Yudovich solutions and Vortex Patches

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December 17, 2019

The present lecture notes are the supplementary material of a 4 hours course given at the University of Trieste in December 2019. Most of the material comes from the monographs [4] and [5].

Contents

1 Introduction

The incompressible Euler equations describe the motion of a perfect fluid and consist of the equations

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{E}$$

All along the present lecture notes we suppose $(t, x) \in \mathbb{R} \times \mathbb{R}^d,\ d \geqslant 2,\ u : \mathbb{R}^{1+d} \to \mathbb{R}^d$ and $p : \mathbb{R}^{1+d} \to \mathbb{R}$. It is easy to see that we can suppress the pressure since

$$-\Delta p = \partial_{ij} \left(u_i\ u_j \right),$$

so that if u exists and it is sufficiently regular p can be determined solving the above equation.

The study of the equations (E) has a long tradition, they have been derived at first by L. Euler in 1757 and they are the second PDE ever been derived (the first one is the 1D wave equation derived by D'Alembert in 1747). Despite being a very classical topic in PDE the mathematical understanding of (E) when $d \geqslant 3$ is far from complete, even though remarkable advances have been proved in recent years.

1.1 Preliminaries

Definition 1.1. For any $s \in \mathbb{R}$ let us define the Sobolev space $H^s(\mathbb{R}^d)$ as the closure of Schwartz functions w.r.t. the norm

$$\|u\|_{H^s(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left(1+|\xi|^2\right)^s |\hat{u}(\xi)|\,\mathrm{d}\xi.$$

We denote $H^s = H^s(\mathbb{R}^2)$.

Notation 1.2. With C we denote a strictly positive constant whose value may vary from line to line and which is independent of any parameter of the problem, the explicit value of C may vary from line to line. We use the notation $A \lesssim B$ if $A \leqslant CB$ and $A \sim B$ is $A \lesssim B$ and $B \lesssim A$.

Local solvability of (E) in any dimension is well understood, cf. [4,5], here we propose a statement in the functional framework of Sobolev spaces $H^s(\mathbb{R}^d)$:

Theorem 1.3. *Let $s > \frac{d}{2}+1$ and let $u_0 \in H^s(\mathbb{R}^d)$, there exists a $T \gtrsim 1/\|u_0\|_{H^s(\mathbb{R}^d)}$ and a unique solution u, p of* (E) *in $[0,T]\times\mathbb{R}^d$ such that*

$$u, \nabla p \in \mathcal{C}\left([0,T]; H^s\left(\mathbb{R}^d\right)\right).$$

1.2 The case $d = 2$

When $d = 2$ the mathematical theory of (E) is remarkably better understood, the main reason of such stark difference is that the *vorticity*

$$\Omega = \nabla u - (\nabla u)^{\mathsf{T}} : \mathbb{R}^{1+d} \mapsto \mathbb{R}^{d\times d},$$

when $d = 2$ is the scalar quantity

$$\omega = -\partial_2 u_1 + \partial_1 u_2.$$

An explicit computation shows that Ω evolves accordingly to the law

$$\partial_t \Omega + u\cdot\nabla\Omega + \Omega\cdot\nabla u + (\nabla u)^{\mathsf{T}}\cdot\Omega = 0, \tag{1.1}$$

while ω solves

$$\begin{aligned} \partial_t\omega + u\cdot\nabla\omega + \overbrace{\omega\,\mathrm{div}\,u}^{\equiv 0} &= \quad , \\ \partial_t\omega + u\cdot\nabla\omega &= 0 \end{aligned} \tag{1.2}$$

so that ω solves a *transport* equation, while in the equation for Ω the term

$$\Sigma = \Omega\cdot\nabla u + (\nabla u)^{\mathsf{T}}\cdot\Omega,$$

known as *stretching term* allows for accumulations of vorticity. This is a key feature for the incompressible Euler equations and it is the mechanism with which (it is conjectured) finite-time singularities are formed. In such direction we state the much celebrated *Beale-Kato-Majda* criterion

Theorem 1.4. *Let u the unique solution of* (E) *identified in Theorem 1.3, and assume $\nabla u_0 \in L^1(\mathbb{R}^d)$, if T is finite and such that*

$$\int_0^T \|\Omega(t)\|_{L^\infty}\,\mathrm{d}t < \infty,$$

then u can be continued beyond T to a $H^s(\mathbb{R}^d)$ solution of (E).

Using Theorem (1.4) and the fact that ω solves a transport equation it is possible to prove the following global result in the 2D case:

Theorem 1.5. *Let $d = 2$ and let u be the unique solution stemming from $u_0 \in H^s \cap W^{1,1}$, $s > 2$, then $u \in \mathcal{C}(\mathbb{R}; H^s)$ and there exists a $c_0 = c_0(u_0) > 0$ such that*

$$\|u(t)\|_{H^s}^2 \leqslant \|u_0\|_{H^s}^2\, e^{c_0 e^{c_0 t}},$$

for all $t > 0$.

2 Yudovich solutions

It is clear that 1.5 asserts that, given a *sufficiently regular* initial velocity field, there exists a unique global solution of (E) which at all times it is as regular as the initial data. Such solution hence do not exhibit *loss of regularity*, and the main ingredient in the above theory is the requirement of a initial velocity flow that is in $\mathcal{C}^{1,1}$. Let us now consider a smooth subdomain D_0 of $\mathbb{R}^2$ (w.l.o.g. we may assume D_0 to be the unit disk), and let us assume

$$\omega_0 = \nabla \times u_0 = \mathbb{1}_{D_0}.$$

It clear that the gradient of the initial velocity flow ∇u_0 is *not* $\mathcal{C}^{0,1}$ and in fact it is discontinuous in the normal direction of the interface, thus the theory above does not apply in such setting.

The construction of global-in-time weak solutions for (E) under the very mild assumption that $\omega_0 \in L^1 \cap L^\infty$ is a classical result due to Yudovich (see [6]) and it is the main goal of the present section.

2.1 Weak vorticity-stream formulation of the Euler equations

Notation 2.1. Given a scalar smooth function $f = f(t,x)$ and a smooth vector field $u = u(t,x)$ we denote

$$D_t f = \partial_t f + u \cdot \nabla f.$$

The evolution equation for ω writes hence as

$$\begin{cases} D_t \omega = 0, \\ \omega|_{t=0} = \omega_0, \end{cases}$$

moreover since the velocity flow is isochoric we know that there exists a scalar function ψ known as the *stream function* such that

$$u = \nabla^\perp \psi = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix} \psi,$$

thus $\Delta\psi = \omega$ hence formally we have that

$$\begin{aligned} u &= \nabla^\perp \psi \\ &= \nabla^\perp \Delta^{-1} \omega \\ &= \frac{1}{2\pi} \nabla^\perp \int \log|x-y| \, \omega(y) \, \mathrm{d}y, \\ &= \frac{1}{2\pi} \int \frac{(x-y)^\perp}{|x-y|^2} \, \omega(y) \, \mathrm{d}y, \\ &= K \star \omega. \end{aligned} \tag{2.1}$$

The relation (2.1) is known as *Biot-Savart law*

We are now in condition to give a suitable definition for weak solution for the Euler equations:

Definition 2.2. Let $\omega_0 \in L^1 \cap L^\infty$, we say that the couple (ω, u) is a weak solution of (E) if

- ⋄ $\omega \in L^\infty([0,T]; L^1 \cap L^\infty)$ for any $T > 0$,
- ⋄ $u = K \star \omega$ and $\omega = \nabla \times u$,
- ⋄ For any $\phi \in \mathcal{C}^1([0;T]; \mathcal{C}^1_0)$ the following equality holds true

$$\int \omega(T,x) \ \phi(T,x) \, \mathrm{d}x - \int \omega_0(x) \ \phi(0,x) \, \mathrm{d}x = \int_0^T \int \omega(t,x) \ D_t \phi(t,x) \, \mathrm{d}x \, \mathrm{d}t. \tag{2.2}$$

2.2 Existence of weak solutions

The result we want to prove is hence the following one:

Theorem 2.3. *Let $\omega_0 \in L^1 \cap L^\infty$, then there exits a unique solution of* (E) *in the sense of Definition 2.2.*

Let us denote with η a smooth, positive function supported in $B(0,1)$ with unitary total mass, and let us define the mollification

$$\omega_0^\varepsilon(x) = \frac{1}{\varepsilon^2} \int \eta\left(\frac{x-y}{\varepsilon}\right) \omega(y)\, \mathrm{d}y.$$

Let us denote with $\eta^\varepsilon(\cdot) = \varepsilon^{-2}\eta(\cdot/\varepsilon)$, for any $p \in [1,\infty]$ we have indeed that

$$\left\|\omega_0^\varepsilon\right\|_{L^p} = \left\|\eta^\varepsilon \star \omega_0\right\|_{L^p} \leqslant \left\|\eta^\varepsilon\right\|_{L^1} \|\omega_0\|_{L^p} = \left\|\eta\right\|_{L^1} \|\omega_0\|_{L^p} = \|\omega_0\|_{L^p}, \tag{2.3}$$

moreover

$$\left\|\omega_0^\varepsilon - \omega_0\right\|_{L^1} \xrightarrow{\varepsilon\to 0} 0. \tag{2.4}$$

Let us prove (2.4), since $\omega_0 \in L^1$ we know that there exists a sequence of smooth functions $\tilde{\omega}_0^\delta$ which converges to ω_0 in L^1 and a.e. (up to non-relabeled subsequence) when $\delta \to 0$ so that

$$\left\|\omega_0^\varepsilon - \omega_0\right\|_{L^1} \leqslant \left\|\omega_0^\varepsilon - \tilde{\omega}_0^\delta\right\|_{L^1} + \left\|\tilde{\omega}_0^\delta - \omega_0\right\|_{L^1},$$

so that for any ε_0 there exists a $\delta = \delta(\varepsilon_0)$ s.t.

$$\left\|\tilde{\omega}_0^\delta - \omega_0\right\|_{L^1} \leqslant \frac{\varepsilon_0}{3}.$$

Next, since η has mass one and is identically zero outside $B(0,1)$ we have that

$$\begin{aligned}
\omega_0^\varepsilon(x) - \tilde{\omega}_0^\delta(x) &= \frac{1}{\varepsilon^2}\int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right)\left(\omega_0(y) - \tilde{\omega}_0^\delta(x)\right)\mathrm{d}y,\\
&= \frac{1}{\varepsilon^2}\int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right)\left(\omega_0(y) - \tilde{\omega}_0^\delta(y)\right)\mathrm{d}y + \frac{1}{\varepsilon^2}\int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon}\right)\left(\tilde{\omega}_0^\delta(y) - \tilde{\omega}_0^\delta(x)\right)\mathrm{d}y,\\
&= I_1(x) + I_2(x),
\end{aligned}$$

using Young inequality it is immediate to prove that

$$\|I_1\|_{L^1} \leqslant \left\|\eta^\varepsilon\right\|_{L^1}\left\|\omega_0 - \tilde{\omega}_0^\delta\right\| \leqslant \frac{\varepsilon_0}{3},$$

while since the function $\tilde{\omega}_0^\delta$ for δ fixed, is uniformly continuous we have that there exists a $\varepsilon = \varepsilon(\varepsilon_0) > 0$ such that

$$\left|\tilde{\omega}_0^\delta(y) - \tilde{\omega}_0^\delta(x)\right| \leqslant \frac{\varepsilon_0}{3},$$

thus we proved (2.4).

Theorem 1.5 assures us that there exists an ω^ε stemming from ω_0^ε that solves globally $D_t\omega^\varepsilon = 0$, moreover $u^\varepsilon = K \star \omega^\varepsilon$ solves (E) with initial data $u_0^\varepsilon = K \star \omega_0^\varepsilon$ globally-in-time. The couple $(\omega^\varepsilon, u^\varepsilon)$ indeed solves (2.2).

Proposition 2.4. *Let $(\omega^\varepsilon, u^\varepsilon)$ the global solution stemming from the mollified initial data ω_0^ε, the following uniform bounds hold true for any $t > 0$*

1. $$\left\|u^\varepsilon(t,\cdot)\right\|_{L^\infty} \lesssim \left\|\omega^\varepsilon(t,\cdot)\right\|_{L^1\cap L^\infty} \lesssim \|\omega_0\|_{L^1\cap L^\infty}, \tag{2.5}$$

2. *There exists a $\omega(t,\cdot) \in L^1 \cap L^\infty$ and $u = K \star \omega$ such that*

$$\omega^\varepsilon(t,\cdot) \xrightarrow{\varepsilon\to 0} \omega(t,\cdot) \qquad \text{in } L^1, \tag{2.6}$$

$$u^\varepsilon(t,\cdot) \xrightarrow{\varepsilon\to 0} u(t,\cdot) \qquad \text{in } L^\infty_{\text{loc}}. \tag{2.7}$$

Using the result stated in Proposition 2.4 it is a simple matter to prove that the limit functions (ω, u) are indeed weak solutions of the incompressible Euler equations. In particular given a test function ϕ we want to prove that

$$\int_0^T \int \left(\omega^\varepsilon(t,x)\ u^\varepsilon(t,x) - \omega(t,x)\ u(t,x)\right)\cdot\nabla\phi(t,x)\ \mathrm{d}x\,\mathrm{d}t \xrightarrow{\varepsilon\to 0} 0,$$

thus the above limit can be proved using the results of Proposition 2.4 and we conclude.

2.3 Proof of Proposition 2.4

In the present section we prove the key technical results stated in Proposition 2.4.

2.3.1 Proof of of Proposition 2.4, point 1

The proof of the inequality

$$\left\|\omega^{\varepsilon}(t,\cdot)\right\|_{L^1\cap L^\infty} \lesssim \left\|\omega_0^{\varepsilon}\right\|_{L^1\cap L^\infty} \lesssim \|\omega_0\|_{L^1\cap L^\infty},$$

follows from the fact that the stransport equation conserves L^p, $p\in[1,\infty]$ norms and the properties of mollifiers proved in (2.3). Let us now denote with χ a $\mathcal{C}_0^\infty$ radial cutoff which is supported in $B(0,2)$ and $\chi(x)\equiv 1$, $\forall\ x\in B(0,1)$. Next, since $u^\varepsilon = K\star\omega^\varepsilon$ we write is as

$$u^{\varepsilon}(t,x) = u_1^{\varepsilon}(t,x) + u_2^{\varepsilon}(t,x),$$

where

$$\begin{aligned} u_1^{\varepsilon} &= \left(\chi K\right)\star\omega^{\varepsilon}, \\ u_2^{\varepsilon} &= \left(\left(1-\chi\right)K\right)\star\omega^{\varepsilon}, \end{aligned}$$

we have that

$$\left\|u_\varepsilon^1\right\|_{L^\infty} \leqslant \left\|\chi K\right\|_{L^1}\left\|\omega^\varepsilon\right\|_{L^\infty}, \qquad \left\|u_2^\varepsilon\right\|_{L^\infty} \leqslant \left\|\left(1-\chi\right)K\right\|_{L^\infty}\left\|\omega^\varepsilon\right\|_{L^1},$$

from which the inequality

$$\left\|u^{\varepsilon}\right\|_{L^\infty} \lesssim \left\|\omega^{\varepsilon}\right\|_{L^1\cap L^\infty},$$

is immediate.

2.3.2 Proof of of Proposition 2.4, point 2

The proof of the second point is longer and more involved. Let us denote with X_ε the particle-trajectory flow generated by u^ε, which is the solution of the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}X_\varepsilon(t,\alpha) = u^\varepsilon\left(t, X_\varepsilon(t,\alpha)\right), \qquad X_\varepsilon(\alpha,0)=\alpha,$$

and let $Y_\varepsilon = Y_\varepsilon(t,\cdot)$ be the inverse flow map at time t, i.e.

$$X_\varepsilon\left(t, Y_\varepsilon(t,x)\right) = x, \qquad Y_\varepsilon\left(t, X_\varepsilon(t,\alpha)\right)=\alpha,$$

which exists since solutions stemming from mollified velocities $u_0^\varepsilon = K\star\left(\eta^\varepsilon\star\omega_0\right)$ exist globally. We can think of $Y_\varepsilon(t,x) = \mathcal{Y}_\varepsilon\left(t',x;t\right)\big|_{t'=t}$ where

$$\mathcal{Y}_\varepsilon\left(t',x;t\right) = X_\varepsilon\left(t-t',\alpha\right), \qquad X_\varepsilon(t,\alpha)=x.$$

Additionally for any $t\in[0,T]$ the backward trajectories solve the ODE

$$\frac{\mathrm{d}}{\mathrm{d}t'}\mathcal{Y}_\varepsilon\left(t',x;t\right) = -u^\varepsilon\left(t-t', \mathcal{Y}_\varepsilon\left(t',x;t\right)\right), \qquad \mathcal{Y}_\varepsilon(0,x;t)=x. \tag{2.8}$$

Since the solution is unique we can write ω^ε along the flow as

$$\omega^\varepsilon(t,x) = \omega_0^\varepsilon\left(Y_\varepsilon(t,x)\right).$$

Assuming there exist a limit (in ε) inverse flow map Y it would be natural to define

$$\begin{aligned} \omega(t,x) &= \omega_0\left(Y(t,x)\right), \\ u(t,x) &= K\star\omega(t,x), \end{aligned} \tag{2.9}$$

and check that the convergence stated in Proposition 2.4 hold for these limit functions.

Before we proceed in such direction let us state the following potential-theoretic estimates for the velocity flow

Lemma 2.5. *Let $\omega^\varepsilon(t,\cdot)\in L^1\cap L^\infty$ for any $t\in[0,T]$ and $u^\varepsilon = K\star\omega^\varepsilon$, then u^ε is quasi-Lipschitz, i.e.*

$$\left|u^\varepsilon(t,x_1)-u^\varepsilon(x_2,t)\right|\leqslant C\|\omega_0\|_{L^1\cap L^\infty}|x_1-x_2|\left(1-\min\left\{0,\ \log|x_1-x_2|\right\}\right).$$

The following result is a consequence of Lemma 2.5:

Lemma 2.6. *Let X_ε and Y_ε be respectively the forward and backward particle flow generated by u^ε, let*

$$\beta(t)=\exp\left\{-C\|\omega_0\|_{L^1\cap L^\infty}\,t\right\},$$

the following hold true for any $\varepsilon>0$ and $t\in[0,T]$

$$\begin{aligned}|X_\varepsilon(t,\alpha_1)-X_\varepsilon(t,\alpha_2)|&\leqslant C|\alpha_1-\alpha_2|^{\beta(t)},\\|Y_\varepsilon(t,x_1)-Y_\varepsilon(t,x_2)|&\leqslant C|x_1-x_2|^{\beta(t)},\end{aligned}\tag{2.10}$$

while for any $0\leqslant t_1\leqslant t_2\leqslant t$

$$\begin{aligned}|X_\varepsilon(t_1,\alpha)-X_\varepsilon(t_2,\alpha)|&\leqslant C|t_1-t_2|^{\beta(t)},\\|Y_\varepsilon(t_1,x)-Y_\varepsilon(t_2,x)|&\leqslant C|t_1-t_2|^{\beta(t)}.\end{aligned}\tag{2.11}$$

The proofs of lemma 2.5 and 2.6 are postponed for the sake of readability.

Let us hence at fist prove that the limit flow map $\lim_\varepsilon Y_\varepsilon$ exists; we use (2.5) and the fact that for any $t\in[0,T]$ the application $x\mapsto Y_\varepsilon(t,x)$ is measure-preserving to argue that

$$|Y_\varepsilon(t,x)-x|=|X_\varepsilon(t,\alpha)-\alpha|=\left|\int_0^t u^\varepsilon\left(X_\varepsilon\left(t',\alpha\right),t'\right)\mathrm{d}t'\right|\leqslant CT,$$

uniformly in ε, hence $(t,x)\mapsto Y(t,x)-x$ (i.e. the flow deformation) is uniformly bounded. From (2.10) and (2.11) we deduce equicontinuity (globally) in space-time, i.e.

$$\begin{aligned}|Y_\varepsilon(x_1,t_1)-Y_\varepsilon(x_2,t_2)|&\leqslant|Y_\varepsilon(x_1,t_1)-Y_\varepsilon(x_1,t_2)|+|Y_\varepsilon(x_1,t_2)-Y_\varepsilon(x_2,t_2)|,\\&\lesssim|t_1-t_2|^{\beta(t)}+|x_1-x_2|^{\beta(t)},\end{aligned}$$

thus we can invoke Ascoli-Arzelà theorem to assert that for any $R>0$

$$Y_\varepsilon\xrightarrow{\varepsilon\to0}Y,\qquad\qquad \text{in } L^\infty\left([0,T]\times\overline{B(0,R)}\right).$$

Claim: For all $t\in[0,T]$ the inverse flow map $Y(t,\cdot)$ is a measure-preserving map, i.e. for any $f\in L^1$ $\int f(Y(x,t))\,\mathrm{d}x=\int f(x)\,\mathrm{d}x$. †

For a proof of the claim we refer the reader to [4, p. 316].

We can now define the limit vorticity and velocity via the relations provided in (2.9), then we have that

$$\begin{aligned}\left\|\omega^\varepsilon(t,\cdot)-\omega(t,\cdot)\right\|_{L^1}&\leqslant\left\|\omega_0^\varepsilon(Y_\varepsilon(t,\cdot))-\omega_0(Y_\varepsilon(t,\cdot))\right\|_{L^1}+\|\omega_0(Y_\varepsilon(t,\cdot))-\omega_0(Y(t,\cdot))\|_{L^1},\\&=J_1^\varepsilon(t)+J_2^\varepsilon(t),\end{aligned}$$

and since Y_ε is volume-preserving and thanks to (2.4) we obtain that $J_1^\varepsilon(t)\xrightarrow{\varepsilon\to0}0$. Since $\omega_0\in L^1$ we know that there exists a sequence of smooth $\left(\tilde\omega_0^\delta\right)_{\delta>0}$ s.t. $\left\|\omega_0-\tilde\omega_0^\delta\right\|_{L^1}\leqslant\delta$, hence

$$J_2^\varepsilon(t)=\sum_{j=1,2,3}J_{2,j}^{\varepsilon,\delta}(t),$$

and

$$\begin{aligned}J_{2,1}^{\varepsilon,\delta}(t)&=\left\|\omega_0(Y_\varepsilon(t,\cdot))-\tilde\omega_0^\delta(Y_\varepsilon(t,\cdot))\right\|_{L^1},\\J_{2,2}^{\varepsilon,\delta}(t)&=\left\|\tilde\omega_0^\delta(Y_\varepsilon(t,\cdot))-\tilde\omega_0^\delta(Y(t,\cdot))\right\|_{L^1},\\J_{2,3}^{\varepsilon,\delta}(t)&=\left\|\tilde\omega_0^\delta(Y(t,\cdot))-\omega_0(Y(t,\cdot))\right\|_{L^1}.\end{aligned}$$

The first and third term tend to zero as $\delta \to 0$, so the only term to study is $J_{2,2}^{\varepsilon,\delta}(t)$, since $\tilde{\omega}_0^\delta$ is continuous for $\delta > 0$ we have that $\tilde{\omega}_0^\delta(Y_\varepsilon(t,\cdot)) \xrightarrow{\varepsilon\to 0} \tilde{\omega}_0^\delta(Y(t,\cdot))$ point-wise, an application of Lebesgue dominated convergence concludes hence that

$$\left\|\omega_0^\varepsilon(Y_\varepsilon(t,\cdot)) - \omega_0(Y(t,\cdot))\right\|_{L^1} \xrightarrow{\varepsilon\to 0} 0.$$

It remains now to prove only that $u^\varepsilon \to u$ in L^∞_{loc}, let us denote with χ the standard radial cutoff and with $\chi_\delta(x) = \chi(x/\delta)$, so that we have the following

$$\left|u^\varepsilon(t,x) - u(t,x)\right| \leqslant \left|\left(\chi_\delta K\right) \star \left(\omega^\varepsilon - \omega\right)(t,x)\right| + \left|\left(\left(1-\chi_\delta\right)K\right) \star \left(\omega^\varepsilon - \omega\right)(t,x)\right|,$$

and

$$\begin{aligned}
\left|\left(\chi_\delta K\right) \star \left(\omega^\varepsilon - \omega\right)(t,x)\right| &\leqslant \left\|\chi_\delta K\right\|_{L^1} \left\|\omega^\varepsilon - \omega\right\|_{L^\infty} \leqslant C \left\|\chi_\delta K\right\|_{L^1} \xrightarrow{\delta\to 0} 0,\\
\left|\left(\left(1-\chi_\delta\right)K\right) \star \left(\omega^\varepsilon - \omega\right)(t,x)\right| &\leqslant \left\|\left(1-\chi_\delta\right)K\right\|_{L^\infty} \left\|\omega^\varepsilon - \omega\right\|_{L^1} \leqslant C \left\|\omega^\varepsilon - \omega\right\|_{L^1} \xrightarrow{\varepsilon\to 0} 0,
\end{aligned}$$

thus we conclude.

2.3.3 Proof of Lemma 2.5

Recall that

$$\begin{aligned}
u^\varepsilon(x) &= K \star \omega^\varepsilon(x),\\
&= \frac{1}{2\pi}\int \frac{(x-z)^\perp}{|x-z|^2}\,\omega^\varepsilon(z)\,\mathrm{d}z,
\end{aligned}$$

and let us select $x \neq y$ s.t. $d = \left|x-y\right| < 1$, we have that

$$\left|u^\varepsilon(x) - u^\varepsilon(y)\right| \leqslant \left(\int_{\mathbb{R}^2\setminus B(x,2)} + \int_{B(x,2)\setminus B(x,2d)} + \int_{B(x,2d)}\right)\left|K(x-z) - K(y-z)\right|\left|\omega^\varepsilon(z)\right|\mathrm{d}z = J_1 + J_2 + J_3.$$

We use the identity

$$\left|K(x) - K(y)\right|^2 \sim \left|\frac{x}{|x|^2} - \frac{y}{|y|^2}\right|^2 = \frac{|x-y|^2}{|x|^2\,|y|^2},$$

so that

$$J_1 \lesssim |x-y| \int_{\mathbb{R}^2\setminus B(x,2)} \frac{|\omega^\varepsilon(z)|}{|x-z|\,|y-z|}\,\mathrm{d}z \lesssim \left\|\omega_0^\varepsilon\right\|_{L^1} |x-y|.$$

Next we study J_2, we use the mean value theorem in order to argue that

$$\left|K(x-z) - K(y-z)\right| \leqslant \sup_{\xi\in(0,1)} \left|\nabla K\left(x-z+\xi(y-x)\right)\right| |x-y| \lesssim \frac{|x-y|}{|x-z|^2},$$

so that

$$J_2 \lesssim \left\|\omega_0^\varepsilon\right\|_{L^\infty} |x-y| \int_{B(x,2)\setminus B(x,2d)} \frac{\mathrm{d}z}{|x-z|^2} \lesssim \left\|\omega_0^\varepsilon\right\|_{L^\infty} |x-y| \int_{2d}^{2} \frac{\mathrm{d}r}{r} \lesssim \left\|\omega_0^\varepsilon\right\|_{L^\infty} |x-y| \left(1 - \log|x-y|\right).$$

The last term can be bounded as

$$J_3 \lesssim \left\|\omega_0^\varepsilon\right\|_{L^\infty} \int_{B(x,2d)} \left(\frac{1}{|x-z|} + \frac{1}{|y-z|}\right)\mathrm{d}z \lesssim \left\|\omega_0^\varepsilon\right\|_{L^\infty} \int_0^{3d} \mathrm{d}r \lesssim \left\|\omega_0^\varepsilon\right\|_{L^\infty} |x-y|,$$

the bounds are uniform in ε.

2.3.4 Proof of Lemma 2.6

Let us recall that $Y_\varepsilon(t,x) = \mathcal{Y}_\varepsilon(t,x;t')\big|_{t'=t}$ and that

$$\mathcal{Y}_\varepsilon(t,x;t') = X_\varepsilon(t-t',\alpha), \qquad\qquad x = X_\varepsilon(t,\alpha).$$

So that we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t'}\left(\mathcal{Y}_\varepsilon(t',\alpha_1;t) - \mathcal{Y}_\varepsilon(t',\alpha_2;t)\right) = -\left[u^\varepsilon\left(t-t',\mathcal{Y}_\varepsilon(t',\alpha_1;t)\right) - u^\varepsilon\left(t-t',\mathcal{Y}_\varepsilon(t',\alpha_2;t)\right)\right],$$

so that we obtain the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|\mathcal{Y}_\varepsilon(t',\alpha_1;t) - \mathcal{Y}_\varepsilon(t',\alpha_2;t)\right| = \left|u^\varepsilon\left(t-t',\mathcal{Y}_\varepsilon(t',\alpha_1;t)\right) - u^\varepsilon\left(t-t',\mathcal{Y}_\varepsilon(t',\alpha_2;t)\right)\right|.$$

Let us denote with $\varrho_\varepsilon(t') = \left|\mathcal{Y}_\varepsilon(t',\alpha_1;t) - \mathcal{Y}_\varepsilon(t',\alpha_2;t)\right|$, using Lemma 2.5 we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_\varepsilon(t') \leqslant C\left\|\omega_0\right\|_{L^1\cap L^\infty}\varrho_\varepsilon(t')\left(1-\min\left\{0,\varrho_\varepsilon(t')\right\}\right).$$

Let us now assume w.l.o.g. that $\varrho_\varepsilon(t') \in [0,1]$ so that the above differential inequality simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_\varepsilon \leqslant C\left\|\omega_0\right\|\ \varrho_\varepsilon\left(1+\log\left(\frac{1}{\varrho_\varepsilon}\right)\right).$$

Setting $z_\varepsilon = \log\varrho_\varepsilon$ we deduce the linear ODE

$$\frac{\mathrm{d}z_\varepsilon}{\mathrm{d}t} \leqslant C\left\|\omega_0\right\|_{L^\infty}(1-z_\varepsilon),$$

which can be solved providing the bound

$$\varrho_\varepsilon(t') \leqslant e\,\varrho_\varepsilon(0)^{\exp\left\{-C\|\omega_0\|_{L^\infty}t'\right\}}.$$

Let us now prove (2.11), set $\alpha, x \in \mathbb{R}^2$ and $0 \leqslant t_1 \leqslant t_2 \leqslant t$ such that $x = X_\varepsilon(t_2,\alpha)$, we want to estimate

$$\left|Y_\varepsilon(t_1,x) - Y_\varepsilon(t_2,x)\right|.$$

Let us define $\alpha^\star = \mathcal{Y}_\varepsilon(t_2,x;t_2-t_1)$, we indeed have that

$$\alpha = \mathcal{Y}_\varepsilon\left(t_2-t_1,\alpha^\star;t_2-t_1\right) = Y_\varepsilon\left(t_2-t_1,\alpha^\star\right),$$

so that we can use (2.10) to deduce that

$$\begin{aligned}\left|Y_\varepsilon(t_1,x) - Y_\varepsilon(t_2,x)\right| = \left|Y_\varepsilon(t_1,x) - Y_\varepsilon\left(t_1,\alpha^\star\right)\right| \leqslant \left|x-\alpha^\star\right|^{\beta(t)} &= \left|X_\varepsilon(t_2,\alpha) - X_\varepsilon(t_1,\alpha)\right|^{\beta(t)} \\ &= \left|\int_{t_1}^{t_2} u^\varepsilon\left(t',X_\varepsilon(t',\alpha)\right)\mathrm{d}\alpha\right|^{\beta(t)} \lesssim \left|t_2-t_1\right|^{\beta(t)}.\end{aligned}$$

2.4 Uniqueness of weak solutions

Before starting to prove uniqueness of weak solutions let us remark that that since $u(t,\cdot) \in L^\infty$ for each $t \in [0,T]$ there exists a $\mathsf{L} = \mathsf{L}(T)$ s.t.

$$\bigcup_{t\in[0,T]} \operatorname{supp}\omega(t,\cdot) \subset B(0,\mathsf{L}).$$

We will use the following technical lemma whose proof is omitted:

Lemma 2.7. *Let u be weak solution of* (E) *constructed in Theorem 2.3, then for each $p \in (1,\infty)$*

$$\left\|\nabla u\right\|_{L^p} \leqslant C_0\left(\left\|\omega_0\right\|_{L^\infty}\right)\ p.$$

Let u_1, u_2 be the solutions of (E) stemming from the same initial data u_0 and let us define

$$E(t) = \| u_1(t,\cdot) - u_2(t,\cdot)\|_{L^2}^2,$$

since the evolution equation for $w = u_1 - u_2$ is

$$\partial_t w + w\cdot\nabla w + u_2\cdot\nabla w + w\cdot\nabla u_2 + \nabla\left(p_1 - p_2\right) = 0,$$

we easily deduce the differential inequality

$$E'(t) \lesssim \int |\nabla u_2(t,\cdot)|\,|w(t,\cdot)|^2\,\mathrm{d}x \leqslant C_0 p \left(\| w(t,\cdot)\|_{L^\infty}^{\frac{2}{p-1}} \int |w|^2\,\mathrm{d}x\right)^{\frac{p-1}{p}} \leqslant Mp\,E(t)^{1-\frac{1}{p}}.$$

It is well known that the differential equality $f' = f^\alpha,\ \alpha\in(0,1)$ has no unique solution stemming from zero, but we know that $\bar{E}(t) = (Mt)^p$ si a solution that is maximal in the sense that $E(t)\leqslant\bar{E}(t)$. Set $t^\star$ s.t. $Mt^\star \leqslant 1/2$ and we obtain that

$$E(t)\leqslant 2^{-p} \xrightarrow{p\to\infty} 0, \quad \text{in } \left[0, t^\star\right],$$

concluding.

3 Vortex patches

Let us at first introduce the problem we want to study: Let us consider a domain $D_0\subset\mathbb{R}^2$ such that ∂D_0 is a bounded, simple $\mathcal{C}^{1,\gamma} = \mathcal{C}^{1,\gamma}\left(\mathbb{S}^1\right)$, $\gamma>0$ curve. Let X be the (unique) Yudovich flow stemming from the initial vorticity $\omega_0 = \mathbb{1}_{D_0}$, accordingly to Theorem 2.3 we have that if $D(t) = X(t, D_0)$ then the vorticity at time t will be simply the deformation of ω_0 by the flow $X(t)$, i.e. $\omega(t) = \mathbb{1}_{D(t)}$, so we ask the following question:

Question. Is $D(t)$ a $\mathcal{C}^{1,\gamma}$ curve for any $t>0$?

Such problem has a long history and it was considered to be *false* due to some numerical simulations pointing in that direction, Majda in [3] proposed the vourtex patch problem as a model of inviscid small scale creation and finite-time singularity formation.

The question has been definitively settled by J.-Y. Chemin in [2] (we refer as well to the more geometrical approach of Bertozzi and Constantin [1] which is the one adopted in the present notes) which proved, by means of paradifferential calculus tools, that despite ∇u is discontinuous in the normal direction of the interface it is in fact continuous in the *tangential* direction to the interface, thus exploiting such observation in order to prove that the curvature of the interface grows, at most, as a double exponential in time.

3.1 Global regularity for Vortex Patches

A fundamental quantity in order to understand the evolution of an Euler flow is ∇u, where indeed due to the Biot-Savart law $u = K\star\omega$ where K is -1 homogeneous function. Let us recall again that the velocity flow u is at most quasi-Lispschitz (or log-Lipschitz) in the regularity setting of Yudovich solutions, so that ∇u has to be understood in the sense of distributions. We will compute explicitly ∇u in the seeting required.

Definition 3.1. Let $f\in\mathcal{C}^\infty\left(\mathbb{R}^d\setminus\{0\}\right)\cap L^1\left(\mathbb{R}^d\right)$ the *Cauchy principal value integral* is

$$\text{p.v.}\int f\,\mathrm{d}x = \lim_{\varepsilon\to 0^+}\int_{|x|\geqslant\varepsilon} f\,\mathrm{d}x.$$

Let now $\mathcal{K}$ be an homogeneous $1-d$ function defined in $\mathbb{R}^d$, i.e. $\mathcal{K}(\lambda x) = \lambda^{1-d}\mathcal{K}(x)$. Let us remark that $\mathcal{K}\in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$, let us consider a test function ϕ and let us compute the distributional derivative of $\mathcal{K}$, since $\mathcal{K}\in L^1_{\text{loc}}\left(\mathbb{R}^d\right)$ we can apply Lebesgue dominated convergence and Green formula in order to obtain that

$$\begin{aligned}
\left\langle\mathcal{K}, \partial_j\phi\right\rangle &= \lim_{\varepsilon\to 0^+}\int_{|x|\geqslant\varepsilon}\mathcal{K}\,\partial_j\phi\,\mathrm{d}x\\
&= \lim_{\varepsilon\to 0^+}\left[-\int_{|x|\geqslant\varepsilon}\partial_j\mathcal{K}\,\phi\,\mathrm{d}x + \int_{|x|=\varepsilon}\mathcal{K}\phi\,\frac{x_j}{|x|}\mathrm{d}\mathcal{H}^1(s)\right].
\end{aligned}$$

A change of variables moreover shows that

$$\lim_{\varepsilon\to 0^+}\int_{|x|=\varepsilon}\mathcal{K}\phi\,\frac{x_j}{|x|}\mathrm{d}\mathcal{H}^1(s)=\phi(0)\underbrace{\int_{|x|=1}\mathcal{K}(x)\,x_j\,\mathrm{d}\mathcal{H}^1(s)}_{=c_j}.$$

we have hence computed the derivative in the sense of distributions and we have that

$$\left\langle\partial_j\mathcal{K},\phi\right\rangle=-\left\langle\mathcal{K},\partial_j\phi\right\rangle=\text{p.v.}\int\partial_j\mathcal{K}\,\phi\,\mathrm{d}x-c_j\left\langle\delta_0,\phi\right\rangle. \tag{3.1}$$

Definition 3.2. Let $\mathcal{J}$ be a $-d$ homogeneous function smooth outside zero with zero mean value on the unit sphere $\mathbb{S}^{d-1}$. Let us formally define the operator

$$\mathcal{J}\phi(x)=\text{p.v.}\int\mathcal{J}(x-y)\,\phi(y)\,\mathrm{d}y. \tag{3.2}$$

An operator of the form (3.2) will be always referred as *Singular Integral Operator* (SIO).

We apply (3.1) and we compute ∇u (recall that $K(x)=x^\perp/|x|$):

$$\nabla u(x)=\frac{1}{2\pi}\text{p.v.}\int\frac{\sigma(x-y)}{|x-y|}\mathrm{d}y+\frac{1}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathbb{1}_{D(t)}(x). \tag{3.3}$$

The 2×2 symmetric matrix σ has the explicit form

$$\sigma(z)=\frac{1}{|z|^2}\begin{pmatrix}2z_1z_2&z_2^2-z_1^2\\z_2^2-z_1^2&-2z_1z_2\end{pmatrix}$$

We will never use the explicit formulation of σ, but only the following properties:

- It is a smooth function homogeneous of degree zero,
- it has zero mean on the unit circle,
- $\sigma(-z)=\sigma(z)$.

Let hence D_0 be a $\mathcal{C}^{1,\gamma}$ patch, this means that there exists a $\varphi_0\in\mathcal{C}^{1,\gamma}\left(\mathbb{R}^2\right)$ function such that

$$D_0=\left\{x\,:\,\varphi_0(x)>0\right\},\qquad\qquad \partial D_0=\left\{x\,:\,\varphi_0(x)=0\right\}. \tag{3.4}$$

Let us now φ_0 be advected by the flow, this defines a time-dependent function φ satisfying the evolution equation

$$\begin{aligned}\partial_t\varphi+u\cdot\nabla\varphi&=0,\\ \varphi\big|_{t=0}&=\varphi_0.\end{aligned} \tag{3.5}$$

The existence of weak solutions for the equation (3.5) is guaranteed by Theorem 2.3, and in fact

$$\varphi(t,x)=\varphi_0(Y(t,x)),$$

where Y is the backward flow generated by $\omega_0=\mathbb{1}_{D_0}$.

Since φ is initially equal to φ_0 the patch $D(t)$ will be defined at later times as a level set of the function φ, i.e.

$$D(t)=\left\{x\,:\,\varphi(t,x)>0\right\},\qquad\qquad \partial D(t)=\left\{x\,:\,\varphi(t,x)=0\right\}.$$

Let us remark that the evolution equation for φ (3.5) can be expressed in terms of φ only since using the Biot-Savart law

$$u(t,x)=\frac{1}{2\pi}\int_{D(t)}\frac{(x-y)^\perp}{|x-y|^2}\,\mathrm{d}y. \tag{3.6}$$

Let us now denote with $W = W(t,x) = \nabla^\perp \varphi(t,x)$ and $W_0(x) = W(0,x)$. We remark that $W_0|_{\partial D_0}$ is a divergence-free vector field tangent at the boundary of D_0, moreover W solves the equation

$$\partial_t W + u\cdot\nabla W = \nabla u\, W. \tag{3.7}$$

Let $z(\cdot,0) \in \mathcal{C}^{1,\gamma}\left(\mathbb{S}^1;\mathbb{R}^2\right)$ be a parametrization of ∂D_0, i.e.

$$\partial D_0 = \left\{x\in\mathbb{R}^2 \ : \ x = z(\alpha,0)\,,\ \alpha\in\mathbb{S}^1\right\},$$

and let us denote with $z(\alpha,t) = X(t,z(\alpha,0))$ the Lagrangian parametrization of the boundary of the patch. remark that $z_\alpha(\alpha,0) = W(z(\alpha,0),0)$ and $z_\alpha(\alpha,t) = W(z(\alpha,t),t)$, so it is sufficient to provide bounds for the Eulerian field W in order to control z

Notation 3.3. ▹ Let us recall that the velocity flow of Yudovich solutions is L^∞, so that if $\omega_0 = \mathbb{1}_{D_0}$, and since the vorticity is transported by the velocity flow, then $D(t)$ is bounded set for any $t>0$. We denote with

$$\mathsf{L} = \mathsf{L}(T) = \inf\left\{R>0 \ \middle|\ D(t)\subset B(0,R)\,,\ \forall\ t\in[0,T]\right\} < \infty,$$

▹ We use the notation

$$\left|\nabla\varphi\right|_{\inf} = \inf_{x\in\partial D}\left|\nabla\varphi(x)\right|,$$

▹ We use the notation $\left|\nabla\varphi\right|_\gamma = \left|\nabla\varphi\right|_{\dot{\mathcal{C}}^\gamma} = \sup_{x\neq y}\dfrac{\left|\nabla\varphi(x)-\nabla\varphi(y)\right|}{\left|x-y\right|^\gamma}$,

▹ Given $A,B\subset\mathbb{R}^2$ and $c\in\mathbb{R}^2$ we denote with

$$d(c,A) = \inf_{a\in A} d(c,a)\,, \qquad\qquad d(A,B) = \inf_{(a,b)\in A\times B} d(a,b)\,.$$

The following result is the main result result of the present section:

Theorem 3.4 (Global regularity for Vortex Patches)**.** *Let D_0 be a $\mathcal{C}^{1,\gamma}$ patch and $\phi_0\in\mathcal{C}^{1,\gamma}\left(\mathbb{R}^2\right)$ function satisfying* (3.4) *which is regular on ∂D_0, i.e.*

$$\left|\nabla\varphi_0\right|_{\inf} \geqslant m > 0,$$

then there exists a

$$C = C\left(\mathsf{L}, \left|\nabla\varphi_0\right|_\gamma, \left\|\nabla\varphi_0\right\|_{L^\infty}, \left|\nabla\varphi_0\right|_{\inf}\right) > 0,$$

and a $C_0>0$ such that (3.5) *has a unique solution defined in $\mathbb{R}_+\times\mathbb{R}^2$ and such that*

$$\begin{aligned}
\left\|\nabla u(t,\cdot)\right\|_{L^\infty} &\leqslant \left\|\nabla u(0,\cdot)\right\|_{L^\infty} e^{Ct},\\
\left|\nabla\varphi(t,\cdot)\right|_\gamma &\leqslant \left|\nabla\varphi_0\right|_\gamma e^{(C_0+\gamma)e^{Ct}},\\
\left\|\nabla\varphi(t,\cdot)\right\|_{L^\infty} &\leqslant \left\|\nabla\varphi_0\right\|_{L^\infty} e^{e^{Ct}},\\
\left|\nabla\varphi(t,\cdot)\right|_{\inf} &\geqslant \left|\nabla\varphi_0\right|_{\inf} e^{-e^{Ct}}.
\end{aligned}$$

The main technical result required in order to prove Theorem 3.4 is the following control of the L^∞ norm of ∇u in terms of the $\mathcal{C}^{1,\gamma}$ norm of φ:

Proposition 3.5 (Key technical Proposition)**.** *Let u be given by* (3.6) *and let D be the level set of φ, the following bound holds true*

$$\left\|\nabla u\right\|_{L^\infty} \leqslant C\left[1+\log\left(\frac{\mathsf{L}\left|\nabla\varphi\right|_\gamma}{\left|\nabla\varphi\right|_{\inf}}\right)\right]. \tag{3.8}$$

Let us recall that we can decompose ∇u is symmetric and anti-symmetric part as it was done in (3.3), thus we only need to estimate the symmetric part and we only need to control the case in which x is close to the boundary of D as often happens in computation of SIO.

Let us select a $x_0 \in \mathbb{R}^2 \setminus \partial D$ and let us denote

$$d(x_0) = d(x_0, \partial D),$$

let us define moreover the cutoff distance

$$\delta = \frac{|\nabla\varphi|_{\inf}}{|\nabla\varphi|_{\gamma}},$$

and let us define the tubular neighborhood of ∂D

$$\mathcal{T}_{\delta/2} = \left\{ x_0 \in \mathbb{R}^2 \ : \ d(x_0) \leqslant \frac{\delta}{2} \right\}.$$

Let $x_0 \in \mathcal{T}_{\delta/2}$ and let us define for any $\rho \geqslant d(x_0)$ the set

$$S_\rho(x_0) = \left\{ s \in \mathbb{S}^1 \ : \ x_0 + \rho s \in D \right\},$$

i.e. the set of directions s.t. $x_0 + \rho s \in D$. Let now $\tilde{x} = \tilde{x}(x_0) \in \partial D$ be s.t. $|x_0 - \tilde{x}| = d(x_0)$, and let us define the semicircle

$$\Sigma(x_0) = \left\{ s \in \mathbb{S}^1 \ : \ \nabla\varphi(\tilde{x}) \cdot s \geqslant 0 \right\},$$

and the symmetric difference

$$R_\rho(x_0) = \left[S_\rho(x_0) \setminus \Sigma(x_0) \right] \cup \left[\Sigma(x_0) \setminus S_\rho(x_0) \right].$$

The key technical point is that, denoting with $\mathcal{H}^1$ the Lebesgue measure on $\mathbb{S}^1$, as $d(x_0) \to 0$ the quantity $\mathcal{H}^1\left(R_\rho(x_0)\right) \to 0$ at a controlled rate, i.e.

Lemma 3.6 (Geometric Lemma)**.** *The following estimate holds true*

$$\mathcal{H}^1\left(R_\rho(x_0)\right) \leqslant 2\pi \left[\left(1 + 2^\gamma\right) \frac{d(x_0)}{\rho} + 2^\gamma \left(\frac{\rho}{\delta}\right)^\gamma \right],$$

for $\rho \geqslant d(x_0)$ *and* $d(x_0) < \delta/2$.

We postpone at the moment the proof of Lemma 3.6 and we show to use it in order to prove the technical estimate (3.8).

Recall that we want to provide a bound for

$$I(x_0) = \frac{1}{2\pi} \text{p.v.} \int_D \frac{\sigma(x_0 - y)}{|x_0 - y|^2} \mathrm{d}y,$$

and we suppose $x_0 \in D$. We split the integration sets in points "close" to x_0 and far ones, i.e.

$$\begin{aligned} I(x_0) &= I_1(x_0) + I_2(x_0), \\ I_1(x_0) &= \frac{1}{2\pi} \text{p.v.} \int_{D \cap \{|x_0 - y| \leqslant \delta\}} \frac{\sigma(x_0 - y)}{|x_0 - y|^2} \mathrm{d}y, \\ I_2(x_0) &= \frac{1}{2\pi} \text{p.v.} \int_{D \cap \{|x_0 - y| \geqslant \delta\}} \frac{\sigma(x_0 - y)}{|x_0 - y|^2} \mathrm{d}y. \end{aligned}$$

Let us at first bound I_2, recalling that we suppose that $D \subset B(0, \mathsf{L})$ we have, passing to polar coordinates

$$\begin{aligned} |I_2(x_0)| &= \frac{1}{2\pi} \left| \int_{D \cap \{|x_0 - y| \geqslant \delta\}} \frac{\sigma(x_0 - y)}{|x_0 - y|^2} \mathrm{d}y \right|, \\ &\leqslant C \int_\delta^{\mathsf{L}} \frac{\mathrm{d}\rho}{\rho} \leqslant C \log\left(\frac{\mathsf{L}}{\delta}\right). \end{aligned} \tag{3.9}$$

We study now I_1 and we write it in polar coordinated centered in x_0 obtaining

$$I_1(x_0) = \int_{\tilde{R}} \int_{\mathbb{S}^1} \frac{\sigma\left(\rho e^{is}\right)}{\rho} \, \mathrm{d}\mathcal{H}^1(s) \, \mathrm{d}\rho,$$

where the integration set in the radial direction is an unspecified and irrelevant set which in bounded in $B(0,\delta)$ since we are integrating in the tubular neighborhood $\mathcal{T}_{\delta/2}$. Let us suppose now that $\rho_0 < d(x_0)$, this means that $B\left(x_0, \rho_0\right) \subset D$, so since σ has zero average on $\mathbb{S}^1$ we have that

$$\int_{\mathbb{S}^1} \sigma\left(\rho_0 e^{is}\right) \mathrm{d}\mathcal{H}^1(s) = 0,$$

so I_1 becomes

$$I_1(x_0) = \int_{d(x_0)}^{\tilde{\rho}} \left[\int_{S_\rho(x_0)} \sigma\left(\rho e^{is}\right) \, \mathrm{d}\mathcal{H}^1(s) \right] \frac{\mathrm{d}\rho}{\rho},$$

where again $\tilde{\rho} < \delta$ is irrelevant in our context. Next let us consider the semicircle $\Sigma(x_0)$, since we are considering $x_0 \in D$ we have that $\Sigma(x_0) \subset R_\rho(x_0)$, moreover since $\sigma(z) = \sigma(-z)$

$$\int_{\Sigma(x_0)} \sigma\left(\rho e^{is}\right) \, \mathrm{d}\mathcal{H}^1(s) = \frac{1}{2} \int_{\Sigma(x_0)} \sigma\left(\rho e^{is}\right) + \sigma\left(-\rho e^{is}\right) \, \mathrm{d}\mathcal{H}^1(s) = \int_{B(x_0,\rho)} \sigma \, \mathrm{d}\mathcal{H}^1(s) = 0,$$

so we obtained that

$$I_1(x_0) = \int_{d(x_0)}^{\tilde{\rho}} \left[\int_{S_\rho(x_0) \setminus \Sigma(x_0)} \sigma\left(\rho e^{is}\right) \, \mathrm{d}\mathcal{H}^1(s) \right] \frac{\mathrm{d}\rho}{\rho}.$$

Since σ is homogeneous of order zero the bound

$$|I_1(x_0)| \leqslant \int_{d(x_0)}^{\delta} \frac{\mathcal{H}^1\left(S_\rho(x_0) \setminus \Sigma(x_0)\right)}{\rho} \, \mathrm{d}\rho.$$

The above estimate is true when $x_0 \in D$, if $x_0 \notin D$ we have

$$|I_1(x_0)| \leqslant \int_{d(x_0)}^{\delta} \frac{\mathcal{H}^1\left(\Sigma(x_0) \setminus S_\rho(x_0)\right)}{\rho} \, \mathrm{d}\rho,$$

so that the bound

$$|I_1(x_0)| \leqslant \int_{d(x_0)}^{\delta} \frac{\mathcal{H}^1\left(R_\rho(x_0)\right)}{\rho} \, \mathrm{d}\rho,$$

covers both cases. We use now the estimate provided in Lemma 3.6 and we have that

$$|I_1(x_0)| \leqslant \int_{d(x_0)}^{\delta} \frac{1}{\rho} \left(\frac{d(x_0)}{\rho} + \left(\frac{\rho}{\delta}\right)^\gamma \right) \mathrm{d}\rho \leqslant \frac{1}{2} + \frac{1}{\gamma} \left(1 - \frac{1}{2^\gamma}\right) \leqslant C,$$

which we combine it now with (3.9) and we obtain

$$|I(x_0)| \leqslant C \left[1 + \log\left(\frac{\mathsf{L} \left|\nabla\varphi\right|_\gamma}{\left|\nabla\varphi\right|_{\inf}} \right) \right].$$

Proposition 3.7. *Let W be a divergence-free vector field tangent to ∂D and let u be give by the Biot-savart law* (3.6)*, then*

$$\nabla u(x) \; W = \frac{1}{2\pi} \text{p.v.} \int \frac{\sigma(x-y)}{\left|x-y\right|^2} \left(W(x) - W(y)\right) \mathrm{d}y.$$

Proof. Let us exploit the following identity

$$\nabla_y \left(\nabla_y^\perp \log\left|x-y\right| \right) = \frac{\sigma(x-y)}{\left|x-y\right|^2},$$

so that integration by parts give

$$\frac{1}{2\pi}\text{p.v.}\int \nabla_y\left(\nabla_y^\perp \log|x-y|\right)W(y)\,\mathrm{d}y = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{D\cap|x-y|\geqslant\varepsilon}\partial_{y_i}\left(\partial_{y_j}^\perp \log|x-y|\right)W_i(y)\,\mathrm{d}y,$$
$$= -\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{D\cap\{|x-y|=\varepsilon\}}\nabla_y^\perp \log|x-y|\;W(y)\cdot\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}y$$
$$-\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{D\cap\{|x-y|\geqslant\varepsilon\}}\partial_{y_j}^\perp \log|x-y|\underbrace{\partial_{y_i}W_i(y)}_{=0}\mathrm{d}y,$$

thus since W is tangent to ∂D we obtain that

$$-\frac{1}{2\pi}\lim_{\varepsilon\to 0}\int_{D\cap\{|x-y|=\varepsilon\}}\nabla_y^\perp \log|x-y|\;W(y)\cdot\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}y = -\frac{1}{2}\mathbb{1}_D(x)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}W(x),$$

which in turn implies that

$$\nabla u(x)\,W(x) = \left(\frac{1}{2\pi}\text{p.v.}\int\frac{\sigma(x-y)}{|x-y|}\mathrm{d}y + \frac{1}{2}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\mathbb{1}_D(x)\right)W(x),$$
$$= \frac{1}{2\pi}\text{p.v.}\int\frac{\sigma(x-y)}{|x-y|}\left(W(x)-W(y)\right)\mathrm{d}y$$
$$+\underbrace{\frac{1}{2\pi}\text{p.v.}\int\frac{\sigma(x-y)}{|x-y|}W(y)\,\mathrm{d}y + \frac{1}{2}\mathbb{1}_D(x)\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}W(x)}_{=0},$$

concluding. □

We need the following commutator estimates in Hölder spaces before proceeding

Lemma 3.8. *Let $\psi\in L^\infty$, $f\in\dot{\mathcal{C}}^\gamma$, $\mathcal{K}$ a Calderon-Zygmund kernel homogeneous of degree $-d$ with zero mean on $\mathbb{S}^{d-1}$ and such that $|\nabla\mathcal{K}|\lesssim|x|^{-(d+1)}$, let*

$$G(x) = \text{p.v.}\int\mathcal{K}(x-y)\left(f(x)-f(y)\right)\psi(y)\,\mathrm{d}y,$$

then

$$|G|_\gamma \leqslant C_0(\gamma,d)\left(\|\mathcal{K}\star\psi\|_{L^\infty}+\|\psi\|_{L^\infty}\right)|f|_\gamma.$$

An application of Lemma 3.8 gives the following result

Corollary 3.9. *Let u and W be as in Proposition 3.7, then*

$$|\nabla u\;W|_\gamma \leqslant C_0\|\nabla u\|_{L^\infty}|W|_\gamma. \tag{3.10}$$

We can now finally prove the required global bounds

Proposition 3.10. *Let φ be the solution of (3.5) in $[0,T]$, $T<\infty$, then we have*

$$\left|\nabla^\perp\varphi(t,\cdot)\right|_\gamma \leqslant \left|\nabla^\perp\varphi_0\right|_\gamma \exp\left\{(C_0+\gamma)\int_0^t\left\|\nabla u(t',\cdot)\right\|_{L^\infty}\mathrm{d}t'\right\}, \tag{3.11}$$

$$\left\|\nabla^\perp\varphi(t,\cdot)\right\|_{L^\infty} \leqslant \left\|\nabla^\perp\varphi_0\right\|_{L^\infty}\exp\left\{\int_0^t\left\|\nabla u(t',\cdot)\right\|_{L^\infty}\mathrm{d}t'\right\}, \tag{3.12}$$

$$\left|\nabla^\perp\varphi(t,\cdot)\right|_{\inf} \geqslant \left|\nabla^\perp\varphi_0\right|_{\inf}\exp\left\{-\int_0^t\left\|\nabla u(t',\cdot)\right\|_{L^\infty}\mathrm{d}t'\right\}. \tag{3.13}$$

Once Proposition 3.10 is proved Theorem 3.4 follows; we plug the estimates (3.11) and (3.13) in (3.8) and apply Gronwall lemma deducing the bound

$$\|\nabla u(t,\cdot)\|_{L^\infty} \leqslant \|\nabla u_0\|_{L^\infty}e^{Ct},$$

which we apply to (3.11)–(3.13) obtaining that

$$\left\|\nabla^{\perp}\varphi(t,\cdot)\right\|_{C^\gamma}\leqslant\left\|\nabla^{\perp}\varphi_0\right\|_{C^\gamma}e^{(C_0+\gamma)e^{Ct}},\qquad\qquad \left|\nabla^{\perp}\varphi(t,\cdot)\right|_{\inf}\geqslant\left|\nabla^{\perp}\varphi_0\right|_{\inf}e^{-e^{Ct}}.$$

Let us denote with $w(t,x)=W(t,X(t,x))=\nabla^{\perp}\varphi(t,X(t,x))$ and we rewrite (3.7) in Lagrangian coordinates we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t,x)=\nabla u(t,X(t,x))\;w(t,x),$$

thus multiplying the above equation for $w/|w|$ and applying Gronwall inequality we obtain that

$$\exp\left\{-\int_0^t\left|\nabla u\left(t',X\left(t',x\right)\right)\right|\mathrm{d}t'\right\}\leqslant\frac{|w(t,x)|}{|w(0,x)|}\leqslant\exp\left\{\int_0^t\left|\nabla u\left(t',X\left(t',x\right)\right)\right|\mathrm{d}t'\right\},$$

proving (3.12) and (3.13).

We have to prove now only (3.12), we have that

$$W(t,x)=W_0(Y(t,x))+\int_0^t(\nabla u\,W)\left(t',Y\left(t-t',x\right)\right)\mathrm{d}t',$$

so that if $x\neq y$

$$\begin{aligned}\left|W(t,x)-W\left(t,y\right)\right|&\leqslant\left|W_0(Y(t,x))-W_0\left(Y\left(t,y\right)\right)\right|+\left|\int_0^t(\nabla u\,W)\left(t',Y\left(t-t',x\right)\right)-(\nabla u\,W)\left(t',Y\left(t-t',y\right)\right)\mathrm{d}t'\right|,\\&\leqslant|W_0|_\gamma\left\|\nabla Y(t,\cdot)\right\|_{L^\infty}^{\gamma}\left|x-y\right|^\gamma+\int_0^t\left|\nabla u\,W\left(t',\cdot\right)\right|_\gamma\left\|\nabla Y\left(t-t',\cdot\right)\right\|_{L^\infty}^{\gamma}\left|x-y\right|^\gamma\mathrm{d}t'.\end{aligned}$$

Since for any $t'\leqslant t$ we have that $\frac{\mathrm{d}}{\mathrm{d}t'}\mathcal{Y}\left(t',x;t\right)=-u\left(t-t',\mathcal{Y}\left(t',x;t\right)\right)$, $Y=\mathcal{Y}|_{t'=t}$ an application of Gronwall inequality gives

$$\begin{aligned}\left|\nabla Y(t,x)\right|&\leqslant\exp\left\{\int_0^t\left\|\nabla u\left(t',\cdot\right)\right\|_{L^\infty}\mathrm{d}t'\right\},\\\left|\nabla Y\left(t-t',x\right)\right|&\leqslant\exp\left\{\int_{t'}^t\left\|\nabla u\left(t'',\cdot\right)\right\|_{L^\infty}\mathrm{d}t''\right\},\end{aligned}$$

so we obtain that

$$\begin{aligned}\left|W(t,x)-W\left(t,y\right)\right|\leqslant|W_0|_\gamma\exp\left\{\gamma\int_0^t\left\|\nabla u\left(t',\cdot\right)\right\|_{L^\infty}\mathrm{d}t'\right\}\left|x-y\right|^\gamma&\\+\int_0^t\left|\nabla u\,W\left(t',\cdot\right)\right|_\gamma\exp\left\{\gamma\int_{t'}^t\left\|\nabla u\left(t'',\cdot\right)\right\|_{L^\infty}\mathrm{d}t''\right\}\left|x-y\right|^\gamma\mathrm{d}t',&\end{aligned}$$

thus

$$|W(t,\cdot)|_\gamma\leqslant|W_0|_\gamma\exp\left\{\gamma\int_0^t\left\|\nabla u\left(t',\cdot\right)\right\|_{L^\infty}\mathrm{d}t'\right\}+\int_0^t\left|\nabla u\,W\left(t',\cdot\right)\right|_\gamma\exp\left\{\gamma\int_{t'}^t\left\|\nabla u\left(t'',\cdot\right)\right\|_{L^\infty}\mathrm{d}t''\right\}\mathrm{d}t'.$$

Let us denote with $Q\left(t'\right)=\left\|\nabla u\left(t',\cdot\right)\right\|_{L^\infty}$, we use the commutator estimate (3.10) and we obtain that

$$|W(t,\cdot)|_\gamma\leqslant|W_0|_\gamma\exp\left\{\gamma\int_0^tQ\left(t'\right)\mathrm{d}t'\right\}+\int_0^tQ\left(t'\right)\left|W\left(t',\cdot\right)\right|_\gamma\exp\left\{\gamma\int_{t'}^tQ\left(t''\right)\mathrm{d}t''\right\}\mathrm{d}t',$$

we multiply both sides for $\exp\left\{-\gamma\int_0^tQ\left(t'\right)\mathrm{d}t'\right\}$ and copute the evolution equation for

$$G(t)=|W(t,\cdot)|_\gamma\exp\left\{\gamma\int_0^tQ\left(t'\right)\mathrm{d}t'\right\},$$

which is

$$G(t)\leqslant|W_0|_\gamma+C_0\int_0^tQ\left(t'\right)G\left(t'\right)\mathrm{d}t',$$

so that an application of Gronwall lemma concludes the proof.

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