}) \left[1 - g(\mathbf{k}') \right] - W_{\mathbf{k}'\mathbf{k}} g(\mathbf{k}')[1 - g(\mathbf{k})] \right\} \tag{18}$$

4 Relaxation time approximation

The simplest expression for the collision term is:

$$\left(\frac{\partial g}{\partial t}\right)_{coll} \approx -\frac{g(\mathbf{k}) - g^0(\mathbf{k})}{\tau(\mathbf{k})} \tag{19}$$

where $\tau(\mathbf{k})$ is some *relaxation time*. Consequently, the Boltzmann equation can be written as:

$$\frac{\partial}{\partial t}g + \frac{\partial}{\partial \mathbf{r}}g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial \mathbf{k}}g \cdot \frac{\mathbf{F}}{\hbar} = -\frac{\delta g}{\tau(\mathbf{k})} \tag{20}$$

suggests that collisions tend to restore the equilibrium, balancing the effect of the drift terms.

5 Scattering from homogeneous materials

In case of *homogeneous materials* and *stationary state* and keeping the relaxation time approximation, it is convenient to focus on the *variation* of $g(\mathbf{k})$, by defining:

$$\delta g(\mathbf{k}) = g(\mathbf{k}) - g^0(\mathbf{k}) \tag{21}$$

In this case the equilibrium distribution function $g^0(\mathbf{k})$ *does not dependent on* $\mathbf{r}$ *nor on* t, but only on $\mathbf{k}$. Therefore, since:

$$\frac{\partial \delta g}{\partial t} = \frac{\partial g}{\partial t} \tag{22}$$

$$\frac{\partial \delta g}{\partial \mathbf{r}} = \frac{\partial g}{\partial \mathbf{r}} \tag{23}$$

$$\frac{\partial \delta g}{\partial \mathbf{k}} = \frac{\partial (g - g^0)}{\partial \mathbf{k}} \tag{24}$$

we finally get that the Boltzmann equation reduces to:

$$\frac{\partial}{\partial t}\delta g + \frac{\partial}{\partial \mathbf{r}}\delta g \cdot \mathbf{v}(\mathbf{k}) + \frac{\partial}{\partial \mathbf{k}}(g^0 + \delta g) \cdot \frac{\mathbf{F}}{\hbar} = -\frac{\delta g}{\tau(\mathbf{k})} \tag{25}$$

If a perturbation not breaking the translational symmetry $\left(\frac{\partial \delta g}{\partial \mathbf{r}} = 0\right)$ has been applied and has driven the system out of equilibrium for $t < 0$ and then it is stopped at $t = 0$ (i.e., $\mathbf{F}(t \geq 0)=0$ and $\delta g(t = 0) = \delta g_0 \neq 0$), Eq. 25 for $t \geq 0$ reduces to:

$$\frac{\partial}{\partial t}\delta g = -\frac{\delta g}{\tau(\mathbf{k})} \tag{26}$$

that gives $\delta g(t) = \delta g_0 \exp(-t/\tau(\mathbf{k}))$, indicating that the system goes back to equilibrium, as expected.

6 Applications and examples

Remaining within the assumption of SMALL FIELDS, we will discuss the following cases:

- Isotropic perturbations $\Longrightarrow$ we can use Eq. 25

    - D.C., static and uniform $\mathbf{E}$ (stationary state)
    - A.C., using Linear Response Theory

- Materials with $\nabla_{\mathbf{r}}T \neq 0$ and $\mu = \mu(\mathbf{r}) \Longrightarrow$ we must use Eq. 20

    - D.C., static and uniform $\mathbf{E}$ (stationary state)
    - $\nabla T = 0$ but $\mu = \mu(\mathbf{r})$; applied static and uniform $\mathbf{E}$ and $\mathbf{H}$

6.1 Isotropic perturbation: static and uniform E applied (stationary state)

In case of stationary state with static and uniform $\mathbf{E}$ applied, Eq. 25 further simplifies, since $\dfrac{\partial}{\partial t}\delta g{=}0$ (stationary state) and $\dfrac{\partial}{\partial \mathbf{r}}\delta g{=}0$ (deviations from the equilibrium cannot depend on a specific point $\mathbf{r}$). Furthermore, $\dfrac{\partial}{\partial \mathbf{k}}\delta g \cdot \mathbf{E}$ must be neglected, being infinitesimal of second order in $\mathbf{E}$, so that the gradient in $\mathbf{k}$ contains only the contribution:

$$\frac{\partial}{\partial \mathbf{k}}g^0 = \nabla_{\mathbf{k}}\mathcal{E}(\mathbf{k})\frac{\partial g^0}{\partial \mathcal{E}} = \hbar \mathbf{v}(\mathbf{k})\frac{\partial g^0}{\partial \mathcal{E}} \tag{27}$$

Therefore we get:

$$\delta g = e\mathbf{E}\cdot\mathbf{v}(\mathbf{k})\tau(\mathbf{k})\frac{\partial g^0}{\partial \mathcal{E}} \tag{28}$$

The equation that relates current and electron velocity, which is $\mathbf{j} = -ne\mathbf{v}$ in the Drude model, becomes:

$$\mathbf{j} = -e\sum_{\mathbf{k}}\delta g(\mathbf{k})\mathbf{v}(\mathbf{k}) = -e^2\int\frac{d\mathbf{k}}{(2\pi)^3}\left(\mathbf{E}\cdot\mathbf{v}(\mathbf{k})\right)\mathbf{v}(\mathbf{k})\tau(\mathbf{k})\frac{\partial g^0}{\partial \mathcal{E}} \tag{29}$$

We can see from the previous eq. the tensorial character of the conductivity:

$$j_i = \sum_{j}\sigma_{ij}E_j \tag{30}$$

with

$$\sigma_{ij} = e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \tau(\mathbf{k}) v_i(\mathbf{k}) v_j(\mathbf{k}) \left[-\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})} \right] \qquad (31)$$

Note:

- **Anisotropy:** in general $\mathbf{j}$ is NOT parallel to $\mathbf{E}$, at variance with free electron case. In case of *cubic* materials, however, the expression of σ is simplified: $\sigma_{ij} = \delta_{ij}\sigma$.

- **Filled bands are inert:** only deviations from full filling are important

- **Importance of Fermi surfaces:** Although the integral is done over the whole BZ, the term $\left[\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})} \right]$ under the integral is non zero only around the Fermi surface (since it is basically $\propto \delta(\mathcal{E} - \mathcal{E}_F)$). The conductivity is determined by the conduction band in an interval of size $k_B T$ around $\mathcal{E}_F$.

- **Particular case: parabolic band and effective mass.** Considering that: (i) $-\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})} = \delta(\mathcal{E} - \mathcal{E}_F)$; (ii) averaging over all the electrons, $\langle v_i v_j \rangle = \langle v_i^2 \rangle \delta_{ij} = \frac{1}{3} v^2 \delta_{ij}$; (iii) accounting for a spin factor of 2 if the two spin channels equally contribute to the current; we have:

$$
\begin{aligned}
\sigma &= 2e^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \tau(\mathbf{k}) \frac{1}{3} v^2(\mathbf{k}) \delta(\mathcal{E} - \mathcal{E}_F) \\
&= \frac{e^2}{12\pi^3} \int_{\substack{Fermi \\ surf}} \tau(\mathbf{k}) v^2(\mathbf{k}) \frac{dS}{|\nabla_\mathbf{k} \mathcal{E}(\mathbf{k})|} \\
&= \frac{e^2}{12\pi^3} \int_{\substack{Fermi \\ surf}} \tau(\mathbf{k}) v(\mathbf{k}) \frac{1}{\hbar} dS \qquad (32)
\end{aligned}
$$

Furthermore, considering that in case of parabolic and isotropic band, the Fermi surface is a Fermi sphere with radius k_F and the integral of dS over such surface simply gives the surface area, we finally get:

$$\sigma = \frac{e^2}{12\pi^3} \tau_F v_F \frac{1}{\hbar} 4\pi k_F^2 = \frac{ne^2 \tau_F}{m^*} \qquad (33)$$

which resembles the well known result for free electrons, BUT with $m \to m^*$ and $\tau \to \tau_F$. Attention: in the final result, the fact that ONLY the electrons around $\mathcal{E}_F$ do contribute to the current is hidden, but clear in the derivation!

_____**END OF THE EXAMPLES DISCUSSED IN CLASS IN 2018/19** _____

6.2 Isotropic perturbation: A.C. conductivity, using Linear Response Theory

A&M, Ch 13, p. 252

We start again from the linearized eq. 25 for δg, but since now $\mathbf{E}$ is space and time dependent, we must keep all the terms in $\dfrac{\partial}{\partial t}$, and $\delta g = \delta g(\mathbf{r}, \mathbf{k}, t)$. As in the previous case, $\frac{\partial}{\partial \mathbf{k}}\delta g \cdot \mathbf{E}$ must be neglected, being infinitesimal of second order in $\mathbf{E}$, so that we get:

$$\frac{\partial}{\partial t}\delta g + \frac{\partial}{\partial \mathbf{r}}\delta g \cdot \mathbf{v}(\mathbf{k}) - \frac{e\mathbf{E}}{\hbar}\cdot\frac{\partial g^0}{\partial \mathbf{k}} = -\frac{\delta g}{\tau(\mathbf{k})} \tag{34}$$

Consider $\mathbf{E}(\mathbf{r}, t) = \mathbf{E_0}e^{i(\mathbf{q}\cdot\mathbf{r}-wt)}$. In case of linear response regime, we have: $\delta g(\mathbf{r}, \mathbf{k}, t) = \Phi(\mathbf{k})e^{i(\mathbf{q}\cdot\mathbf{r}-wt)}$. Substituting in eq. 34, we get:

$$-iw\Phi + \mathbf{v}\cdot i\mathbf{q}\Phi - \frac{e}{\hbar}\mathbf{E_0}\frac{\partial g^0}{\partial \mathbf{k}} = -\frac{\Phi}{\tau} \tag{35}$$

from which, using eq. 27, we obtain:

$$\Phi(\mathbf{k}) = \frac{e\tau\mathbf{E_0}\cdot\mathbf{v}}{1 - i\tau(w - \mathbf{q}\cdot\mathbf{v})}\frac{\partial g^0}{\partial \mathcal{E}} \tag{36}$$

Considering the expression for the current in terms of the electron velocity (always remind the analogous in the Drude model, $\mathbf{j} = -ne\mathbf{v}$):

$$
\begin{aligned}
\mathbf{j} &= -e\int\frac{d\mathbf{k}}{(2\pi)^3}\delta g(\mathbf{k})\mathbf{v}(\mathbf{k}) \\
&= -e\int\frac{d\mathbf{k}}{(2\pi)^3}\Phi(\mathbf{k})e^{i(\mathbf{q}\cdot\mathbf{r}-wt)}\mathbf{v}(\mathbf{k}) \\
&= -e\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{e\tau\mathbf{E}\cdot\mathbf{v}}{1 - i\tau(w - \mathbf{q}\cdot\mathbf{v})}\frac{\partial g^0}{\partial \mathcal{E}}\mathbf{v}(\mathbf{k})
\end{aligned}
\tag{37}
$$

and the expression for the current in terms of conductivity and electric field (eq. 30), we get:

$$\sigma_{ij} = e^2\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{\tau(\mathbf{k})v_i(\mathbf{k})v_j(\mathbf{k})}{1 - i\tau(\mathbf{k})\,(w - \mathbf{q}\cdot\mathbf{v}(\mathbf{k}))}\left[-\frac{\partial g^0}{\partial \mathcal{E}(\mathbf{k})}\right] \tag{38}$$

which reduces to the static case previously discussed (eq. 31) when $\mathbf{q} \to 0$ (long wavelengths, field uniform in space) and $w \to 0$ (static limit).

6.3 Anisotropic material ($\nabla T \neq 0$, $\mu = \mu(\mathbf{r})$) static and uniform E (stationary state)

In this case also g^0 depends on $\mathbf{r}$, so that we must use eq. 20. We have:

$$\frac{\partial g}{\partial \mathbf{r}} = \frac{\partial g}{\partial \mu}\cdot\nabla\mu + \frac{\partial g}{\partial T}\cdot\nabla T \tag{39}$$

and since, being consistent with the linear expansion in the perturbative terms:

$$\frac{\partial g}{\partial \mu} \approx \frac{\partial g^0}{\partial \mu} = \frac{g^0}{k_B T} e^{\frac{\mathcal{E}-\mu}{k_B T}}$$

$$\frac{\partial g}{\partial T} \approx \frac{\partial g^0}{\partial T} = \frac{\mathcal{E}-\mu}{(k_B T)^2}(g^0)^2 k_B e^{\frac{\mathcal{E}-\mu}{k_B T}} \tag{40}$$

Then, since $\dfrac{\partial g}{\partial \mathcal{E}} \approx \dfrac{\partial g^0}{\partial \mathcal{E}} = -\dfrac{(g^0)^2}{k_B T} e^{\frac{\mathcal{E}-\mu}{k_B T}}$, we can express the right-hand side terms of eq. 40 in terms of $\dfrac{\partial g^0}{\partial \mathcal{E}}$, obtaining:

$$\frac{\partial g}{\partial \mu} \approx -\frac{\partial g^0}{\partial \mathcal{E}}$$

$$\frac{\partial g}{\partial T} \approx -\frac{\mathcal{E}-\mu}{T}\frac{\partial g^0}{\partial \mathcal{E}} \tag{41}$$

Similarly, also $\dfrac{\partial g}{\partial \mathbf{k}}$ can be expressed in terms of $\dfrac{\partial g^0}{\partial \mathcal{E}}$, obtaining:

$$\frac{\partial g}{\partial \mathbf{k}} = \frac{\partial g}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathbf{k}} \approx \frac{\partial g^0}{\partial \mathcal{E}}\frac{\partial \mathcal{E}}{\partial \mathbf{k}} = \frac{\partial g^0}{\partial \mathcal{E}}\ \hbar \mathbf{v}(\mathbf{k}). \tag{42}$$

Substituting in eq. 20 and considering the stationary regime $\left(\dfrac{\partial g}{\partial t} = 0\right)$, we get, a part from terms of higher order in $\nabla_{\mathbf{r}}T$ and $\mathbf{E}$:

$$\tau(\mathcal{E}(\mathbf{k}))\ \mathbf{v}(\mathbf{k}) \cdot \left(\nabla_{\mathbf{r}}\mu + \frac{\mathcal{E}-\mu}{T}\ \nabla_{\mathbf{r}}T + e\mathbf{E}\right)\left(-\frac{\partial g^0}{\partial \mathcal{E}}\right) = g^0(\mathbf{k}) - g(\mathbf{k}) \tag{43}$$

which is eq. (13.43) of A&M book (Ch. 13.2).

6.4 Anisotropic material ($\nabla T = 0$ but $\mu = \mu(\mathbf{r})$) static and uniform E and H (stationary state)

For the sake of simplicity, we consider now $\nabla T = 0$. We can start from the last eq. 43. The external force term, $e\mathbf{E}$, must be substituted with $e\mathbf{E} + \dfrac{1}{c}\mathbf{v} \times \mathbf{H}$. It is interesting to notice that $\mathbf{v} \cdot \mathbf{v} \times \mathbf{H} = 0$, so that the external applied magnetic field has NO EFFECT! Furthermore, if we consider the case of a parabolic and isotropic band, so that $\mathcal{E}(\mathbf{k}) = \dfrac{\hbar^2 k^2}{2m^*}$, we have that $\mathbf{v} = \dfrac{\hbar \mathbf{k}}{m^*}$ (note that only in this approximation $\mathbf{v}$ is parallel to $\mathbf{k}$!), and eq. 43 becomes:

$$\underbrace{\tau(\mathcal{E}(\mathbf{k}))\ \left(-\frac{\partial g^0}{\partial \mathcal{E}}\right)(\nabla_{\mathbf{r}}\mu + e\mathbf{E})\frac{\hbar}{m^*}} \cdot \mathbf{k} = g^0(\mathbf{k}) - g(\mathbf{k}). \tag{44}$$

The part $\underbrace{}$ in the left-hand side term is dependent only on $\mathcal{E}$, so that we can rewrite the last equation in a very simplified form:

$$\mathbf{a}(\mathcal{E}) \cdot \mathbf{k} = g^0(\mathbf{k}) - g(\mathbf{k}). \tag{45}$$

which is eq. (16.26) of A&M book.