1 Fourier transform

Definition 1.1 (Fourier transform)**.** For $f \in L^1(\mathbb{R}^d, \mathbb{C})$ we call its Fourier transform the function defined by the following formula

$$\widehat{f}(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} f(x) dx. \tag{1.1}$$

We use also the notation $\mathcal{F}f(\xi) = \widehat{f}(\xi)$.

Example 1.2. We have for any $\varepsilon > 0$

$$e^{-\varepsilon\frac{|\xi|^2}{2}} = (2\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x} e^{-\frac{|x|^2}{2\varepsilon}} dx. \tag{1.2}$$

We set also

$$\mathcal{F}^* f(\xi) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x} f(x) dx. \tag{1.3}$$

We have what follows.

Theorem 1.3. *The following facts hold.*

(1) We have $|\widehat{f}(\xi)| \le (2\pi)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d,\mathbb{C})}$. *So in particular we have*

$$\|\mathcal{F}f\|_{L^\infty(\mathbb{R}^d,\mathbb{C})} \le (2\pi)^{-\frac{d}{2}} \|f\|_{L^1(\mathbb{R}^d,\mathbb{C})}. \tag{1.4}$$

(2) (Riemann– Lebesgue Lemma) We have $\lim_{\xi\to\infty} \widehat{f}(\xi) = 0$.

(3) The bounded linear operator $\mathcal{F} : L^1(\mathbb{R}^d, \mathbb{C}) \to L^\infty(\mathbb{R}^d, \mathbb{C})$ *has values in the following space* $C_0(\mathbb{R}^d, \mathbb{C}) \subset L^\infty(\mathbb{R}^d, \mathbb{C})$

$$C_0(\mathbb{R}^d, \mathbb{C}) := \{g \in C^0(\mathbb{R}^d, \mathbb{C}) : \lim_{x\to\infty} g(x) = 0\}. \tag{1.5}$$

(4) $\mathcal{F}$ *defines an isomorphism of the space of Schwartz functions* $\mathcal{S}(\mathbb{R}^d, \mathbb{C})$ *into itself.*

(5) $\mathcal{F}$ *defines an isomorphism of the space of tempered distributions* $\mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ *into itself. We have* $\mathcal{F}[\partial_{x_j} f] = -\mathrm{i}\xi_j \mathcal{F}f$.

(6) For $f, g \in L^1(\mathbb{R}^d, \mathbb{C})$ *we have*

$$\widehat{f * g}(\xi) = (2\pi)^{\frac{d}{2}} \widehat{f}(\xi)\widehat{g}(\xi).$$

□

Theorem 1.4 (Fourier transform in L^2)**.** *The following facts hold.*

(1) For a function $f \in L^1(\mathbb{R}^d, \mathbb{C}) \cap L^2(\mathbb{R}^d, \mathbb{C})$ *we have that* $\widehat{f} \in L^2(\mathbb{R}^d, \mathbb{C})$ *and* $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$. *An operator*

$$\mathcal{F} : L^2(\mathbb{R}^d, \mathbb{C}) \to L^2(\mathbb{R}^d, \mathbb{C}) \tag{1.6}$$

remains defined. For $f \in L^2(\mathbb{R}^d, \mathbb{C})$ *for any function* $\varphi \in C_c(\mathbb{R}^d, \mathbb{C})$ *with* $\varphi = 1$ *near* 0 *set*

$$\begin{aligned}\mathcal{F}f(\xi) &:= \lim_{\lambda \nearrow \infty} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\mathrm{i}\xi \cdot x} f(x) \varphi(x/\lambda) dx \\ &= \lim_{\lambda \nearrow \infty} (2\pi)^{-\frac{d}{2}} \int_{|x| \le \lambda} e^{-\mathrm{i}\xi \cdot x} f(x) dx.\end{aligned} \tag{1.7}$$

Then (1.7) *defines an isometric isomorphism inside* $L^2(\mathbb{R}^d, \mathbb{C})$, *so in particular we have*

$$\|\mathcal{F}f\|_{L^2(\mathbb{R}^d, \mathbb{C})} = \|f\|_{L^2(\mathbb{R}^d, \mathbb{C})}. \tag{1.8}$$

(2) The inverse map is defined by

$$\begin{aligned}\mathcal{F}^* f(x) &= \lim_{\lambda \nearrow \infty} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} f(\xi) \varphi(\xi/\lambda) d\xi \\ &= \lim_{\lambda \nearrow \infty} (2\pi)^{\frac{d}{2}} \int_{|\xi| \le \lambda} e^{\mathrm{i}\xi \cdot x} f(\xi) d\xi.\end{aligned} \tag{1.9}$$

(3) For $f \in L^1(\mathbb{R}^d, \mathbb{C}) \cap L^2(\mathbb{R}^d, \mathbb{C})$ *the two definitions* (1.1) *and* (1.7) *of* $\mathcal{F}$ *coincide (by dominated convergence). Similarly, for* $f \in L^1(\mathbb{R}^d, \mathbb{C}) \cap L^2(\mathbb{R}^d, \mathbb{C})$ *the two definitions* (1.3) *and* (1.9) *of* $\mathcal{F}^*$ *coincide.*

The above notions extend naturally to vector fields. So we have a Fourier transform $f \to \widehat{f}$ from $(L^1(\mathbb{R}^d))^d \to (C_0(\mathbb{R}^d))^d$, from $(L^2(\mathbb{R}^d))^d \to (L^2(\mathbb{R}^d))^d$, from $(\mathcal{S}(\mathbb{R}^d))^d \to (\mathcal{S}(\mathbb{R}^d))^d$ and more generally from $(\mathcal{S}'(\mathbb{R}^d))^d \to (\mathcal{S}'(\mathbb{R}^d))^d$. Notice that all these maps except the 1st are isomorphisms, and all are one to one maps.

The Fourier transform extends to the spaces $L^p(\mathbb{R}^d, \mathbb{C})$ for $p \in [1, 2]$.

Theorem 1.5 (Hausdorff–Young). *For* $p \in [1, 2]$ *and* $f \in L^p(\mathbb{R}^d, \mathbb{C})$ *then* (1.7) *defines a function* $\mathcal{F}f \in L^{p'}(\mathbb{R}^d, \mathbb{C})$ *where* $p' = \frac{p}{p-1}$ *and an operator remains defined which satisfies*

$$\|\mathcal{F}f\|_{L^{p'}(\mathbb{R}^d, \mathbb{C})} \le (2\pi)^{-d\left(\frac{1}{2} - \frac{1}{p'}\right)} \|f\|_{L^p(\mathbb{R}^n, \mathbb{C})}. \tag{1.10}$$

We know already cases $p = 2$ and $p = 1$. This implies that Theorem 1.5 is a consequence of the Marcel Riesz interpolation Theorem, which we discuss now.

Theorem 1.6 (Riesz–Thorin). *Let* T *be a linear map from* $L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d)$ *to* $L^{q_0}(\mathbb{R}^d) \cap L^{q_1}(\mathbb{R}^d)$ *satisfying*

$$\|Tf\|_{L^{q_j}} \le M_j \|f\|_{L^{p_j}} \text{ for } j = 0, 1.$$

Then for $t \in (0,1)$ and for p_t and q_t defined by

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad , \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

we have

$$\|Tf\|_{L^{q_t}} \le (M_0)^{1-t}(M_1)^t \|f\|_{L^{p_t}} \text{ for } f \in L^{p_0}(\mathbb{R}^d) \cap L^{p_1}(\mathbb{R}^d).$$

Proof of the Hausdorff–Young's Theorem. We have $\frac{1}{p} = \frac{1-t}{2} + t$ for $t = \frac{2}{p} - 1$. Hence $1 - t = 2(1 - 1/p) = \frac{2}{p'}$ and $\frac{1}{p'} = \frac{1}{2}\frac{2}{p'} + \frac{t}{\infty}$ and

$$\|\mathcal{F}\|_{L^p \to L^{p'}} \le (2\pi)^{-\frac{d}{2}t} = (2\pi)^{-\frac{d}{2}\left(\frac{2}{p}-1\right)} = (2\pi)^{d\left(\frac{1}{p}-\frac{1}{2}\right)} = (2\pi)^{d\left(\frac{1}{2}+\frac{1}{p}-1\right)} = (2\pi)^{-d\left(\frac{1}{2}-\frac{1}{p'}\right)}.$$

Proof of Riesz–Thorin's Interpolation Theorem. First of all notice that if $f \in L^a \cap L^b$ with $a < b$ then $f \in L^c$ for any $c \in (a,b)$. To see this recall Hölder

$$\|fg\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q} \text{ for } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$$

Then, since $\frac{1}{c} = \frac{t}{a} + \frac{1-t}{b}$ for $t \in (0,1)$ from $|f| = |f|^t|f|^{1-t}$ we have

$$\|f\|_{L^c} = \||f|^t|f|^{1-t}\|_{L^c} \le \||f|^t\|_{L^{\frac{a}{t}}}\||f|^{1-t}\|_{L^{\frac{b}{1-t}}} = \|f\|_{L^a}^t\|f\|_{L^b}^{1-t}.$$

For $p_t = p_0 = p_1 = \infty$ (in fact we can repeat a similar argument for $p_t = p_0 = p_1$ any fixed value in $[1,\infty]$) we then have

$$\|Tf\|_{L^{q_t}} \le \|Tf\|_{L^{q_1}}^t\|Tf\|_{L^{q_0}}^{1-t} \le (M_0)^{1-t}(M_1)^t\|f\|_{L^\infty}.$$

So let us suppose $p_t < \infty$. Then it is enough to prove

$$|\int Tfg dx| \le (M_0)^{1-t}(M_1)^t\|f\|_{L^{p_t}}\|g\|_{L^{q_t'}} = (M_0)^{1-t}(M_1)^t$$

considering only $\|f\|_{L^{p_t}} = \|g\|_{L^{q_t'}} = 1$ for simple functions $f = \sum_{j=1}^m a_j\chi_{E_j}$ where we can take the E_j to be finite measure sets mutually disjoint. If $q_t' < \infty$ we can also reduce to simple functions $g = \sum_{k=1}^N b_k\chi_{F_k}$ where the F_j are finite measure sets mutually disjoint. The case $q_t' = \infty$ reduces to the case $p_t = \infty$ by duality. In fact, see Remark 16 p. 44 [2],

$$\|T\|_{\mathcal{L}(L^{p_t},L^1)} = \|T^*\|_{\mathcal{L}(L^\infty,L^{p'_t})}.$$

Notice that if both $p_0 < \infty$ and $p_1 < \infty$ and since we are treating $q_0 = q_1 = 1$, then $\|T\|_{\mathcal{L}(L^{p_j},L^1)} = \|T^*\|_{\mathcal{L}(L^\infty,L^{p'_j})} \le M_j$ and so one reduces to the case $p_t = \infty$. If, say, $p_0 = \infty$, then $\|T\|_{\mathcal{L}(L^{p_1},L^1)} = \|T^*\|_{\mathcal{L}(L^\infty,L^{p'_1})} \le M_1$ since $p_1 < \infty$, but $\|T\|_{\mathcal{L}(L^{p_0},L^1)} = \|T^*\|_{\mathcal{L}(L^\infty,(L^\infty)')} \le M_0$, so in other words, we don't get a Lebesgue space. However, the

issue is to bound for $f \in L^{p_0} \cap L^\infty$ a $T^* f \in L^1 \cap (L^\infty)' = L^1$ where $\|T^* f\|_{(L^\infty)'} = \|T^* f\|_{L^1}$, so that one can still apply the above argument used for $p_t = \infty$.
Let us turn to the case $p_t < \infty$ and $q_t' < \infty$. For $a_j = e^{\mathrm{i}\theta_j}|a_j|$ and $b_k = e^{\mathrm{i}\psi_k}|b_k|$ the polar representations, set

$$f_z := \sum_{j=1}^{m} |a_j|^{\frac{\alpha(z)}{\alpha(t)}} e^{\mathrm{i}\theta_j} \chi_{E_j} \text{ with } \alpha(z) := \frac{1-z}{p_0} + \frac{z}{p_1}$$
$$g_z := \sum_{k=1}^{N} |b_k|^{\frac{1-\beta(z)}{1-\beta(t)}} e^{\mathrm{i}\psi_k} \chi_{F_k} \text{ with } \beta(z) := \frac{1-z}{q_0} + \frac{z}{q_1}.$$

Notice that since we are assuming $q_t' < \infty$, then $q_t > 1$ and so $\beta(t) = \frac{1}{q_t} < 1$, so that g_z is well defined. Similarly, since $p_t < \infty$ we have $\alpha(t) = \frac{1}{p_t} > 0$, so also f_z is well defined.
We consider now the function

$$F(z) = \int T f_z g_z dx.$$

Our goal is to prove $|F(t)| \le M_0^{1-t} M_1^t$.
$F(z)$ is holomorphic in $0 < \operatorname{Re} z < 1$, continuous and bounded in $0 \le \operatorname{Re} z \le 1$. Boundedness follows from estimates like

$$\left||a_j|^{\frac{\alpha(z)}{\alpha(t)}}\right| = |a_j|^{\frac{\alpha(\operatorname{Re} z)}{\alpha(t)}} \text{ which is bounded for } 0 \le \operatorname{Re} z \le 1.$$

We have $F(t) = \int T f g dx$ since $f_t = f$ and $g_t = g$.
By the 3 lines lemma, see below, which yields $|F(z)| \le M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$ if the two estimates below are true, our theorem is a consequence of the following two inequalities

$$|F(z)| \le M_0 \text{ for } \operatorname{Re} z = 0 \; ;$$
$$|F(z)| \le M_1 \text{ for } \operatorname{Re} z = 1 \; .$$

For $z = \mathrm{i}y$ we have for $p_0 < \infty$

$$|f_{\mathrm{i}y}|^{p_0} = \sum_{j=1}^{m} \left| |a_j|^{\frac{\alpha(\mathrm{i}y)}{\alpha(t)}} \right|^{p_0} \chi_{E_j} = \sum_{j=1}^{m} \left| |a_j|^{\frac{\frac{1}{p_0} + \mathrm{i}y\left(\frac{1}{p_1} - \frac{1}{p_0}\right)}{\frac{1}{p_t}}} \right|^{p_0} \chi_{E_j}$$
$$= \sum_{j=1}^{m} \left| |a_j|^{\mathrm{i}y p_t \left(\frac{1}{p_1} - \frac{1}{p_0}\right)} |a_j|^{\frac{p_t}{p_0}} \right|^{p_0} \chi_{E_j} = \sum_{j=1}^{m} |a_j|^{p_t} \chi_{E_j} = |f|^{p_t}.$$

This implies

$$\|f_{\mathrm{i}y}\|_{p_0} = \left(\int_{\mathbb{R}^d} |f_{\mathrm{i}y}|^{p_0} dx \right)^{\frac{1}{p_0}} = \left(\int_{\mathbb{R}^d} |f|^{p_t} dx \right)^{\frac{1}{p_0}} = 1. \tag{1.11}$$

Notice that we have also $\|f_{\mathrm{i}y}\|_\infty = 1$ when $p_0 = \infty$.

Proceeding similarly, using $1-\beta(z)=\frac{1-z}{q_0'}+\frac{z}{q_1'}$, for $z=\mathrm{i}y$ and $q_0'<\infty$ we have

$$|g_{\mathrm{i}y}|^{q_0'}=\sum_{k=1}^{N}\left||b_k|^{\frac{1-\beta(\mathrm{i}y)}{1-\beta(t)}}\right|^{q_0'}\chi_{F_k}=\sum_{k=1}^{N}\left||b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'}-\frac{1}{q_0'}\right)}{\frac{1}{q_t'}}}|b_k|^{\frac{\frac{1}{q_0'}}{\frac{1}{q_t'}}}\right|^{q_0'}\chi_{F_k}=\sum_{j=1}^{N}|b_k|^{q_t'}\chi_{F_k}=|g|^{q_t'}.$$

This implies

$$\|g_{\mathrm{i}y}\|_{q_0'}=\left(\int_{\mathbb{R}^d}|g_{\mathrm{i}y}|^{q_0'}dx\right)^{\frac{1}{q_0'}}=\left(\int_{\mathbb{R}^d}|g|^{q_t'}dx\right)^{\frac{1}{q_0'}}=1. \tag{1.12}$$

Notice that we have also $\|g_{\mathrm{i}y}\|_\infty=1$ when $q_0'=\infty$.
Then

$$|F(\mathrm{i}y)|\le\|Tf_{\mathrm{i}y}\|_{q_0}\|g_{\mathrm{i}y}\|_{q_0'}\le M_0\|f_{\mathrm{i}y}\|_{p_0}\|g_{\mathrm{i}y}\|_{q_0'}=M_0.$$

By a similar argument

$$|f_{1+\mathrm{i}y}|^{p_1}=|f|^{p_t}$$
$$|g_{1+\mathrm{i}y}|^{q_1'}=|g|^{q_t'}.$$

Indeed by $\alpha(1+\mathrm{i}y)=\frac{1+\mathrm{i}y}{p_1}-\frac{\mathrm{i}y}{p_0}$

$$\begin{aligned}|f_{1+\mathrm{i}y}|^{p_1}&=\sum_{j=1}^{m}\left||a_j|^{\frac{\alpha(1+\mathrm{i}y)}{\alpha(t)}}\right|^{p_1}\chi_{E_j}=\sum_{j=1}^{m}\left||a_j|^{\frac{\frac{1}{p_1}+\mathrm{i}y\left(\frac{1}{p_1}-\frac{1}{p_0}\right)}{\frac{1}{p_t}}}\right|^{p_1}\chi_{E_j}\\&=\sum_{j=1}^{m}\left||a_j|^{\frac{p_t}{p_1}}\right|^{p_1}\chi_{E_j}=\sum_{j=1}^{m}|a_j|^{p_t}\chi_{E_j}=|f|^{p_t}\end{aligned}$$

and by $1-\beta(1+\mathrm{i}y)=\frac{1+\mathrm{i}y}{q_1'}-\frac{\mathrm{i}y}{q_0'}$

$$|g_{1+\mathrm{i}y}|^{q_1'}=\sum_{k=1}^{N}\left||b_k|^{\frac{1-\beta(1+\mathrm{i}y)}{1-\beta(t)}}\right|^{q_1'}\chi_{F_k}=\sum_{k=1}^{N}\left||b_k|^{\frac{\mathrm{i}y\left(\frac{1}{q_1'}-\frac{1}{q_0'}\right)}{\frac{1}{q_t'}}}|b_k|^{\frac{\frac{1}{q_1'}}{\frac{1}{q_t'}}}\right|^{q_1'}\chi_{F_k}=\sum_{j=1}^{N}|b_k|^{q_t'}\chi_{F_k}=|g|^{q_t'}.$$

Finally

$$|F(1+\mathrm{i}y)|\le\|Tf_{1+\mathrm{i}y}\|_{q_1}\|g_{1+\mathrm{i}y}\|_{q_1'}\le M_1\|f_{1+\mathrm{i}y}\|_{p_1}\|g_{1+\mathrm{i}y}\|_{q_1'}=M_1\|f\|_{p_t}^{\frac{p_t}{p_1}}\|g\|_{q_t'}^{\frac{q_t'}{q_1'}}=M_1.$$

□

Here we have used the following lemma.

Lemma 1.7 (Three Lines Lemma)**.** *Let $F(z)$ be holomorphic in the strip $0 < \operatorname{Re} z < 1$, continuous and bounded in $0 \le \operatorname{Re} z \le 1$ and such that*

$$|F(z)| \le M_0 \text{ for } \operatorname{Re} z = 0 \;;$$
$$|F(z)| \le M_1 \text{ for } \operatorname{Re} z = 1 \;.$$

Then we have $|F(z)| \le M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$ for all $0 < \operatorname{Re} z < 1$.

Proof. Let us start with the special case $M_0 = M_1 = 1$ and set $B := \|F\|_{L^\infty}$. Set $h_\epsilon(z) := (1+\epsilon z)^{-1}$ with $\epsilon > 0$. Since $\operatorname{Re}(1+\epsilon z) = 1 + \epsilon x \ge 1$ it follows $|h_\epsilon(z)| \le 1$ in the strip. Furthermore $\operatorname{Im}(1+\epsilon z) = \epsilon y$ implies also $|h_\epsilon(z)| \le |\epsilon y|^{-1}$. Consider now the two horizontal lines $y = \pm B/\epsilon$ and let R be the rectangle $0 \le x \le 1$ and $|y| \le B/\epsilon$. In $|y| \ge B/\epsilon$ we have

$$|F(z)h_\epsilon(z)| \le \frac{B}{|\epsilon y|} \le \frac{B}{|\epsilon B/\epsilon|} = 1.$$

On the other hand by the maximum modulus principle

$$\sup_R |F(z)h_\epsilon(z)| = \sup_{\partial R} |F(z)h_\epsilon(z)| \le 1,$$

where on the horizontal sides the last inequality follows from the previous inequality and on the vertical sides follows from $|F(z)| \le 1$ for $\operatorname{Re} z = 0, 1$ and from $|h_\epsilon(z)| \le 1$.
Hence in the whole strip $0 \le x \le 1$ we have $|F(z)h_\epsilon(z)| \le 1$ for any $\epsilon > 0$. This implies

$$\lim_{\epsilon \searrow 0} |F(z)h_\epsilon(z)| = |F(z)| \le 1$$

in the whole strip $0 \le x \le 1$.
In the general case $(M_0, M_1) \ne (1,1)$ set $g(z) := M_0^{1-z} M_1^z$. Notice that

$$g(z) = e^{(1-z)\log M_0} e^{z \log M_1} \Rightarrow |g(z)| = M_0^{1-x} M_1^x \Rightarrow$$
$$\min(M_0, M_1) \le |g(z)| \le \max(M_0, M_1).$$

So $F(z)g^{-1}(z)$ satisfies the hypotheses of the case $M_0 = M_1 = 1$ and so $|F(z)| \le |g(z)| = M_0^{1-\operatorname{Re} z} M_1^{\operatorname{Re} z}$

□

We consider now for $\triangle := \sum_j \frac{\partial^2}{\partial x_j^2}$ and for $f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ the heat equation

$$u_t - \triangle u = 0\,, \quad u(0,x) = f(x). \tag{1.13}$$

By applying $\mathcal{F}$ we transform the above problem into

$$\widehat{u}_t + |\xi|^2 \widehat{u} = 0\,, \quad \widehat{u}(0,\xi) = \widehat{f}(\xi).$$

This yields $\widehat{u}(t,\xi) = e^{-t|\xi|^2} \widehat{f}(\xi)$. Notice that since $\widehat{f} \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C})$ and $e^{-t|\cdot|^2} \in \mathcal{S}(\mathbb{R}^d, \mathbb{C})$ for any $t > 0$, the last product is well defined. Furthermore, we have $\widehat{u}(t,\cdot) \in C^0([0,+\infty), \mathcal{S}'(\mathbb{R}^d, \mathbb{C}))$

and, as a consequence, since $\mathcal{F}$ is an isomorphism of $\mathcal{S}'(\mathbb{R}^d,\mathbb{C})$ also $u(t,\cdot)\in C^0([0,+\infty),\mathcal{S}'(\mathbb{R}^n,\mathbb{C}))$. We have $e^{-t|\xi|^2}=\widehat{G}(t,\xi)$ with $G(t,x)=(2t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$. Then, from $\widehat{u}(t,\xi)=\widehat{G}(t,\xi)\widehat{f}(\xi)$ it follows $u(t,x)=(2\pi)^{-\frac{d}{2}}G(t,\cdot)*f(x)$. In particular, for $f\in L^p(\mathbb{R}^d,\mathbb{C})$, we have

$$u(t,x)=(4\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\frac{|x-y|^2}{4t}}f(y)dy.$$

Notice that by (1.2) we have

$$(4\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\frac{|x|^2}{4t}}dx=1.$$

We will write

$$e^{t\triangle}f(x):=(4\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\frac{|x-y|^2}{4t}}f(y)dy. \tag{1.14}$$

Notice that for $p\ge 1$ we have $\|e^{t\triangle}f\|_{L^p(\mathbb{R}^d)}\le\|f\|_{L^p(\mathbb{R}^d)}$ and for $f\in L^1(\mathbb{R}^d)$ and any $x\in\mathbb{R}^d$

$$|e^{t\triangle}f(x)|\le(4\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{-\frac{|x-y|^2}{4t}}|f(y)|dy\le(4\pi t)^{-\frac{d}{2}}\int_{\mathbb{R}^d}|f(y)|dy=(4\pi t)^{-\frac{d}{2}}\|f\|_{L^1(\mathbb{R}^d)}.$$

We set also $K_t(x):=(4\pi t)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4t}}$. Then $e^{t\triangle}f=K_t*f$. $K_t(x-y)$ is the *Heath Kernel*. As a corollary to the Riesz–Thorin Theorem we obtain the following result.

Corollary 1.8. *For any $q\ge p\ge 1$ and any $f\in L^p(\mathbb{R}^d)$ we have*

$$\|e^{t\triangle}f\|_{L^q(\mathbb{R}^d)}\le(4\pi t)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\|f\|_{L^p(\mathbb{R}^d)}. \tag{1.15}$$

Proof. Notice that (1.15) is true for $p=q$ and for $q=\infty$ and $p=1$. For $q>p=1$ Riesz–Thorin and $\dfrac{1}{q}=\dfrac{1-\frac{1}{q}}{\infty}+\dfrac{\frac{1}{q}}{1}$ yields

$$\|e^{t\triangle}\|_{L^1\to L^q}\le\|e^{t\triangle}\|^{1-\frac{1}{q}}_{L^1\to L^\infty}\|e^{t\triangle}\|^{\frac{1}{q}}_{L^1\to L^1}\le(4\pi t)^{-\frac{d}{2}\left(1-\frac{1}{q}\right)}=(4\pi t)^{-\frac{d}{2q'}}\text{ with }q'=\frac{q}{q-1}.$$

Next, for $1<p<q$ we have $\frac{1}{p}=\alpha+\frac{1-\alpha}{q}=\frac{1}{q}+\frac{\alpha}{q'}$ s.t. $\alpha=q'\left(\frac{1}{p}-\frac{1}{q}\right)$. Then

$$\|e^{t\triangle}\|_{L^p\to L^q}\le\|e^{t\triangle}\|^{\alpha}_{L^1\to L^q}\|e^{t\triangle}\|^{1-\alpha}_{L^q\to L^q}\le(4\pi t)^{-\frac{d}{2q'}\alpha}=(4\pi t)^{-\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}.$$

□

Another application of M. Riesz's Theorem is the following useful tool.

Lemma 1.9 (Young's Inequality)**.** *Let*

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy$$

where

$$\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|dy<C,\ \sup_{y\in\mathbb{R}^n}\int_{\mathbb{R}^n}|K(x,y)|dx<C. \tag{1.16}$$

Then

$$\|Tf\|_{L^p(\mathbb{R}^n)}\le C\|f\|_{L^p(\mathbb{R}^n)} \text{ for all } p\in[1,\infty]$$

Proof. The case $p=1,\infty$ follow immediately from (1.16). The intermediate cases from Riesz's Theorem. □

Theorem 1.10. *$\rho\in L^1(\mathbb{R}^d)$ be s.t. $\int\rho(x)dx=1$. Set $\rho_\epsilon(x):=\epsilon^{-d}\rho(x/\epsilon)$. Consider $C_c(\mathbb{R}^d,\mathbb{C})$ and for each $p\in[1,\infty]$ let $\overline{C_c(\mathbb{R}^d,\mathbb{C})}_p$ be the closure of $C_c(\mathbb{R}^d,\mathbb{C})$ in $L^p(\mathbb{R}^d,\mathbb{C})$, so that $\overline{C_c(\mathbb{R}^d,\mathbb{C})}_p=L^p(\mathbb{R}^d,\mathbb{C})$ for $p<\infty$ and $\overline{C_c(\mathbb{R}^d,\mathbb{C})}_\infty=C_0(\mathbb{R}^d,\mathbb{C})\subsetneqq L^\infty(\mathbb{R}^d,\mathbb{C})$. Then for any $f\in\overline{C_c(\mathbb{R}^d,\mathbb{C})}_p$ we have*

$$\lim_{\epsilon\searrow 0}\rho_\epsilon * f=f \text{ in } L^p(\mathbb{R}^d,\mathbb{C}). \tag{1.17}$$

In particular we have

$$\lim_{t\searrow 0}e^{t\triangle}f=f \text{ in } L^p(\mathbb{R}^d,\mathbb{C}). \tag{1.18}$$

Proof. Clearly, (1.18) is a special case of (A.10) setting $\epsilon=\sqrt{t}$ and $\rho(x)=(4\pi)^{-\frac{d}{2}}e^{-\frac{|x|^2}{4}}$. To prove (A.10) we start with $f\in C_c(\mathbb{R}^d,\mathbb{C})$. In this case

$$\rho_\epsilon * f(x)-f(x)=\int_{\mathbb{R}^d}(f(x-\epsilon y)-f(x))\rho(y)dy$$

so that, by Minkowski inequality and for $\Delta(y):=\|f(\cdot-y)-f(\cdot)\|_{L^p}$, we have

$$\|\rho_\epsilon * f(x)-f(x)\|_{L^p}\le\int|\rho(y)|\Delta(\epsilon\, y)dy.$$

Now we have $\lim_{y\to 0}\Delta(y)=0$ and $\Delta(y)\le 2\|f\|_{L^p}$. So, by dominated convergence we get

$$\lim_{\epsilon\searrow 0}\|\rho_\epsilon * f(x)-f(x)\|_{L^p}=\lim_{\epsilon\searrow 0}\int|\rho(y)|\Delta(\epsilon\, y)dy=0.$$

So this proves (A.10) for $f\in C_c(\mathbb{R}^d,\mathbb{C})$. The general case is proved by a density argument. □

2 First part on Sobolev Spaces

We recall some basic definitions of Sobolev Spaces. We will redefine them later.

We consider an open set $\Omega\subseteq\mathbb{R}^d$. Recall that $L^p(\Omega)\subset\mathcal{D}'(\Omega)$ for any $p\in[1,\infty]$. For $u\in L^p(\Omega)$ and any multi-index $\alpha\in(\mathbb{N}\cup\{0\})^d$ we can consider the derivatives $\partial^\alpha u\in\mathcal{D}'(\Omega)$ where $\partial^\alpha=\partial_1^{\alpha_1}...\partial_n^{\alpha_n}$. We set $|\alpha|=\sum_{j=1}^d\alpha_j$.

Definition 2.1. Given an open subset $\Omega \subseteq \mathbb{R}^d$ and and $m \in \mathbb{N}$ we set

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq m\}. \tag{2.1}$$

In $W^{m,p}(\Omega)$ we introduce the norm

$$\|u\|_{W^{m,p}(\Omega)} := \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_{L^p(\Omega)}. \tag{2.2}$$

In the special case $p = 2$ we use also the notation $H^m(\Omega) := W^{m,2}(\Omega)$. In this space it is possible to introduce the following inner product:

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \langle \partial^\alpha u, \partial^\alpha v\rangle_{L^2(\Omega)} \tag{2.3}$$

where $\langle u, v\rangle_{L^2(\Omega)} = \operatorname{Re} \int_{\mathbb{R}^d} u(x)\overline{v}(x)dx$. Notice that the corresponding norm,

$$\sqrt{\sum_{|\alpha|\leq m} \|\partial^\alpha u\|^2_{L^2(\Omega)}}$$

is equivalent to (2.2) when $p = 2$.

We recall the following result. We have the following form of the Sobolev Embedding Theorem

Theorem 2.2. *For $d \geq 2$ let $1 \leq p < d$. Then there exists a constant $C(p, d)$ s.t. for any $u \in W^{1,p}(\mathbb{R}^d)$ we have*

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C(p, d)\|\nabla u\|_{L^p(\mathbb{R}^d)} \text{ where} \tag{2.4}$$

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}. \tag{2.5}$$

Remark 2.3. For (2.4) to be true relation (2.5) is necessary by scaling. Indeed if we set $u_\lambda(x) = u(\lambda x)$ for $\lambda > 0$ then

$$\|u_\lambda\|_{L^{p^*}(\mathbb{R}^d)} = \lambda^{-\frac{d}{p^*}}\|u\|_{L^{p^*}(\mathbb{R}^d)}$$
$$\|\nabla u_\lambda\|_{L^p(\mathbb{R}^d)} = \lambda^{1-\frac{d}{p}}\|\nabla u\|_{L^p(\mathbb{R}^d)}$$

hence (2.4) applied to u_λ implies

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C(p, d)\lambda^{1-\frac{d}{p}+\frac{d}{p^*}}\|\nabla u\|_{L^p(\mathbb{R}^d)}.$$

But since this must be valid for any $u \in W^{1,p}(\mathbb{R}^d)$ and any $\lambda > 0$, if the exponent of λ is different from 0, by considering either $\lambda \to 0^+$ or $\lambda \to +\infty$ we obtain $\|u\|_{L^q(\mathbb{R}^d)} = 0$, that is $u = 0$ for any $u \in W^{1,p}(\mathbb{R}^d)$, which is absurd.

Remark 2.4. For $p = d > 1$ and $p^* = \infty$ the statement would be false, contrary to the somewhat special case $p = d = 1$ and $p^* = \infty$. See later Example 2.7.

Before proving the general case in Theorem 2.2 we consider the special case $p = 1$ and $p^* = \frac{d}{d-1}$.

Proposition 2.5. *For any $u \in W^{1,1}(\mathbb{R}^d)$ with $d \geq 2$ we have*

$$\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \frac{1}{2d} \sum_{j=1}^{d} \|\partial_j u\|_{L^1(\mathbb{R}^d)}. \tag{2.6}$$

The proof of Proposition 2.5 is based on the following lemma.

Lemma 2.6. *For a point $x = (x_1, ..., x_{j-1}, x_j, x_{j+1}, .., x_d) \in \mathbb{R}^d$ set $\widetilde{x}_j := (x_1, ..., x_{j-1}, x_{j+1}, .., x_n) \in \mathbb{R}^{d-1}$. Let*

$$u(x) = \prod_{j=1}^{d} u_j(\widetilde{x}_j).$$

Then we have

$$\|u\|_{L^1(\mathbb{R}^d)} \leq \prod_{j=1}^{d} \|u_j\|_{L^{d-1}(\mathbb{R}^{d-1})}. \tag{2.7}$$

Proof. The case $d = 2$ reduces to $\|u_1(x_1)u_2(x_2)\|_{L^1(\mathbb{R}^2)} = \prod_{j=1}^{2} \|u_j\|_{L^1(\mathbb{R})}$.
Let us see the case $d = 3$, that is $\|u\|_{L^1(\mathbb{R}^3)} \leq \prod_{j=1}^{3} \|u_j\|_{L^2(\mathbb{R}^2)}$. We have

$$u(x) = u_1(x_2, x_3)u_2(x_1, x_3)u_3(x_1, x_2)$$

Keeping x_3 fixed we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |u(x)|dx_1 dx_2 = \int_{\mathbb{R}} |u_1(x_2, x_3)| \ |u_2(x_1, x_3)| \ |u_3(x_1, x_2)|dx_1 dx_2 \\
&\leq \|u_3\|_{L^2(\mathbb{R}^2)} \left(\int_{\mathbb{R}^2} dx_1 dx_2 |u_1(x_2, x_3)|^2 |u_2(x_1, x_3)|^2 \right)^{\frac{1}{2}} \\
&= \|u_3\|_{L^2(\mathbb{R}^2)} \left(\int_{\mathbb{R}} dx_2 |u_1(x_2, x_3)|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} dx_1 |u_2(x_1, x_3)|^2 \right)^{\frac{1}{2}}
\end{aligned}$$

Hence

$$\int_{\mathbb{R}^3} |u(x)|dx_1 dx_2 dx_3 \leq \|u_3\|_{L^2(\mathbb{R}^2)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} dx_2 |u_1(x_2, x_3)|^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} dx_1 |u_2(x_1, x_3)|^2 \right)^{\frac{1}{2}} dx_3$$

yields by Schwartz inequality

$$\|u\|_{L^1(\mathbb{R}^3)} \leq \prod_{j=1}^{3} \|u_j\|_{L^2(\mathbb{R}^2)}.$$

Assume the result known for d and let us prove it for $d+1$. We have fixing x_{d+1}

$$\begin{aligned}
&\int_{\mathbb{R}^d} |u(x)|dx_1..dx_d = \int_{\mathbb{R}^d} |u_1(\widetilde{x}_1)...u_1(\widetilde{x}_d)|\ |u_{d+1}(\widetilde{x}_{d+1})|dx_1..dx_d \\
&\le \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |u_1(\widetilde{x}_1)...u_d(\widetilde{x}_d)|^{\frac{d}{d-1}} dx_1..dx_d\right)^{\frac{d-1}{d}} \\
&= \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |u_1(\widetilde{x}_1)|^{\frac{d}{d-1}}...|u_d(\widetilde{x}_d)|^{\frac{d}{d-1}} dx_1..dx_d\right)^{\frac{d-1}{d}} \\
&\le \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\prod_{j=1}^d \||u_j|^{\frac{d}{d-1}}\|_{L^{d-1}(\mathbb{R}^{d-1})}\right)^{\frac{d-1}{d}} \\
&= \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \left(\prod_{j=1}^d \|u_j\|^{\frac{d}{d-1}}_{L^d(\mathbb{R}^{d-1})}\right)^{\frac{d-1}{d}} = \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{j=1}^d \|u_j\|_{L^d(\mathbb{R}^{d-1})},
\end{aligned}$$

where in the last inequality we used induction. So, applying $\int_{\mathbb{R}} dx_{d+1}$,

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}} |u(x)|dx = \int_{\mathbb{R}} dx_{d+1} \int_{\mathbb{R}^d} |u(x)|dx_1..dx_d \\
&\le \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \int_{\mathbb{R}} \prod_{j=1}^d \|u_j\|_{L^d(\mathbb{R}^{d-1})} dx_{d+1} \\
&\le \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{j=1}^d \|\|u_j\|_{L^d(\mathbb{R}^{d-1})}\|_{L^d_{x_{d+1}}} = \|u_{d+1}\|_{L^d(\mathbb{R}^d)} \prod_{j=1}^d \|u_j\|_{L^d(\mathbb{R}^d)}.
\end{aligned}$$

□

Proof of Proposition 2.5. We assume $u \in C^\infty_c(\mathbb{R}^d)$. Then we have

$$|u(x)| \le \frac{1}{2} \int_{\mathbb{R}} |\partial_j u(x_1, ..., x_{j-1}, t, x_{j+1}, .., x_n)|dt =: \frac{1}{2} u_j(\widetilde{x}_j) \text{ where } \widetilde{x}_j := (x_1, ..., x_{j-1}, , x_{j+1}, .., x_n).$$

Then

$$|u(x)|^{\frac{d}{d-1}} \le \frac{1}{2^{\frac{d}{d-1}}} \prod_{j=1}^d |u_j(\widetilde{x}_j)|^{\frac{1}{d-1}}.$$

Hence, by Lemma 2.6

$$\begin{aligned}
\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} dx &\le \frac{1}{2^{\frac{d}{d-1}}} \prod_{j=1}^d \|\ |u_j(\widetilde{x}_j)|^{\frac{1}{d-1}}\|_{L^{d-1}(\mathbb{R}^{d-1})} \frac{1}{2^{\frac{d}{d-1}}} \prod_{j=1}^d \|u_j(\widetilde{x}_j)\|^{\frac{1}{d-1}}_{L^1(\mathbb{R}^{d-1})} \\
&= \frac{1}{2^{\frac{d}{d-1}}} \prod_{j=1}^d \|\partial_j u\|^{\frac{1}{d-1}}_{L^1(\mathbb{R}^d)} = \frac{1}{2^{\frac{d}{d-1}}} \left(\prod_{j=1}^d \|\partial_j u\|_{L^1(\mathbb{R}^d)}\right)^{\frac{1}{d-1}}.
\end{aligned}$$

Then using $(\prod_{j=1}^{d} a_j)^{\frac{1}{d}} \leq \frac{1}{d}\sum_{j=1}^{d} a_j$ (follows by the concavity of log) we get

$$\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \frac{1}{2}\left(\prod_{j=1}^{d} \|\partial_j u\|_{L^1(\mathbb{R}^d)}\right)^{\frac{1}{d}} \leq \frac{1}{2d}\sum_{j=1}^{d} \|\partial_j u\|_{L^1(\mathbb{R}^d)} \leq \frac{1}{2d}\|\nabla u\|_{L^1(\mathbb{R}^d)}.$$

By density this estimate extends to all $u \in W^{1,1}(\mathbb{R}^d)$.

□

Proof of Theorem 2.2. We know already case $p = 1$, so we will consider $1 < p < d$. For $t > 1$ set $v = |u|^{t-1}u$. Notice that if $u \in C_c^\infty(\mathbb{R}^d)$ then $v \in C_c^1(\mathbb{R}^d)$. Indeed $v = G \circ u$ with $G(s) := |s|^{t-1}s$ which is in $C^1(\mathbb{R})$. We have $G'(s) = t|s|^{t-1}$ and by the chain rule

$$\partial_j v = G'(u)\partial_j u = t|u|^{t-1}\partial_j u.$$

We apply (2.6) to v. Then for $p' = \frac{p}{p-1}$

$$\|u\|^t_{L^{t\frac{d}{d-1}}(\mathbb{R}^d)} \leq \frac{1}{2d}\sum_{j=1}^{d} \|t|u|^{t-1}\partial_j u\|_{L^1(\mathbb{R}^d)} \leq t\frac{1}{2d}\|u\|^{t-1}_{L^{(t-1)p'}(\mathbb{R}^d)} \sum_{j=1}^{d} \|\partial_j u\|_{L^p(\mathbb{R}^d)}. \tag{2.8}$$

We look for $t\frac{d}{d-1} = (t-1)p'$. This is equivalent to

$$\begin{aligned} &td(p-1) = (d-1)pt - (d-1)p \iff ((\not{d}-1)p - d(\not{p}-1))t = (d-1)p \iff \\ &(d-p)t = (d-1)p \iff d\not{p}\left(\frac{1}{p} - \frac{1}{d}\right)t = (d-1)\not{p} \iff t = \frac{d-1}{d}p^*. \end{aligned}$$

Notice that $1 < p < d$ implies

$$t = \frac{d-1}{d}p^* = \frac{d-1}{d}\ \frac{pd}{d-p} = \frac{d-1}{d-p}p > 1.$$

Since $(t-1)p' = t\frac{d}{d-1} = p^*$, inequality (2.8) becomes

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq \frac{d-1}{2d^2}p^*\sum_{j=1}^{d} \|\partial_j u\|_{L^p(\mathbb{R}^d)} = \frac{(d-1)p}{2d(d-p)}\sum_{j=1}^{d} \|\partial_j u\|_{L^p(\mathbb{R}^d)} \leq C(p,d)\|\nabla u\|_{L^p(\mathbb{R}^d)}. \tag{2.9}$$

□

We show with an example that $W^{1,d}(\mathbb{R}^d) \not\subset L^\infty(\mathbb{R}^d)$

Example 2.7. Fix $R > 0$ and consider a function $u \in C^\infty(\mathbb{R}^d\backslash\{0\})$ such that

$$u(x) = \begin{cases} v(x) := \log\left(\log\left(\frac{4R}{|x|}\right)\right) \text{ for } |x| < R \\ 0 \text{ for } |x| > 2R\ . \end{cases} \tag{2.10}$$

Clearly $u \notin L^\infty(D_{\mathbb{R}^d}(0,4R))$. On the other hand, for $|x|<R$

$$\nabla v = \frac{1}{\log\left(\frac{4R}{|x|}\right)} \frac{|x|}{4R}\left(-\frac{4R}{|x|^2}\right)\frac{x}{|x|} = -\frac{1}{\log\left(\frac{4R}{|x|}\right)}\frac{x}{|x|^2}.$$

Then

$$\begin{aligned}\int_{D_{\mathbb{R}^d}(0,R)} |\nabla v|^d &= c\int_0^R \frac{1}{\log^d\left(\frac{4R}{r}\right)}\frac{1}{r^d} r^{d-1}dr = c\int_0^R \frac{1}{\log^d\left(\frac{4R}{r}\right)}\frac{1}{r}dr\\ &= c\int_0^R \frac{1}{\left(\log(4R)-\log(r)\right)^d}\frac{1}{r}dr = c\int_{-\infty}^{\log(R)} \frac{1}{\left(\log(4R)-s\right)^d}ds\\ &= \frac{c}{d-1}\left(\log(4R)-s\right)^{-(d-1)}\Big]_{-\infty}^{\log(R)} = \frac{c}{d-1}\left(\log(4)\right)^{-(d-1)}.\end{aligned}$$

So it is easy to conclude that $u \in W^{1,d}(\mathbb{R}^d)$. □

We recall the following important theorem.

Theorem 2.8 (Rellich–Kondrakov)**.** *Let $\Omega \subset \mathbb{R}^d$ be bounded and with $\partial\Omega$ a C^1 sub–manifold of $\mathbb{R}^d$. Then we have the following statements.*

1. *For $1 \le p < d$ and $q \in [1, p^*)$ the embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.*
2. *For $q \in [1,\infty)$ the embedding $W^{1,d}(\Omega) \hookrightarrow L^q(\Omega)$ is compact.*
3. *For $p > d$ the embedding $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ is compact.*

We skip the proof. We will need later a special case of this theorem, and we will prove that case. □

Remark 2.9.

1. Recall that we have seen $W^{1,d}(\mathbb{R}^d) \not\subset L^\infty(\mathbb{R}^d)$ in Example 2.7. The discussion therein shows that always $W^{1,d}(\Omega) \not\subset L^\infty(\Omega)$.
2. For $1 \le p < d$ the embedding $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ is **not** compact. This is related to the scaling argument in Remark 2.3. Indeed, pick $\varphi \in \mathcal{D}(D_{\mathbb{R}^d}(0,1))$ and for $\varepsilon \in (0,1)$ consider $\varphi_\varepsilon(x) := \varepsilon^{-\frac{d}{p^*}}\varphi\left(\frac{x}{\varepsilon}\right)$. Then we have

$$\|\varepsilon^{-\frac{d}{p^*}}\varphi\left(\frac{x}{\varepsilon}\right)\|_{L^{p^*}(D_{\mathbb{R}^d}(0,1))} = \|\varepsilon^{-\frac{d}{p^*}}\varphi\left(\frac{x}{\varepsilon}\right)\|_{L^{p^*}(D_{\mathbb{R}^d}(0,\varepsilon))} = \|\varphi\|_{L^{p^*}(D_{\mathbb{R}^d}(0,1))},$$

$$\|\varepsilon^{-\frac{d}{p^*}}\varphi\left(\frac{x}{\varepsilon}\right)\|_{L^{p}(D_{\mathbb{R}^d}(0,1))} = \varepsilon^{\frac{d}{p}-\frac{d}{p^*}}\|\varphi\|_{L^{p}(D_{\mathbb{R}^d}(0,1))} = \varepsilon\|\varphi\|_{L^{p}(D_{\mathbb{R}^d}(0,1))} \overset{\varepsilon\to 0}{\to} 0$$

and

$$\|\nabla\varphi_\varepsilon\|_{L^{p}(D_{\mathbb{R}^d}(0,1))} = \varepsilon^{-\frac{d}{p^*}-1}\|(\nabla\varphi)\left(\frac{x}{\varepsilon}\right)\|_{L^{p}(D_{\mathbb{R}^d}(0,\varepsilon))} = \|\nabla\varphi\|_{L^{p}(D_{\mathbb{R}^d}(0,1))}.$$

On one hand we see that $\varphi_\varepsilon \overset{\varepsilon\to 0}{\rightharpoonup} 0$ weakly but it cannot obviously converge to 0 in norm in $L^{p^*}(D_{\mathbb{R}^d}(0,1))$.

3 Some spaces of functions on L^2 based Sobolev Spaces

We will introduce the *homogeneous* Sobolev spaces $\dot{H}^k(\mathbb{R}^d)$ and we will generalize the standard Sobolev spaces $H^k(\mathbb{R}^d)$. For $\xi \in \mathbb{R}^d$ let $\langle \xi \rangle = \sqrt{1+|\xi|^2}$ be the *Japanese bracket*. For a tempered distribution u we denote by $\widehat{u}$ its Fourier transform. We consider for $s \in \mathbb{R}$ the space formed by the tempered distributions u

$$H^s(\mathbb{R}^d) \text{ with norm } \|u\|_{H^s(\mathbb{R}^d)} := \|\langle \xi \rangle^s \widehat{u}\|_{L^2(\mathbb{R}^d)} < \infty \ . \tag{3.1}$$

We consider for $s \in \mathbb{R}$ the space formed by the tempered distributions u s.t. $\widehat{u} \in L^1_{loc}(\mathbb{R}^d)$

$$\dot{H}^s(\mathbb{R}^d) \text{ with norm } \|u\|_{\dot{H}^s(\mathbb{R}^d)} := \||\xi|^s \widehat{u}\|_{L^2(\mathbb{R}^d)} < \infty \ . \tag{3.2}$$

Exercise 3.1. Check that for $s \in \mathbb{N}$, the definition of $H^s(\mathbb{R}^d)$ in (3.1) and the definition in Sec. 2 are equivalent.

The following lemma is elementary.

Lemma 3.2. *The following statements are true.*

- $L^2(\mathbb{R}^d) \to H^s(\mathbb{R}^d)$ *defined by* $f \to \mathcal{F}^*\left(\frac{\widehat{f}}{\langle \xi \rangle^s}\right)$ *is an isometric isomorphism and all the* $H^s(\mathbb{R}^d)$ *are Hilbert spaces with inner product* $\langle f, g \rangle_{H^s} = \langle \langle \xi \rangle^s \widehat{f}, \langle \xi \rangle^s \widehat{g} \rangle_{L^2}$.
- *We have* $\mathcal{S}(\mathbb{R}^d) \subseteq \dot{H}^s(\mathbb{R}^d)$ *if and only if* $s > -d/2$.
- *The* $\dot{H}^s(\mathbb{R}^d)$ *have an inner product defined by* $\langle f, g \rangle_{\dot{H}^s} = \langle |\xi|^s \widehat{f}, |\xi|^s \widehat{g} \rangle_{L^2}$.

While the $\dot{H}^s(\mathbb{R}^d)$ have an inner product, in general they are not complete topological vector spaces and the following will be important to us.

Proposition 3.3. *For* $s < d/2$ *the space* $\dot{H}^s(\mathbb{R}^d)$ *is complete and the Fourier transform establishes an isometric isomorphism* $\mathcal{F} : \dot{H}^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$.

The above proposition is a consequence of the following lemma.

Lemma 3.4. *Let* $s < \frac{d}{2}$. *Then we have the following facts.*

- $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi) \subset L^1_{loc}(\mathbb{R}^d, d\xi)$
- $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi) \subset \mathcal{S}'(\mathbb{R}^d)$
- *The Fourier transform* $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ *is s.t.* $\mathcal{F}\left(\dot{H}^s(\mathbb{R}^d)\right) = L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ *and establishes an isometry between these two spaces.*

Proof. Let $g \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$. Obviously $g \in L^1_{loc}(\mathbb{R}^d \backslash \{0\}, d\xi)$. Let now $B = \{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. Then

$$\int_B |g(\xi)| d\xi \leq \left(\int_B |\xi|^{2s} |g(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_B |\xi|^{-2s} d\xi \right)^{\frac{1}{2}}$$
$$\leq \sqrt{\mathrm{vol}(S^{d-1})} (\int_0^1 r^{d-1-2s} dr)^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)} = \sqrt{\frac{\mathrm{vol}(S^{d-1})}{d-2s}} \|g\|_{L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)}.$$

Next, we check that $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi) \subset \mathcal{S}'(\mathbb{R}^d)$. We split $g = \chi_B g + \chi_{B^c} g$. Then $\chi_B g \in L^1(\mathbb{R}^d, d\xi)$ implies $\chi_B g \in \mathcal{S}'(\mathbb{R}^d)$. On the other hand we have $\chi_{B^c} g \in L^2(\mathbb{R}^d, \langle\xi\rangle^{2s} d\xi)$. This in turn implies $\chi_{B^c} g \in \mathcal{S}'(\mathbb{R}^d)$, where the embedding $L^2(\mathbb{R}^d, \langle\xi\rangle^{2\sigma} d\xi) \subset \mathcal{S}'(\mathbb{R}^d)$ for any $\sigma \in \mathbb{R}$ follows from

$$\int_{\mathbb{R}^d} f(\xi)\varphi(\xi) d\xi = \int_{\mathbb{R}^d} \langle\xi\rangle^{\sigma} f(\xi) \langle\xi\rangle^{-\sigma} \varphi(\xi) d\xi \leq \|f\|_{L^2(\mathbb{R}^d, \langle\xi\rangle^{2\sigma} d\xi)} (\int_{\mathbb{R}^d} \langle\xi\rangle^{-2\sigma} \varphi(\xi) d\xi)^{\frac{1}{2}}$$
$$\leq \|f\|_{L^2(\mathbb{R}^d, \langle\xi\rangle^{2\sigma} d\xi)} (\int_{\mathbb{R}^d} \langle\xi\rangle^{-2\sigma-m} d\xi)^{\frac{1}{2}} \|\langle\xi\rangle^m \varphi\|_{L^\infty(\mathbb{R}^d)}$$

for m chosen s.t. $2\sigma + m > d$. □

Remark 3.5. For $s \geq \frac{d}{2}$ the space $\dot{H}^s(\mathbb{R}^d)$ is not a complete space for the norm indicated. In particular, the Fourier trasform defines an embedding $\dot{H}^s(\mathbb{R}^d) \overset{\mathcal{F}}{\hookrightarrow} L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ with image which is strictly contained and dense in $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$. The fact that the image is dense can be seen observing that $C_c^\infty(\mathbb{R}^d \backslash \{0\})$ is dense in $L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ and we have $\mathcal{F}\dot{H}^s(\mathbb{R}^d) \supseteq C_c^\infty(\mathbb{R}^d \backslash \{0\})$.

For $s = \frac{d}{2} + \varepsilon_0$ with $\varepsilon_0 > 0$, if we pick $f \in C_c^\infty(\mathbb{R}^d)$ with $f(0) \neq 0$, then $\dfrac{f(\xi)}{|\xi|^{d+\frac{\varepsilon_0}{2}}}$ is a Borel function not contained in $L^1_{loc}(\mathbb{R}^d, d\xi)$. But $|\xi|^{2s} \left| \dfrac{f(\xi)}{|\xi|^{d+\frac{\varepsilon_0}{2}}} \right|^2 = \dfrac{|f(\xi)|^2}{|\xi|^{d-\varepsilon_0}} \in L^1(\mathbb{R}^d, d\xi)$ implies that $\dfrac{f(\xi)}{|\xi|^{d+\frac{\varepsilon_0}{2}}} \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$.

For $s = \frac{d}{2}$ consider $f(\xi) = \displaystyle\sum_{k=1}^{\infty} \frac{2^{kd}}{k} \chi_{[3/4,5/4]}(2^k|\xi|)$. Notice that for each ξ, at most one term of the sum is non zero, because $[2^{-k}3/4, 2^{-k}5/4] \cap [2^{-j}3/4, 2^{-j}5/4] = \emptyset$ for $j \neq k$. Indeed, if $j < k$ then

$$2^{-k}5/4 \leq 2^{-(j-1)}5/4 < 2^{-j}3/4 \text{ where the latter follows from } 5 < 6.$$

Then $|\xi|^{\frac{d}{2}} |f(\xi)| \in L^2(\mathbb{R}^d, d\xi)$ since

$$\int_{\mathbb{R}^d} |\xi|^d |f(\xi)|^2 d\xi = \sum_{k=1}^{\infty} \frac{1}{k^2} 2^{2kd} \int_{\mathbb{R}^d} |\xi|^d \chi_{[3/4,5/4]}(2^k|\xi|) d\xi = \sum_{k=1}^{\infty} \frac{1}{k^2} \int_{\mathbb{R}^d} |\xi|^d \chi_{[3/4,5/4]}(|\xi|) d\xi < \infty$$

but f, which is supported in the ball $B(0, 5/4)$, is not in $L^1(\mathbb{R}^d, d\xi)$ since otherwise we would have

$$\infty > \int_{\mathbb{R}^d} |f(\xi)| d\xi \geq \sum_{k=1}^{n} \frac{1}{k} 2^{kd} \int_{\mathbb{R}^d} \chi_{[3/4,5/4]}(2^k|\xi|) d\xi = \sum_{k=1}^{n} \frac{1}{k} \int_{\mathbb{R}^d} \chi_{[3/4,5/4]}(|\xi|) d\xi \overset{n\to\infty}{\to} \infty.$$

Later on we, when discussing the Navier Stokes Equation, we will deal with vector fields. Given a vector field $u = (u^j)_{j=1}^d \in (\mathcal{S}'(\mathbb{R}^d))^d$ its divergence is

$$\mathrm{div} u = \nabla \cdot u := \sum_{j=1}^{d} \frac{\partial}{\partial x_j} u^j .$$

Notice that $\widehat{\mathrm{div} u} = -\mathrm{i} \sum_{j=1}^d \xi^j \widehat{u}^j$ so that a u is divergence free, that is $\mathrm{div} u = 0$, if and only if $\sum_{j=1}^d \xi^j \widehat{u}^j = 0$.

We define now an operator $\mathbb{P}$ by

$$(\mathcal{F}(\mathbb{P}u))^j = \widehat{u}^j - \frac{1}{|\xi|^2} \sum_{k=1}^{d} \xi_j \xi_k \widehat{u}^k. \tag{3.3}$$

Lemma 3.6. *Let $s < \frac{d}{2}$. Formula (3.3) defines a bounded operator from $(\dot{H}^{s-1}(\mathbb{R}^d))^d)$ into itself.*
$\mathbb{P}$ is a projection with image Range($\mathbb{P}$) represented by the divergence free elements of $(\dot{H}^{s-1}(\mathbb{R}^d))^d)$.
It is the orthogonal projection.
We have $\ker \mathbb{P} = \nabla \dot{H}^s(\mathbb{R}^d)$.

Proof. First of all for $\mathbb{P}$ defined by (3.3) we have

$$\|\mathbb{P}u\|_{\dot{H}^{s-1}} = \sum_{j=1}^{d} \|(\mathbb{P}u)^j\|_{\dot{H}^{s-1}} = \sum_{j=1}^{d} \||\xi|^{s-1} \mathcal{F}(\mathbb{P}u)^j\|_{L^2} = \sum_{j=1}^{d} \||\xi|^{s-1} (\widehat{u}^j - \frac{1}{|\xi|^2} \sum_{k=1}^{d} \xi_j \xi_k \widehat{u}^k)\|_{L^2}$$

$$\leq \sum_{j=1}^{d} \||\xi|^{s-1} \widehat{u}^j\|_{L^2} + \sum_{j,k=1}^{d} \|\frac{\xi_j \xi_k}{|\xi|^2}\|_{L^\infty} \||\xi|^{s-1} \widehat{u}^k\|_{L^2} \leq \sum_{j=1}^{d} \|u^j\|_{\dot{H}^{s-1}} + \sum_{j,k=1}^{d} \|u^k\|_{\dot{H}^{s-1}} \leq (d+1) \|u\|_{\dot{H}^{s-1}}.$$

Hence this is a bounded linear operator from $(\dot{H}^{s-1}(\mathbb{R}^d))^d \to (\dot{H}^{s-1}(\mathbb{R}^d))^d$. In fact it is a projection (so $\|\mathbb{P}u\|_{\dot{H}^{s-1}} \leq \|u\|_{\dot{H}^{s-1}}$) as we will see in a moment. But first observe that

$$\mathcal{F}(\mathrm{div} \mathbb{P}u) = \mathrm{i} \sum_{j=1}^{d} \xi^j (\mathcal{F}(\mathbb{P}u))^j = \mathrm{i} \sum_{j=1}^{d} \xi^j \widehat{u}^j - \frac{\mathrm{i}}{|\xi|^2} \sum_{j=1}^{d} (\xi^j)^2 \sum_{k=1}^{d} \xi_k \widehat{u}^k = 0$$

which shows that the image of $\mathbb{P}$ is formed by divergence free vector fields. Notice also that if $\text{div}u = 0$, and hence $\sum_{j=1}^d \xi^j \widehat{u}^j = 0$, we have

$$(\mathcal{F}(\mathbb{P}u))^j = \widehat{u}^j - \frac{1}{|\xi|^2}\xi_j \underbrace{\sum_{k=1}^d \xi_k \widehat{u}^k}_{0} = \widehat{u}^j,$$

and so $\mathbb{P}u = u$.
Now we check that $\mathbb{P}^2 = \mathbb{P}$. We have

$$(\mathcal{F}(\mathbb{P}^2 u))^j = (\mathcal{F}(\mathbb{P}u))^j - \frac{\xi_j}{|\xi|^2} \underbrace{\sum_{k=1}^d \xi_k (\mathcal{F}(\mathbb{P}u))^k}_{0}$$

where we use the fact checked above that $\text{div}\mathbb{P}u = 0$.
All the above steps show that (3.3) defines a projection in $(\dot{H}^{s-1}(\mathbb{R}^d))^d$ whose image is formed by the divergence free operators in $(\dot{H}^{s-1}(\mathbb{R}^d))^d$.
Pick now $V \in \dot{H}^s$. Then $\nabla V \in (\dot{H}^{s-1}(\mathbb{R}^d))^d$ and we have

$$(\mathcal{F}(\mathbb{P}\nabla V))^j = -\mathrm{i}\left(\xi_j - \sum_{k=1}^d \frac{\xi_j \xi_k^2}{|\xi|^2}\right)\widehat{V}(\xi) = 0.$$

Hence $\ker \mathbb{P} \supseteq \nabla \dot{H}^s(\mathbb{R}^d)$. We now show $\ker \mathbb{P} \subseteq \nabla \dot{H}^s(\mathbb{R}^d)$.
If $\mathbb{P}u = 0$ then

$$\widehat{u}^j = -\mathrm{i}\xi_j \widehat{V}(\xi) \text{ where } \widehat{V}(\xi) := \frac{\mathrm{i}}{|\xi|^2}\sum_{k=1}^d \xi_k \widehat{u}^k$$

It is easy to see that $\widehat{V} \in L^2(\mathbb{R}^d, |\xi|^{2s} d\xi)$ and in view of the identification of this space with $\dot{H}^s(\mathbb{R}^d)$ through the Fourier transform when $s < \frac{d}{2}$ we conclude that $V \in \dot{H}^s(\mathbb{R}^d)$ with $\nabla V = u$.

□

For $u \in \dot{H}^k(\mathbb{R}^d)$ and $\lambda > 0$ let us set $\mathbf{P}_\lambda u := \mathcal{F}^*(\chi_{|\xi|\le\lambda}\mathcal{F}u)$. Notice that this map sends $L^2(\mathbb{R}^d)$ into itself since

$$\|\mathbf{P}_\lambda u\|_{\dot{H}^k(\mathbb{R}^d)} = \||\xi|^k \chi_{|\xi|\le\lambda}\mathcal{F}u\|_{L^2(\mathbb{R}^d)} \le \||\xi|^k \mathcal{F}u\|_{L^2(\mathbb{R}^d)} = \|u\|_{\dot{H}^k(\mathbb{R}^d)}.$$

Notice that $\mathbf{P}_\lambda$ is a projection, that is $\mathbf{P}_\lambda^2 = \mathbf{P}_\lambda$, by

$$\mathbf{P}_\lambda^2 u = \mathbf{P}_\lambda \circ \mathbf{P}_\lambda u = \mathcal{F}^*(\chi_{|\xi|\le\lambda}\mathcal{F}\mathbf{P}_\lambda u) = \mathcal{F}^*(\chi^2_{|\xi|\le\lambda}\mathcal{F}u) = \mathcal{F}^*(\chi_{|\xi|\le\lambda}\mathcal{F}u) = \mathbf{P}_\lambda u.$$

If $\text{div}u = 0$ then also $\text{div}\mathbf{P}_\lambda u = 0$. Indeed

$$(\text{div}u = 0 \Leftrightarrow \sum_{j=1}^d \xi^j\ \widehat{u}^j = 0) \Rightarrow \mathcal{F}(\text{div}\mathbf{P}_\lambda u) = \sum_{j=1}^d \xi^j \chi_{|\xi|\le\lambda}\widehat{u}^j = \chi_{|\xi|\le\lambda}\sum_{j=1}^d \xi^j\ \widehat{u}^j = 0,$$

which in turn implies $\text{div}\mathbf{P}_\lambda u = 0$.

4 L^p based Sobolev Spaces

The following spaces, for $p \in (1,\infty)$ are formed by tempered distributions u s.t. $\widehat{u}$ is in $L^1_{loc}(\mathbb{R}^d)$ for $s \in \mathbb{R}$:

$$\dot{\mathcal{W}}^{s,p}(\mathbb{R}^d) \text{ defined with } \|u\|_{\dot{\mathcal{W}}^{s,p}(\mathbb{R}^d)} := \|(|\xi|^s \widehat{u})^\vee\|_{L^p(\mathbb{R}^d)} \ ; \tag{4.1}$$

$$\mathcal{W}^{s,p}(\mathbb{R}^d) \text{ defined with } \|u\|_{\mathcal{W}^{s,p}(\mathbb{R}^d)} := \|(\langle\xi\rangle^s \widehat{u})^\vee\|_{L^p(\mathbb{R}^d)} \ . \tag{4.2}$$

We will not use the above spaces except for $p = 2$. The following is true.

Theorem 4.1. *We have*

$$W^{k,p}(\mathbb{R}^d) = \mathcal{W}^{k,p}(\mathbb{R}^d) \text{ for all } p \in (1,\infty) \text{ and all } k \in \mathbb{N}. \tag{4.3}$$

Proof. Maybe, later. For this we need the theory of Calderon and Zygmund operators. □

For $p = 1$ and $p = \infty$ (4.3) is not true, see [13].

To generalize the Sobolev Embedding Theorem 2.2, we need information on the Hardy Littlewood maximal function.

4.1 Hardy Littlewood maximal function

Let $f \in L^1_{loc}(\mathbb{R}^d)$ and consider (for $B(x,r)$ the ball of center x and radius r in $\mathbb{R}^d$) averages

$$A_r f(x) = \frac{1}{\text{vol}(B(x,r))} \int_{B(x,r)} f(y)dy.$$

Notice that for any $r > 0$ the function $x \to A_r f(x)$ is continuous. Indeed, fix $\delta_0 > 0$ and consider $\delta x \in B(0,\delta_0)$. Then by the triangular inequality $B(x+\delta x, r) \subset B(x, r+\delta_0)$. So, for $\delta x \in B(0,\delta_0)$

$$A_r f(x) - A_r f(x+\delta x) = \frac{1}{\text{vol}(B(0,1)) r^d} \int_{B(x,r+\delta_0)} \left(\chi_{B(x,r)\backslash B(x+\delta x,r)}(y) - \chi_{B(x+\delta x,r)\backslash B(x,r)}(y)\right) f(y)dy$$

with for any y

$$\left(\chi_{B(x,r)\backslash B(x+\delta x,r)}(y) - \chi_{B(x+\delta x,r)\backslash B(x,r)}(y)\right) \chi_{B(x,r+\delta_0)}(y) f(y) \overset{|\delta x| \to 0}{\to} 0.$$

By dominated convergence $A_r f(x) - A_r f(x+\delta x) \to 0$. We define

$$Mf(x) = \sup_{r>0} A_r |f|(x). \tag{4.4}$$

From the definition we conclude that Mf is lower semi continuous that is $\{x : Mf(x) > a\}$ is open for any a. It also obvious that M is sub additive:

$$M(f+g)(x) \le Mf(x) + Mg(x).$$

We have the following obvious estimate

$$|Mf(x)| \leq |f|_{L^\infty(\mathbb{R}^d)}. \tag{4.5}$$

One important fact is that it is not true that M maps $L^1(\mathbb{R}^d)$ into itself. Indeed if say $K \subset \mathbb{R}^d$ is any compact set and if $B(0, c_0) \supset K$, then since for $|x| > c_0$ we have $B(x, 2|x|) \supset B(0, |x|) \supset K$, we have computing at $r = 2|x|$

$$M\chi_K(x) = \sup_{r>0} \frac{\text{vol}(B(x,r) \cap K)}{\text{vol}(B(0,1))r^d} \geq \frac{\text{vol}(K)}{\text{vol}(B(0,1))2^d|x|^d}$$

which shows that $M\chi_K \notin L^1(\mathbb{R}^d)$.
Notice that each $g \in L^1(\mathbb{R}^d)$ satisfies Chebyshev's inequality:

$$\text{vol}(\{x : |g(x)| > \alpha\}) \leq \frac{|g|_{L^1(\mathbb{R}^d)}}{\alpha} \text{ for any } \alpha > 0 \tag{4.6}$$

Indeed (4.6) follows immediately from.

$$|g|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |g(y)|dy \geq \int_{\{x:|g(x)|>\alpha\}} |g(y)|dy \geq \int_{\{x:|g(x)|>\alpha\}} \alpha dy = \alpha \text{vol}(\{x : |g(x)| > \alpha\})$$

If $T : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ satisfies $\|Tf\|_{L^1(\mathbb{R}^d)} \leq A\|f\|_{L^1(\mathbb{R}^d)}$ for all $f \in L^1(\mathbb{R}^d)$ and for a fixed constant A, from (4.6) it is easy to conclude that

$$\text{vol}(\{x : |Tf(x)| > \alpha\}) \leq \frac{A}{\alpha}|f|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0 \text{ and any } f \in L^1(\mathbb{R}^d).$$

Unfortunately we have seen that M does not map $L^1(\mathbb{R}^d)$ into itself. However we will show that it satisfies the last property. Indeed we will prove now that M *is weak* $(1,1)$ *bounded*, that is there exists a constant $A > 0$ (in fact we will prove $A = 3^d$) s.t.

$$\text{vol}(\{x : Mf(x) > \alpha\}) \leq \frac{A}{\alpha}|f|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0\ . \tag{4.7}$$

To prove this we consider the set $\{x : Mf(x) > \alpha\}$. Then, for any x in this set, there is a ball with center in x, which we denote by B_x, with $\int_{B_x} |f| > \alpha \text{vol}(B_x)$. Pick any compact subset K of the above set, and cover it with such balls B_x. Extract now a finite cover, corresponding to finitely many points x_1, ...x_N. We have the following covering result, which we state without proof.

Theorem 4.2 (Vitali's lemma)**.** *Let B_{x_1},...,B_{x_N} be a finite number of balls in $\mathbb{R}^d$. There exists a subset of balls*

$$\{B_1, ..., B_m\} \subseteq \{B_{x_1}, ..., B_{x_N}\} \tag{4.8}$$

with the B_1...B_m pairwise disjoint, s.t.

$$vol(B_{x_1} \cup \cdots \cup B_{x_N}) \leq 3^d \sum_{j=1}^{m} vol(B_j). \tag{4.9}$$

We consider balls $B_1 ... B_m$ as in (4.8) and from

$$K \subset B_{x_1} \cup \cdots \cup B_{x_N} \Rightarrow \text{vol}(K) < \text{vol}(B_{x_1} \cup \cdots \cup B_{x_N}),$$

from (4.9) and from the definition of the B_{x_j} we get

$$3^{-d}\text{vol}(K) \leq \sum_{j=1}^{m} \text{vol}(B_j) < \sum_{j=1}^{m} \frac{1}{\alpha} \int_{B_j} |f| \leq \frac{|f|_1}{\alpha}. \tag{4.10}$$

(4.10) implies $\text{vol}(K) \leq 3^d \alpha^{-1} |f|_1$. By $\text{vol}(\{x : |Mf(x)| > \alpha\}) = \sup_{K \subset \{x : |Mf(x)| > \alpha\}} \text{vol}(K)$ for compact sets K, then (4.10) implies (4.7).

(4.5) and (4.7) imply by the Marcinkiewicz Interpolation Theorem 4.3, proved below,

$$\|Mf\|_{L^p(\mathbb{R}^d)} < A_p \|f\|_{L^p(\mathbb{R}^d)} \text{ for all } p \in (1, \infty] \ . \tag{4.11}$$

We will use this result in the proof of the Hardy-Littlewood-Sobolev Theorem, and of Sobolev's estimates.

Before introducing the Marcinkiewicz interpolation Theorem, we recall that for a measurable function $g : \mathbb{R}^d \to \mathbb{R}$ the distribution function is

$$\lambda(\alpha) := \text{vol}(\{x \in \mathbb{R}^d : |g(x)| > \alpha\}).$$

Notice that $\lambda : [0, \infty) \to [0, \infty]$ is decreasing. This implies that it is measurable.
For a function $g \in L^p(\mathbb{R}^d)$ with $1 \leq p < \infty$ we have

$$\begin{aligned} &\int_{\mathbb{R}^d} |g(x)|^p dx = \int_{\mathbb{R}^d} dx \int_0^{|g(x)|} p\alpha^{p-1} d\alpha = \int_0^\infty d\alpha p \alpha^{p-1} \int_{\{x \in \mathbb{R}^d : |g(x)| > \alpha\}} dx \\ &= \int_0^\infty p\alpha^{p-1} \lambda(\alpha) d\alpha \end{aligned} \tag{4.12}$$

where the 1st equality is elementary, the last follows immediately by the definition of $\lambda(\alpha)$, and the 2nd follows from Tonelli's Theorem applied to the positive measurable function $F(x, \alpha) := |\alpha|^{p-1} \chi_{\mathbb{R}_+}(|g(x)| - \alpha) \chi_{\mathbb{R}_+}(\alpha)$.

Theorem 4.3 (Marcinkiewicz Interpolation)**.** *Let $T : L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d) \to L^1_{loc}(\mathbb{R}^d)$ be a sublinear operator s.t. for two constants A_1 and A_∞ and for all f*

$$\|Tf\|_{L^\infty(\mathbb{R}^d)} \leq A_\infty \|f\|_{L^\infty(\mathbb{R}^d)} \tag{4.13}$$

$$|\{x : |Tf(x)| > \alpha\}| \leq \frac{A_1}{\alpha} |f|_{L^1(\mathbb{R}^d)} \textit{ for any } \alpha > 0 \ . \tag{4.14}$$

Then for any $p \in (1, \infty)$ there is a constant A_p such that for any $f \in L^p(\mathbb{R}^d)$ we have

$$\|Tf\|_{L^p(\mathbb{R}^d)} \leq A_p \|f\|_{L^p(\mathbb{R}^d)}. \tag{4.15}$$

Proof. Dividing T by a constant, we can assume $A_\infty = 1$. Fix $p \in (1, \infty)$ and $f \in L^p(\mathbb{R}^d)$. For $\alpha > 0$ arbitrary set

$$f_1(x) = \begin{cases} f(x) \text{ if } |f(x)| \geq \frac{\alpha}{2} \\ 0 \text{ otherwise.} \end{cases}$$

Notice that $f_1 \in L^1(\mathbb{R}^d)$ by

$$\int_{\mathbb{R}^d} |f_1(x)| dx = \int_{\{x:|f(x)|\geq \frac{\alpha}{2}\}} |f(x)| dx \leq \frac{2^{p-1}}{\alpha^{p-1}} \int_{\mathbb{R}^d} |f(x)|^p dx.$$

Using (4.13), we get $|Tf(x)| \leq |Tf_1(x)| + \frac{\alpha}{2}$, since $\|f - f_1\|_{L^\infty(\mathbb{R}^d)} \leq \frac{\alpha}{2}$. Then

$$\{x : |Tf(x)| > \alpha\} \subseteq \{x : |Tf_1(x)| > \frac{\alpha}{2}\}.$$

We have, using (4.14),

$$\text{vol}(\{x : |Tf_1(x)| > \frac{\alpha}{2}\}) \leq A_1 \frac{2}{\alpha} \int_{\mathbb{R}^d} |f_1(x)| dx = A_1 \frac{2}{\alpha} \int_{\{x:|f(x)|\geq \frac{\alpha}{2}\}} |f(x)| dx.$$

Substituting $g = Tf$ in (4.12)

$$\begin{aligned}
\int_{\mathbb{R}^d} |Tf(x)|^p dx &= \int_0^\infty p\alpha^{p-1} \text{vol}(\{x : |Tf(x)| > \alpha\}) d\alpha \\
&\leq \int_0^\infty p\alpha^{p-1} \text{vol}(\{x : |Tf_1(x)| > \frac{\alpha}{2}\}) d\alpha \leq 2A_1 \int_0^\infty p\alpha^{p-2} \int_{\{x:|f(x)|\geq \frac{\alpha}{2}\}} |f(x)| dx \\
&= 2pA_1 \int_{\mathbb{R}^d} dx |f(x)| \underbrace{\int_0^{2|f(x)|} \alpha^{p-2} d\alpha}_{\frac{2^{p-1}|f(x)|^{p-1}}{p-1}} = \frac{2^p p}{p-1} A_1 \int_{\mathbb{R}^d} |f(x)|^p dx.
\end{aligned}$$

□

4.2 Back to Sobolev Embedding

We will use the properties of the Hardy Littlewood Maximal function, and specifically the definition and (4.11), to prove the following important theorem.

Theorem 4.4 (Hardy-Littlewood-Sobolev inequality). *For any*

$$\gamma \in (0, d) \text{ and } 1 < p < q < \infty \text{ with } \frac{1}{p} = \frac{1}{q} + \frac{d - \gamma}{d} \tag{4.16}$$

there exists a constant C s.t.

$$\| \int_{\mathbb{R}^d} f(x - y)|y|^{-\gamma} dy \|_{L^q(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}. \tag{4.17}$$

Proof. For an $R > 0$ to be chosen momentarily, we split

$$\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy = \int_{|y|<R} f(x-y)|y|^{-\gamma}dy + \int_{|y|>R} f(x-y)|y|^{-\gamma}dy.$$

We claim that

$$|\int_{|y|<R} f(x-y)|y|^{-\gamma}dy| \le Mf(x)\int_{|y|<R} |y|^{-\gamma}dy = cR^{d-\gamma}Mf(x). \tag{4.18}$$

We assume for a moment this claim and complete the rest of the proof. By Hölder we have

$$|\int_{|y|>R} f(x-y)|y|^{-\gamma}dy| \le \|f\|_{L^p(\mathbb{R}^d)} \||y|^{-\gamma}\chi_{\{y:|y|>R\}}\|_{L^{p'}(\mathbb{R}^d)}.$$

We have $|y|^{-\gamma}\chi_{\{y:|y|>R\}} \in L^{p'}(\mathbb{R}^d)$ exactly if $\gamma p' > d$. The latter inequality is true because

$$\frac{1}{p'} - \frac{\gamma}{d} = -\frac{1}{q} < 0 \Rightarrow \gamma p' - d = \frac{dp'}{q} > 0.$$

In this case

$$\||y|^{-\gamma}\chi_{\{y:|y|>R\}}\|_{L^{p'}(\mathbb{R}^d)} = \left(\mathrm{vol}(\mathbb{S}^{d-1})\int_{r>R} r^{-\gamma p'+d-1}dr\right)^{\frac{1}{p'}} = cR^{\frac{d}{p'}-\gamma} = cR^{-\frac{d}{q}}.$$

Hence

$$|\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy| \lesssim R^{d-\gamma}Mf(x) + \|f\|_{L^p(\mathbb{R}^d)}R^{-\frac{d}{q}}.$$

Now we choose R so that the two terms on the r.h.s. are equal:

$$\frac{Mf(x)}{\|f\|_{L^p}} = R^{\gamma-d-\frac{d}{q}} = R^{-\frac{d}{p}}.$$

Then we get

$$|\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy| \lesssim R^{d-\gamma}Mf(x) + \|f\|_{L^p(\mathbb{R}^d)}R^{-\frac{d}{q}} = 2\|f\|_{L^p(\mathbb{R}^d)}\left(\frac{Mf(x)}{\|f\|_{L^p}}\right)^{\frac{d}{q}\cdot\frac{p}{d}}$$
$$= 2\left(Mf(x)\right)^{\frac{p}{q}}\|f\|_{L^p}^{1-\frac{p}{q}}.$$

Then

$$\|\int_{\mathbb{R}^d} f(x-y)|y|^{-\gamma}dy\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^p}^{1-\frac{p}{q}}\|(Mf)^{\frac{p}{q}}\|_{L^q} = \|f\|_{L^p}^{1-\frac{p}{q}}\|(Mf)\|_{L^p}^{\frac{p}{q}} \lesssim \|f\|_{L^p}.$$

To complete the proof we need the inequality in (4.18). More generally, we prove that if $\Phi \in L^1(\mathbb{R}^d)$ is radial, positive and decreasing, then

$$|\int_{\mathbb{R}^d} f(x-y)\Phi(y)dy| \le Mf(x)\int_{\mathbb{R}^d}\Phi(y)dy. \tag{4.19}$$

Then (4.18) is just (4.19) for $\Phi(y) = |y|^{-\gamma}\chi_{\{y:|y|<R\}}$.
Notice that (4.19) is true for radial functions of the form

$$\Phi = \sum_j a_j \chi_{B_j}$$

for $a_j > 0$, B_j a ball of center 0. Indeed

$$\sum_j a_j \int_{B_j} |f(x-y)|dy = \sum_j a_j \frac{\text{vol}(B_j)}{\text{vol}(B_j)} \int_{B_j} |f(x-y)|dy \leq \sum_j a_j \text{vol}(B_j) Mf(x) = Mf(x) \int \Phi dy.$$

In the general case the result follows from the fact that Φ can be approximated by these functions.

□

For the above proof see [14] p.354, while for the next one see [13] p.73.

Lemma 4.5. *For any $\gamma \in (0,d)$ there exists $c_\gamma > 0$ s.t.*

$$\mathcal{F}(|\cdot|^{-\gamma})(\xi) = c_\gamma |\xi|^{\gamma-d}. \tag{4.20}$$

Proof. It is enough to show that for any $\phi \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |x|^{-\gamma}\phi(x)dx = c_\gamma \int_{\mathbb{R}^d} |\xi|^{\gamma-d}\widehat{\phi}(\xi)d\xi. \tag{4.21}$$

Starting from (1.2) and Plancherel we have

$$\int_{\mathbb{R}^n} \varepsilon^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \phi(x)dx = \int_{\mathbb{R}^d} e^{-\varepsilon\frac{|\xi|^2}{2}} \widehat{\phi}(\xi)d\xi.$$

Now we apply to both sides $\int_0^\infty \frac{d\epsilon}{\epsilon}\epsilon^{\frac{d-\gamma}{2}}$ and commuting order of integration we obtain

$$\int_{\mathbb{R}^d} dx\phi(x) \underbrace{\int_0^\infty \varepsilon^{-\frac{\gamma}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \frac{d\epsilon}{\epsilon}}_{a_\gamma|x|^{-\gamma}} = \int_{\mathbb{R}^d} d\xi\widehat{\phi}(\xi) \underbrace{\int_0^\infty \epsilon^{\frac{d-\gamma}{2}} e^{-\varepsilon\frac{|\xi|^2}{2}} \frac{d\epsilon}{\epsilon}}_{b_\gamma|\xi|^{\gamma-d}}$$

for appropriate constants a_γ and b_γ.

□

Theorem 4.6 (Sobolev Embedding Theorem with fractional derivatives)**.** *Let $p \in (1,\infty)$, $0 < s < \frac{d}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$. Then there exists a C s.t. we have*

$$\|f\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{\dot{\mathcal{W}}^{s,p}(\mathbb{R}^d)} \text{ for any } f \in \mathcal{S}(\mathbb{R}^d). \tag{4.22}$$

Proof. For $f \in \mathcal{S}(\mathbb{R}^d)$ we have for some fixed c

$$f(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi\cdot x} |\xi|^{-s} \left(|\xi|^s \widehat{f}(\xi)\right) d\xi = c \int_{\mathbb{R}^d} |x-y|^{s-d} g(y) dy \text{ where } \widehat{g}(\xi) = |\xi|^s \widehat{f}(\xi)$$

where we used $\widehat{\varphi * T} = (2\pi)^{\frac{d}{2}} \widehat{\varphi}\, \widehat{T}$ which holds for $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $T \in \mathcal{S}'(\mathbb{R}^d)$.
Since $g \in L^p(\mathbb{R}^d)$, by the Hardy-Littlewood-Sobolev Theorem we have that $f \in L^q(\mathbb{R}^d)$ for

$$\frac{1}{q} = \frac{1}{p} - \frac{d-(d-s)}{d} = \frac{1}{p} - \frac{s}{d}$$

□

Notice that for $0 < s < \frac{d}{2}$ we know that $\dot{H}^s(\mathbb{R}^d)$ contains $\mathcal{S}(\mathbb{R}^d)$ as a dense subspace, so (4.22) with $p = 2$ extends to all $f \in \dot{H}^s(\mathbb{R}^d)$.

5 Assorted inequalities

Lemma 5.1 (Interpolation of Sobolev norms)**.** *For any $s \in [0,1]$ and any $k = sk_1 + (1-s)k_2$ we have*

$$\|f\|_{\dot{H}^k(\mathbb{R}^d)} \le \|f\|^s_{\dot{H}^{k_1}(\mathbb{R}^d)} \|f\|^{1-s}_{\dot{H}^{k_2}(\mathbb{R}^d)} \text{ for any } f \in \dot{H}^{k_1}(\mathbb{R}^d) \cap \dot{H}^{k_2}(\mathbb{R}^d). \tag{5.1}$$

In particular, for $s \in [0,1]$ and any $f \in H^1(\mathbb{R}^d)$

$$\|f\|_{\dot{H}^s(\mathbb{R}^d)} \le \|f\|^{1-s}_{L^2(\mathbb{R}^d)} \|f\|^s_{\dot{H}^1(\mathbb{R}^d)} \tag{5.2}$$

Proof. (5.2) follows from (5.1) for $k_1 = 1$ and $k_2 = 0$. So let us turn to (5.1).
Obviously there is nothing to prove for $s = 0, 1$, so we can assume $s \in (0,1)$. Notice that for $p = \frac{1}{s}$ we have $p' := \frac{p}{p-1} = \frac{1}{1-s}$. Now, we have

$$\begin{aligned}
\|f\|^2_{\dot{H}^k(\mathbb{R}^d)} &= \int \left(|\xi|^{2sk_1} |\widehat{f}(\xi)|^{2s}\right) \left(|\xi|^{2(1-s)k_2} |\widehat{f}(\xi)|^{2(1-s)}\right) d\xi \\
&\le \| |\xi|^{2sk_1} |\widehat{f}(\xi)|^{2s} \|_{L^{\frac{1}{s}}(\mathbb{R}^d)} \| |\xi|^{2(1-s)k_2} |\widehat{f}(\xi)|^{2(1-s)} \|_{L^{\frac{1}{1-s}}(\mathbb{R}^d)} \\
&= \| |\xi|^{k_1} |\widehat{f}(\xi)| \|^{2s}_{L^2(\mathbb{R}^d)} \| |\xi|^{k_1} \widehat{f}(\xi) \|^{2(1-s)}_{L^2(\mathbb{R}^d)} = \|f\|^{2s}_{\dot{H}^{k_1}(\mathbb{R}^d)} \|f\|^{2(1-s)}_{\dot{H}^{k_2}(\mathbb{R}^d)}.
\end{aligned}$$

□

Theorem 5.2 (Gagliardo–Nirenberg)**.** *If $p \in [2,\infty)$ is s.t. $\frac{1}{p} > \frac{1}{2} - \frac{1}{d}$ then there exists C s.t.*

$$\|f\|_{L^p(\mathbb{R}^d)} \le C \|f\|^{1-s}_{L^2(\mathbb{R}^d)} \|f\|^s_{\dot{H}^1(\mathbb{R}^d)} \text{ where } s = d\left(\frac{1}{2} - \frac{1}{p}\right). \tag{5.3}$$

Proof. By Sobolev, for $\frac{1}{p} = \frac{1}{2} - \frac{s}{d}$ we have

$$\|f\|_{L^p(\mathbb{R}^d)} \le C\|f\|_{\dot{H}^s(\mathbb{R}^d)}.$$

Here s is like in the statement. Also $s = d\left(\frac{1}{2} - \frac{1}{p}\right) < 1 \Leftrightarrow \frac{1}{2} - \frac{1}{p} < \frac{1}{d}$. Finally, apply (5.2). $\square$

Remark 5.3. For $p = 4$ and $d = 2, 3$ we have $s = d/4$ and $\|f\|_{L^4(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}^{1-d/4}\|f\|_{\dot{H}^1(\mathbb{R}^d)}^{d/4}$.

Lemma 5.4 (Gronwall's inequality)**.** *Let $T > 0$, λ and φ two functions in $L^1(0,T)$, both ≥ 0 a.e., and C_1, C_2 two non negative constants. Let $\lambda\varphi \in L^1(0,T)$ and let*

$$\varphi(t) \le C_1 + C_2\int_0^t \lambda(s)\ \varphi(s)ds \text{ for a.e. } t \in (0,T).$$

Then we have

$$\varphi(t) \le C_1 e^{C_2\int_0^t \lambda(s)ds} \text{ for a.e. } t \in (0,T).$$

Proof. Set

$$\psi(t) := C_1 + C_2\int_0^t \lambda(s)\ \varphi(s)ds.$$

Then $\psi(t)$ is absolutely continuous and so it is differentiable almost everywhere and we have

$$\psi'(t) = C_2\lambda(t)\ \varphi(t) \le C_2\lambda(t)\ \psi(t) \text{ for a.e. } t \in (0,T).$$

Also, the function $\psi(t)e^{-C_2\int_0^t \lambda(s)ds}$ is absolutely continuous with

$$\frac{d}{dt}\left(\psi(t)e^{-C_2\int_0^t \lambda(s)ds}\right) \le 0 \text{ for a.e. } t \in (0,T).$$

Then we have

$$\psi(t) \le e^{C_2\int_0^t \lambda(s)ds}\psi(0) = C_1 e^{C_2\int_0^t \lambda(s)ds} \text{ for all } t \in (0,T).$$

Since $\varphi(t) \le \psi(t)$ a.e., the result follows. $\square$

6 The Calderon–Zygmund theory

We consider Calderon–Zygmund (CZ) kernels. We will use the following definition.

Definition 6.1. In these notes, we well say that a function $K : \mathbb{R}^d \times \mathbb{R}^d \backslash \Delta \to \mathbb{C}$ with Δ the diagonal $\{(x,x) : x \in \mathbb{R}^d\}$, is CZ if there exists a fixed constant C s.t. the following conditions hold:

(C–Z1) we have

$$\begin{aligned} |K(x,y)| &\leq \frac{C}{|x-y|^d} \text{ for any } x \neq y \text{ and} \\ |\nabla_{x,y} K(x,y)| &\leq \frac{C}{|x-y|^{d+1}} \text{ for any } x \neq y. \end{aligned} \tag{6.1}$$

(C–Z2) the operator

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy \text{ for } x \notin \text{supp } f \tag{6.2}$$

extends into a bounded operator $T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ with norm bounded by C.

There are many examples.

(1) Let us consider the operator $R_j = \frac{\partial_j}{\sqrt{-\triangle}}$ which is a well defined bounded operator in $L^2(\mathbb{R}^d)$ since

$$\widehat{R_j f}(\xi) = -\mathrm{i} \frac{\xi_j}{|\xi|} \widehat{f}(\xi).$$

Notice that for $K = \mathcal{F}^* \left(-\mathrm{i} \frac{\xi_j}{|\xi|} \right)$, we have $R_j f(x) = (2\pi)^{-\frac{d}{2}} K * f(x)$ where for $\varphi \in C_c^\infty(\mathbb{R}^d, [0,1])$ any function with $\varphi = 1$ in $B(0,a)$ and $\varphi = 0$ outside $B(0,b)$, for some $0 < a < b$, we have

$$K(x) = -\mathrm{i} \lim_{R \to +\infty} (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} \frac{\xi_j}{|\xi|} \varphi(\xi/R) d\xi.$$

It is easy to see that for any $x \neq 0$ the above limit converges and that $K(x-y)$ satisfies the inequalities (6.1) for a fixed C. For example, the 1st inequality follows splitting

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} \frac{\xi_j}{|\xi|} \varphi(\xi|x|) \varphi(\xi/R) d\xi + \int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} \frac{\xi_j}{|\xi|} \varphi(\xi/R)(1 - \varphi(\xi|x|)) d\xi$$

where we bound the absolute value of the 1st integral by

$$\int_{|\xi| \leq \frac{b}{|x|}} d\xi = \frac{b^d \mathrm{vol}(S^{d-1})}{d} \frac{1}{|x|^d}$$

and the absolute value of the 2nd integral by means of an integration by parts using $Le^{\mathrm{i}\xi \cdot x} = e^{\mathrm{i}\xi \cdot x}$ with $L = \frac{x \cdot \nabla_\xi}{\mathrm{i}|x|^2} e^{\mathrm{i}\xi \cdot x}$, and writing it as

$$\int_{\mathbb{R}^d} e^{\mathrm{i}\xi \cdot x} (L^*)^N \left[\frac{\xi_j}{|\xi|} \varphi(\xi/R)(1 - \varphi(\xi|x|)) \right] d\xi.$$

It is now easy to see that

$$\left|(L^*)^N\left[\frac{\xi_j}{|\xi|}\varphi(\xi/R)(1-\varphi(\xi|x|))\right]\right| \le C_N\frac{1}{|x|^N|\xi|^N}.$$

Hence the absolute of the 2nd integral is bounded by

$$C_N\frac{1}{|x|^N}\int_{|\xi|\ge\frac{a}{|x|}}\frac{1}{|\xi|^N}d\xi \le C_N C_d\frac{|x|^{N-d}}{a^{N-d}|x|^N} = \frac{C_N C_d}{a^{N-d}}\frac{1}{|x|^d}.$$

The 2nd inequality in (6.1) can be obtained noticing that

$$\partial_k K(x) = -\mathrm{i}\lim_{R\to+\infty}(2\pi)^{-\frac{d}{2}}\int_{\mathbb{R}^d}e^{\mathrm{i}\xi\cdot x}\xi_k\frac{\xi_j}{|\xi|}\varphi(\xi/R)d\xi.$$

When one considers the above inequalities with an additional factor ξ_k inside the integral, one gets the upper bound of the 2nd inequality in (6.1).

The operators R_j are called Riesz transforms.

(2) The above discussion works out similarly with operators $\frac{\partial_j}{\sqrt{1-\triangle}}$ and $\frac{\partial^\alpha}{(1-\triangle)^{\frac{k}{2}}}$ with α any multi–index with $|\alpha|\le k$.

(3) Let us consider in $\mathbb{R}$ the Hilbert transform

$$Hf(x) := -\frac{1}{\pi}\lim_{\varepsilon\to0^+}\int_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy = -\frac{1}{\pi}(P.V\frac{1}{x})*f \tag{6.3}$$

with $P.V\frac{1}{x}$ the tempered distribution that acts on a $\phi\in\mathcal{S}(\mathbb{R})$ as $\lim_{\varepsilon\to0^+}\int_{|x|>\varepsilon}\frac{\phi(x)}{x}dx$.

Notice that using the Residue theorem we have

$$\lim_{\varepsilon\to0^+}\int_{|x|>\varepsilon}e^{-\mathrm{i}\xi x}\frac{dx}{x} = -\mathrm{i}\pi\mathrm{sign}(\xi)$$

so that

$$\frac{1}{\pi}\mathcal{F}(P.V.\frac{1}{x}) = -\mathrm{i}(2\pi)^{-\frac{1}{2}}\mathrm{sign}(\xi).$$

Then

$$\mathcal{F}(Hf)(\xi) = -\mathrm{i}\mathrm{sign}(\xi)\widehat{f}(\xi).$$

which implies that (C–Z2) is true. Since (C–Z1) is obvious, we conclude that the Hilbert transform meets the conditions of Definition 6.1.

Remark 6.2. Consider the operator $T_{\mathbb{R}_+} f := \mathcal{F}^* \left[\chi_{\mathbb{R}_+} \widehat{f} \right]$. Then $\chi_{\mathbb{R}_+} = 2^{-1}\mathrm{i}(-\mathrm{isign} - \mathrm{i})$ implies $T_{\mathbb{R}_+} = 2^{-1}(I + \mathrm{i}H)$. Analogously $T_{\mathbb{R}_-} = 2^{-1}(I - \mathrm{i}H)$. Next,

$$T_{(a,+\infty)} = 2^{-1}(I + \mathrm{i}e^{\mathrm{i}ax} H e^{-\mathrm{i}ax}) \text{ and } T_{(-\infty,b)} = 2^{-1}(I - \mathrm{i}e^{\mathrm{i}bx} H e^{-\mathrm{i}bx}).$$

Finally

$$T_{(a,b)} = 2^{-1}(T_{(a,+\infty)} - T_{(b,+\infty)}) = 4^{-1}\mathrm{i}(e^{\mathrm{i}ax} H e^{-\mathrm{i}ax} - e^{\mathrm{i}bx} H e^{-\mathrm{i}bx}).$$

Next, if in $\mathbb{R}^d$ we consider the half–plane $x_1 > 0$, then

$$\mathcal{F}^* \left[\chi_{\{x_1>0\}} \widehat{f} \right] = 2^{-1}(I + \mathrm{i}H_1)f \text{ where}$$
$$(H_1 f)(x_1, x_2, ..., x_d) := H(f(\cdot, x_2, ..., x_d))(x_1).$$

In general, any operator of the form $\mathcal{F}^* \left[\chi_P \widehat{f} \right]$ with P a polygon in $\mathbb{R}^d$ can be expressed in terms of the Hilbert transform.

Remark 6.3. Let $p \in (1, \infty)$ and let $L^p(\mathbb{R}, \mathbb{C}) \ni f = \lim_{y \to 0^+} F(\cdot + \mathrm{i}y)$ where $F : \{x + \mathrm{i}y : x \in \mathbb{R},\ y > 0\} \to \mathbb{C}$ is a holomorphic function with $\sup_{y>0} \int_{\mathbb{R}} |F(x + \mathrm{i}y)|^p dx < \infty$. Then, if $u = \mathrm{Re}\, f$ and $v = \mathrm{Im}\, f$, we have $v = Hu$ (and, by $H^2 = -1$, $u = -Hv$). We give a brief impressionistic and non–rigorous discussion of how this comes about. Notice that if f is the boundary value in $\mathbb{R}$ of F by Cauchy integral formula we have

$$F(x + \mathrm{i}y) = \frac{1}{2\pi\mathrm{i}} \int_{\mathbb{R}} \frac{1}{t - x - \mathrm{i}y} f(t) dt = \frac{1}{2\pi\mathrm{i}} \left(\frac{1}{\cdot - \mathrm{i}y} * f\right)(x)$$

where here we assume $f \in \mathcal{S}(\mathbb{R}, \mathbb{C})$. Then for $y \to 0^+$ by the Sokhotski–Plemelj theorem we get

$$\lim_{y \to 0^+} \frac{1}{t - \mathrm{i}y} = P.V.\frac{1}{t} + \mathrm{i}\pi\delta(t) \text{ in } \mathcal{S}'(\mathbb{R}, \mathbb{C}). \tag{6.4}$$

This implies, assuming here $F \in C^0(\mathbb{R} \times [0, \infty))$, that by $f(x) = \lim_{y \to 0^+} F(x + \mathrm{i}y)$ we have

$$f(x) = \frac{1}{2\pi\mathrm{i}} \left(\lim_{\varepsilon \to 0^+} \int_{|x| > \varepsilon} \frac{f(x)}{x} dx + \mathrm{i}\pi f(x) \right),$$

that is $f = \mathrm{i}Hf$, which is the desired result.

As for (6.4), for $f \in \mathcal{S}(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \frac{f(t)}{t - \mathrm{i}y} dt = \int_{\mathbb{R}} \frac{t}{t^2 + y^2} f(t) dt + \mathrm{i} \int_{\mathbb{R}} \frac{y}{t^2 + y^2} f(t) dt.$$

By a change of variables, by dominated convergence and by the continuity of f in 0 we have

$$\int_{\mathbb{R}} \frac{y}{t^2 + y^2} f(t) dt = \int_{\mathbb{R}} \frac{1}{t^2 + 1} f(ty) dt \overset{y \to 0}{\to} \pi f(0).$$

Next we write

$$\int_{\mathbb{R}} \frac{t}{t^2+y^2} f(t)dt = \int_{|t|\le y} \frac{t}{t^2+y^2} f(t)dt + \int_{|t|\ge y} \frac{t}{t^2+y^2} f(t)dt.$$

We have

$$\left| \int_{|t|\le y} \frac{t}{t^2+y^2} f(t)dt \right| = \left| \int_{|t|\le y} \frac{t}{t^2+y^2} (f(t)-f(0))dt \right| \overset{y\to 0}{\to} 0.$$

Next we write

$$\int_{|t|\ge y} \frac{t}{t^2+y^2} f(t)dt = \int_{|t|\ge y} \left(\frac{t}{t^2+y^2} - \frac{1}{t} \right) f(t)dt + \int_{|t|\ge y} \frac{f(t)}{t} dt$$

and observe that

$$\left| \int_{|t|\ge y} \left(\frac{t}{t^2+y^2} - \frac{1}{t} \right) f(t)dt \right| \le \int_{|t|\ge y} \frac{y^2}{t^2+y^2} |f(t)|dt \overset{y\to 0}{\to} 0$$

by dominated convergence. This proves (6.4).

Theorem 6.4. *Consider an operator T as in Definition 6.1. Then for any $p \in (1,\infty)$ the operator T, initially defined in $L^p(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, extends into a bounded operator $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ with operator norm that depends only on p and C.*

Before proving Theorem 6.4 we need the Calderon–Zygmund decomposition lemma.

Theorem 6.5 (C–Z Decomposition)**.** *For any $f \in L^1(\mathbb{R}^d)$ and any $\alpha > 0$ there exist families of balls B_j, disjoint sets Q_j with $B_j \subseteq Q_j \subseteq 3B_j$ with $\cup_j Q_j = \cup_j 3B_j$ (here $3B_j$ has same center and trice the radius of B_j) functions g and b_j s.t.*

1. $f = g + \sum_j b_j$.
2. $|g(x)| \le 3^d \alpha$ *for a.a.* x, $\|g\|_{L^1(\mathbb{R}^d)} \le \left(1+3^{2d}\right) \|f\|_{L^1(\mathbb{R}^d)}$.
3. *supp* $b_j \subseteq Q_j$, $\int_{\mathbb{R}^d} b_j(x)dx = 0$ *and* $\sum_j \|b_j\|_{L^1(\mathbb{R}^d)} \le \left(1+3^{2d}\right) \|f\|_{L^1(\mathbb{R}^d)}$.
4. $\sum_j vol(B_j) \le \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}$.

Remark 6.6. Notice that in the Calderon–Zygmund decomposition g is the *good* part of f and b_j form the *bad* part of f.

Proof. Define $\Omega = \{x \in \mathbb{R}^d : Mf(x) > \alpha\}$. Here notice that if $\Omega = \emptyset$ then just set $g = f$. For any $x \in \Omega$ there exists a maximal r_x s.t.

$$A_{r_x}|f|(x) := \frac{1}{\text{vol}(B(x, r_x))} \int_{B(x,r_x)} |f(y)|dy = \alpha.$$

Let us consider the family of balls $\{B(x, r_x)\}_{x\in\Omega}$. It contains, by a generalization of Vitali's Lemma, see Theorem 4.2, a maximal family of pairwise disjoint balls $\{B_j\}$ s.t.

$$\Omega \subseteq \cup_{x\in\Omega} B(x, r_x) \subseteq \cup_j(3B_j).$$

Notice that this implies

$$\text{vol}(\cup_{x\in\Omega} B(x, r_x)) \le \sum_j \text{vol}(3B_j) \le 3^d \sum_j \text{vol}(B_j) \le \frac{3^d}{\alpha} \|f\|_{L^1(\mathbb{R}^d)}.$$

It is possible to choose disjoint sets Q_j s.t. $B_j \subseteq Q_j \subseteq 3B_j$ and $\cup_j Q_j = \cup_j(3B_j)$. One way is to choose

$$Q_k = 3B_k \cap C\left(\cup_{j<k} Q_j\right) \cap C\left(\cup_{j>k} B_j\right) \tag{6.5}$$

with CX the complement of X. Notice indeed that obviously for $k > \ell$ we have

$$Q_k \cap Q_\ell \subseteq C\left(\cup_{j<k} Q_j\right) \cap Q_\ell = \left(\cap_{j<k} CQ_j\right) \cap Q_\ell \subseteq CQ_\ell \cap Q_\ell = \emptyset.$$

Obviously $Q_k \subseteq 3B_k$.
We have $B_k \cap (\cup_{j>k} B_j) = \emptyset$ and so $B_k \subseteq C\left(\cup_{j>k} B_j\right)$. We have $B_k \cap (\cup_{j<k} Q_j) = \emptyset$ because, by (6.5), we have $B_k \cap Q_j = \emptyset$ for any $j < k$. Hence we conclude $B_k \subseteq Q_k$.
Finally we show $\cup_k Q_k = \cup_k 3B_k$. Obviously we have $\cup_k Q_k \subseteq \cup_k 3B_k$. Suppose there exists $x \in \cup_k 3B_k$ with $x \notin \cup_k Q_k$. The latter implies $x \notin \cup_k B_k$, and so $x \in C\left(\cup_{j>k} B_j\right)$ for all k, as well as $x \in C\left(\cup_{j<k} Q_j\right)$ for all k. But then, since $x \in 3B_\ell$ for some ℓ, it follows that $x \in Q_\ell$. And so we get a contradiction. Hence $\cup_k Q_k = \cup_k 3B_k$.
Now define

$$b_j(x) := \left(f(x) - \text{average}_{Q_j} f\right) \chi_{Q_j}(x)$$

$$g(x) := \begin{cases} \text{average}_{Q_j} f \text{ for } x \in Q_j, \\ f(x) \text{ for } x \notin \cup_j Q_j \end{cases}$$

Then we claim that the statement of the theorem is satisfied. First of all for any $x \in \mathbb{R}^d$ either $x \notin Q_j$ for all j, and so $f(x) = g(x)$ with $b_j(x) = 0$ for all j, or $x \in Q_{j_0}$ for exactly one j_0, and so $f(x) = g(x) + b_{j_0}(x)$ with $b_j(x) = 0$ for all $j \neq j_0$. This proves the 1st claim.

For $x \notin \cup_j Q_j \supseteq \Omega$ we have $Mf(x) \le \alpha$. Then, since for a.e. x we have

$$|f(x)| = \lim_{r\to 0^+} |A_r f(x)| \le Mf(x)$$

we get $|g(x)| = |f(x)| \le \alpha$ a.e. in the complement of $\cup_j Q_j$. For $x \in Q_j$ we have

$$|g(x)| = |\text{average}_{Q_j} f| \le \frac{1}{\text{vol}(Q_j)} \int_{Q_j} |f(y)|dy \le \frac{1}{\text{vol}(B_j)} \int_{3B_j} |f(y)|dy = \frac{3^d}{\text{vol}(3B_j)} \int_{3B_j} |f(y)|dy < 3^d\alpha.$$

Furthermore we have

$$\|g\|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d \setminus \cup_j Q_j} |f(x)|dx + \sum_j \int_{Q_j} |g(x)|dx \le \|f\|_{L^1(\mathbb{R}^d)} + 3^d \alpha \sum_j \mathrm{vol}(3B_j)$$
$$\le \left(1 + 3^{2d}\right) \|f\|_{L^1(\mathbb{R}^d)}.$$

The fact that supp $b_j \subseteq Q_j$, $\int_{\mathbb{R}^d} b_j(x)dx = 0$ follows immediately by the definition of b_j. We have

$$\sum_j \|b_j\|_{L^1(\mathbb{R}^d)} \le \|f\|_{L^1(\mathbb{R}^d)} + \sum_j \mathrm{vol}(Q_j)|\mathrm{average}_{Q_j} f| \le \|f\|_{L^1(\mathbb{R}^d)} + 3^d \alpha \sum_j \mathrm{vol}(Q_j)$$
$$\le \left(1 + 3^{2d}\right) \|f\|_{L^1(\mathbb{R}^d)}.$$

□

Proof of Theorem 6.4. By duality it is enough to consider only $p \in (1, 2]$. Furthermore, since by hypothesis (C–Z2) we know that the case $p = 2$ is true, by Marcinkiewicz Interpolation the statement of Theorem 6.4 results from proving that T is weak–type $(1, 1)$. We need to prove that there exists an $A > 0$ s.t.

$$\mathrm{vol}(\{x : |Tf(x)| > \alpha\}) \le \frac{A}{\alpha}\|f\|_{L^1(\mathbb{R}^d)} \text{ for any } \alpha > 0 \text{ and any } f \in L^1(\mathbb{R}^d). \tag{6.6}$$

For fixed $\alpha > 0$ and any $f \in L^1(\mathbb{R}^d)$ consider the C–Z decomposition $f = g + \sum_j b_j$. Notice that $|g(x)| \le 3^d\alpha$ a.e. and $\|g\|_{L^1(\mathbb{R}^d)} \le \left(1 + 3^{2d}\right) \|f\|_{L^1(\mathbb{R}^d)}$ imply $g \in L^2(\mathbb{R}^d)$ with

$$\int_{\mathbb{R}^d} |g|^2 dx \le C_d \alpha \int_{\mathbb{R}^d} |f| dx \text{ for } C_d = 3^d \left(1 + 3^{2d}\right)$$

and so by Hypothesis (C–Z2) we have $\|Tg\|^2_{L^2(\mathbb{R}^d)} \le C\alpha \|f\|_{L^1(\mathbb{R}^d)}$.
Then by Chebyshev's inequality (4.6) we have

$$\mathrm{vol}(\{x : |(Tg)(x)| > \alpha/2\}) \le \frac{4\|Tg\|^2_{L^2(\mathbb{R}^d)}}{\alpha^2} \le 4C\frac{\|f\|_{L^1(\mathbb{R}^d)}}{\alpha}.$$

We next consider b_j and consider for $x \notin 3B_j$ and for y_j the center of B_j,

$$Tb_j(x) = \int_{Q_j} K(x, y) b_j(y)dy = \int_{Q_j} \left(K(x, y) - K(x, y_j)\right) b_j(y)dy$$

were we used $\mathrm{average}_{Q_j} b_j = 0$. Then by (6.1) we have

$$|Tb_j(x)| \le \frac{C}{|x - y_j|^{d+1}} \int_{Q_j} |y - y_j|\ |b_j(y)|dy.$$

Then for $r = \text{radius}(B_j)$

$$\int_{\mathbb{R}^d\setminus 3B_j} |Tb_j(x)|dx \le \int_{|x-y_j|\ge 3r} dx \frac{C}{|x-y_j|^{d+1}} \int_{|y-y_j|\le 3r} |y-y_j|\ |b_j(y)|dy$$
$$\le c_d \frac{C}{3r} \int_{|y-y_j|\le 3r} |y-y_j|\ |b_j(y)|dy \le c_d C \|b_j\|_{L^1(\mathbb{R}^d)}.$$

Let now $E = \cup_j (3B_j)$. Then for $b = \sum_j b_j$ we have

$$\int_{\mathbb{R}^d\setminus E} |Tb| \le \sum_j \int_{\mathbb{R}^d\setminus 3B_j} |Tb_j| \le c_d C \sum_j \|b_j\|_{L^1(\mathbb{R}^d)} \le c_d C(1+3^{2d})\|f\|_{L^1(\mathbb{R}^d)}.$$

Hence

$$\text{vol}(\{x \notin E : |(Tb)(x)| > \alpha/2\}) \le \frac{2\|Tb\|_{L^1(\mathbb{R}^d)}}{\alpha} \le c_d C(1+3^{2d}) \frac{\|f\|_{L^1(\mathbb{R}^d)}}{\alpha}.$$

So since

$$\text{vol}(\{x \notin E : |Tf(x)| > \alpha\}) \le \text{vol}(\{x \notin E : |Tg(x)| > \alpha/2\} + \text{vol}(\{x \notin E : |(Tb)(x)| > \alpha/2\})$$
$$\le \left[4C + c_d C(1+3^{2d})\right] \frac{\|f\|_{L^1(\mathbb{R}^d)}}{\alpha}$$

and

$$\text{vol}(E) \le \sum_j \text{vol}(3B_j) \le 3^d \sum_j \text{vol}(B_j) \le \frac{3^d}{\alpha}\|f\|_{L^1(\mathbb{R}^d)}$$

we conclude that (6.6) as been proved with $A = 3^d + 4C + c_d C(1+3^{2d})$.

□

Now we consider the Proof of Theorem 4.1. We follow [13] from p. 136. Preliminarily, we state the following lemma.

Lemma 6.7. *Suppose* $1 < p < \infty$ *and* $s \ge 1$. *Then* $f \in \mathcal{W}^{s,p}(\mathbb{R}^d)$ *if and only if* $f \in \mathcal{W}^{s-1,p}(\mathbb{R}^d)$ *and* $\partial_{x^j} f \in \mathcal{W}^{s-1,p}(\mathbb{R}^d)$ *for all* $j = 1, ..., d$ *and furthermore the norms* $\|f\|_{\mathcal{W}^{s,p}}$ *and* $\|f\|_{\mathcal{W}^{s-1,p}} + \sum_{j=1}^d \|\partial_{x^j} f\|_{\mathcal{W}^{s-1,p}}$ *are equivalent.*

Proof of Theorem 4.1 assuming Lemma 6.7. Obviously for $k = 0$ we have $\mathcal{W}^{0,p} = W^{0,p} = L^p$.
It is obvious that $f \in W^{k,p}(\mathbb{R}^d)$ if and only if $f \in W^{k-1,p}(\mathbb{R}^d)$ and $\partial_{x^j} f \in W^{k-1,p}(\mathbb{R}^d)$ and that the the norms $\|f\|_{W^{k,p}}$ and $\|f\|_{W^{k-1,p}} + \sum_{j=1}^d \|\partial_{x^j} f\|_{W^{k-1,p}}$ are equivalent. But then Lemma (6.7) guarantees that $\mathcal{W}^{1,p} = W^{1,p}$ with equivalent norms, and so on for all $k \in \mathbb{N}$.

□

Proof of Lemma 6.7. Let us start assuming that $f \in \mathcal{W}^{s,p}(\mathbb{R}^d)$. Then setting $\widehat{g}(\xi) := \langle\xi\rangle^s \widehat{f}(\xi)$ we have $g \in L^p(\mathbb{R}^d)$ by definition of $\mathcal{W}^{s,p}(\mathbb{R}^d)$. Then notice that

$$(\langle\xi\rangle^{s-1} \widehat{f})^\vee = (\langle\xi\rangle^{-1} \widehat{g})^\vee = (2\pi)^{-\frac{d}{2}} \mathcal{J}_{-1} * g$$

where $\mathcal{J}_{-s} = (\langle\xi\rangle^{-1})^{\vee}$ is easily seen to be an $L^1(\mathbb{R}^d)$ function: this can be seen by an integration by parts argument like in the discussion of the Riesz transforms above. Hence we have

$$\|f\|_{\mathcal{W}^{s-1,p}} \leq (2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1}\|g\|_{L^p} = (2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1}\|f\|_{\mathcal{W}^{s,p}}.$$

Next we consider

$$\langle\xi\rangle^{s-1}\,\widehat{\partial_j f}(\xi) = -\mathrm{i}\,\langle\xi\rangle^{s-1}\,\xi_j\widehat{f}(\xi) = -\mathrm{i}\frac{\xi_j}{\langle\xi\rangle}\widehat{g}(\xi) = \widehat{R_j g}(\xi),$$

where R_j is the Riesz transform considered earlier. But then, since the Riesz transforms are CZ operators, it follows that

$$\|\partial_j f\|_{W^{k-1,p}} \leq \|R_j\|_{L^p\to L^p}\|g\|_{L^p} = \|R_j\|_{L^p\to L^p}\|g\|_{L^p}\|f\|_{\mathcal{W}^{s,p}}.$$

Summing up, we obtained

$$\|f\|_{\mathcal{W}^{s-1,p}} + \sum_{j=1}^{d}\|\partial_{x^j} f\|_{\mathcal{W}^{s-1,p}} \leq \Big((2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1} + d\|R_1\|_{L^p\to L^p}\Big)\,\|f\|_{\mathcal{W}^{s,p}},$$

where we used the fact, easy to show, that $\|R_j\|_{L^p\to L^p}$ is constant in j, so that one implication is proved.

Now we consider the opposite implication, assuming $f\in\mathcal{W}^{s-1,p}(\mathbb{R}^d)$ and $\partial_{x^j} f \in \mathcal{W}^{s-1,p}(\mathbb{R}^d)$ for all $j=1,...,d$. Then $\widehat{g}(\xi) := \langle\xi\rangle^{s-1}\,\widehat{f}(\xi)$ is $g\in L^p(\mathbb{R}^d)$ and, from $\widehat{\partial_{x_j} g}(\xi) = \langle\xi\rangle^{s-1}\,\widehat{\partial_{x_j} f}(\xi)$, $\partial_{x_j} g \in L^p(\mathbb{R}^d)$ for any j. Now we have

$$\langle\xi\rangle^{s}\,\widehat{f} = \langle\xi\rangle\,\widehat{g} = \langle\xi\rangle^{2}\,\frac{1}{\langle\xi\rangle}\widehat{g} = \frac{1}{\langle\xi\rangle}\widehat{g} - \sum_{j=1}^{d}\frac{-\mathrm{i}\xi_j}{\langle\xi\rangle}(-\mathrm{i}\xi_j)\widehat{g}.$$

This means that

$$(\langle\xi\rangle^{s}\,\widehat{f})^{\vee} = (2\pi)^{-\frac{d}{2}}\mathcal{J}_{-1} * g - \sum_{j=1}^{d} R_j\partial_{x_j} g$$

and so

$$\begin{aligned}\|f\|_{\mathcal{W}^{s,p}} &\leq (2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1}\|g\|_{L^p} + \sum_{j=1}^{d}\|R_j\|_{L^p\to L^p}\|\partial_{x_j} g\|_{L^p}\\ &= (2\pi)^{-\frac{d}{2}}\|\mathcal{J}_{-1}\|_{L^1}\|f\|_{\mathcal{W}^{s-1,p}} + \sum_{j=1}^{d}\|R_j\|_{L^p\to L^p}\|\partial_{x_j} f\|_{\mathcal{W}^{s-1,p}},\end{aligned}$$

which obviously roves the opposite implication and completes the proof of Lemma 6.7.

□

7 Linear heat equation

For Sections 7–8 see [5].

Let $T \in \mathbb{R}_+$ and $f : [0,T] \to \dot{H}^{s-1}(\mathbb{R}^d, \mathbb{R}^d)$, for $d = 2, 3$, be an external force s.t. $f = \mathbb{P}f$ and consider the following heat equation:

$$\begin{cases} u_t - \nu \triangle u = f \\ \nabla \cdot u = 0 \qquad\qquad (t,x) \in [0,T] \times \mathbb{R}^d \\ u(0) = u_0 \in \mathbb{P}\dot{H}^s(\mathbb{R}^d, \mathbb{R}^d) \end{cases} \tag{7.1}$$

Definition 7.1. For a fixed $s \in (-d/2, d/2)$ let $f \in L^2([0,T], \dot{H}^{s-1}(\mathbb{R}^d, \mathbb{R}^d))$ with $f = \mathbb{P}f$. Then u is a solution of (7.1) if

$$u \in L^\infty([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d)) \ , \ \nabla u \in L^2([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d \times \mathbb{R}^d)), \tag{7.2}$$

if

$$u \text{ is weakly continuous from } [0,T] \text{ into } \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d) \tag{7.3}$$

(that is, if for any $\psi \in \dot{H}^{-s}(\mathbb{R}^d, \mathbb{R}^d)$ the function $t \to \langle u(t), \psi \rangle$, which is a well defined function in $L^\infty([0,T], \mathbb{R})$, is in fact in $C^0([0,T], \mathbb{R})$)
and if for any $\Psi \in C_c^\infty([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ we have

$$\langle u(t), \Psi(t) \rangle_{L^2} = \int_0^t \left(\nu \langle u(t'), \triangle \Psi(t') \rangle_{L^2} + \langle u(t'), \partial_t \Psi(t') \rangle_{L^2} + \langle f(t'), \Psi(t') \rangle_{L^2} \right) dt' + \langle u_0, \Psi(0) \rangle_{L^2}. \tag{7.4}$$

The following theorem yields existence, uniqueness and energy estimate for (7.1).

Theorem 7.2. *Problem* (7.1) *admits exactly one solution in the sense of the above definition. For any* t *the following energy estimate is satisfied:*

$$\|u(t)\|_{\dot{H}^s}^2 + 2\nu \int_0^t \|\nabla u(t')\|_{\dot{H}^s}^2 dt' = \|u_0\|_{\dot{H}^s}^2 + 2 \int_0^t \langle f(t'), u(t') \rangle_{\dot{H}^s} dt'. \tag{7.5}$$

Furthermore we have

$$u \in C^0([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d)) \tag{7.6}$$

and the formula

$$\widehat{u}(t,\xi) = e^{-t\nu|\xi|^2} \widehat{u}_0(\xi) + \int_0^t e^{-(t-t')\nu|\xi|^2} \widehat{f}(t',\xi) dt'. \tag{7.7}$$

Proof. (**Uniqueness**). It is enough to show that the only solution of the case $u_0 = 0$ and $f = 0$ is $u = 0$. Let u be such a solution. Then

$$\langle u(t), \Psi(t) \rangle_{L^2} = \int_0^t \left(\nu \langle u(t'), \triangle \Psi(t') \rangle_{L^2} + \langle u(t'), \partial_t \Psi(t') \rangle_{L^2} \right) dt'.$$

Let $\Psi(t,x)=\psi(x)$ with $\psi\in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$. Then the above equality reduces to

$$\langle u(t),\psi\rangle_{L^2}=\nu\int_0^t\langle u(t'),\triangle\psi\rangle_{L^2}. \tag{7.8}$$

We claim that this identity holds for all $\psi\in\dot H^{-s}(\mathbb{R}^d,\mathbb{R}^d)\cap\dot H^{-s+1}(\mathbb{R}^d,\mathbb{R}^d)$. First of all, it can be shown that $C_c^\infty(\mathbb{R}^d\backslash\{0\},\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d,\mathbb{R}^d,|\xi|^{-2s}d\xi)\cap L^2(\mathbb{R}^d,\mathbb{R}^d,|\xi|^{2-2s}d\xi)$. Hence $\mathcal{S}(\mathbb{R}^d\backslash\{0\},\mathbb{R}^d)$ is dense in $\dot H^{-s}(\mathbb{R}^d,\mathbb{R}^d)\cap\dot H^{-s+1}(\mathbb{R}^d,\mathbb{R}^d)$. So it is enough to show (7.8) for all $\psi\in\mathcal{S}(\mathbb{R}^d\backslash\{0\},\mathbb{R}^d)$. For $\psi\in\mathcal{S}(\mathbb{R}^d,\mathbb{R}^d)$, $\chi\in\mathbb{C}_c^\infty(\mathbb{R}^d,[0,1])$ a cutoff function with $\chi=1$ near the origin, it is possible to show that $\chi\left(\frac{x}{n}\right)\psi\xrightarrow{n\to+\infty}\psi$ in $H^\sigma(\mathbb{R}^d,\mathbb{R}^d)$ for any $\sigma>-d/2$. Indeed

$$\widehat{f*g}(\xi)=(2\pi)^{\frac d2}\widehat f(\xi)\widehat g(\xi).$$

$$\begin{aligned}\|\chi\left(\frac{x}{n}\right)\psi-\psi\|_{H^\sigma}^2&=\int d\xi|\xi|^{2\sigma}\left|\int(2\pi)^{-\frac d2}n^d\widehat\chi(n\eta)\widehat\psi(\xi-\eta)d\eta-\widehat\psi(\xi)\right|^2\\&=\int d\xi|\xi|^{2\sigma}\left|\int(2\pi)^{-\frac d2}\widehat\chi(\eta)\left(\widehat\psi\left(\xi-\frac{\eta}{n}\right)-\widehat\psi(\xi)\right)d\eta\right|^2.\end{aligned}$$

So

$$\|\chi\left(\frac{x}{n}\right)\psi-\psi\|_{H^\sigma}\le(2\pi)^{-\frac d2}\int d\eta|\widehat\chi(\eta)|\left(\int|\xi|^{2\sigma}\left|\int\left(\widehat\psi\left(\xi-\frac{\eta}{n}\right)-\widehat\psi(\xi)\right)d\eta\right|^2\right)^{\frac12}.$$

We split in the right integrating in $|\eta|\le C$ and in $|\eta|\ge C$. In the integral in $|\eta|\le C$ we get a sequence that, by dominated convergence, converges to 0. Next, we consider the integral in $|\eta|\ge C$. We can bound it from above by

$$(2\pi)^{-\frac d2}\int_{|\eta|\ge C}d\eta|\widehat\chi(\eta)|\left(\left(\int|\xi|^{2\sigma}\left|\int\widehat\psi\left(\xi-\frac{\eta}{n}\right)\right|^2\right)^{\frac12}+\|\psi\|_{H^\sigma}\right). \tag{7.9}$$

Not we claim that for c independent of η we have

$$\int|\xi|^{2\sigma}\left|\int\widehat\psi\left(\xi-\eta\right)\right|^2\le c+c|\eta|^{2\sigma}. \tag{7.10}$$

We spit the integral into regions $|\eta|\ll|\xi|$, $|\eta|\sim|\xi|$ and $|\eta|\gg|\xi|$. We have

$$\int_{|\eta|\gg|\xi|}|\xi|^{2\sigma}\left|\widehat\psi\left(\xi-\eta\right)\right|^2\lesssim\langle\eta\rangle^{-N}\int_{|\eta|\gg|\xi|}|\xi|^{2\sigma}\lesssim1.$$

We have

$$\int_{|\eta|\ll|\xi|}|\xi|^{2\sigma}\left|\widehat\psi\left(\xi-\eta\right)\right|^2\lesssim\int_{\mathbb{R}^d}|\xi-\eta|^{2\sigma}\left|\widehat\psi\left(\xi-\eta\right)\right|^2=\|\psi\|_{H^\sigma}^2.$$

Finally, for $|\eta| \sim |\xi|$

$$\int_{|\eta|\sim|\xi|} |\xi|^{2\sigma} \left|\widehat{\psi}\,(\xi-\eta)\right|^2 \le \int_{\mathbb{R}^d} |\xi-\eta|^{2\sigma} \left|\widehat{\psi}\,(\xi)\right|^2 \lesssim |\eta|^{2\sigma} \int_{\mathbb{R}^d} \left|\widehat{\psi}\,(\xi)\right|^2 + \|\psi\|^2_{\dot{H}^{\sigma}}.$$

So we proved (7.10). Inserting this in (7.9) and taking C sufficiently large we obtain that (7.9) is arbitrarily small.
Hence we can conclude that (7.8) is true for all $\psi \in \dot{H}^{-s}(\mathbb{R}^d,\mathbb{R}^d) \cap \dot{H}^{-s+1}(\mathbb{R}^d,\mathbb{R}^d)$. In particular we can replace ψ by $\mathbf{P}_n\psi$ and get

$$\begin{aligned}
&\langle \mathbf{P}_n u(t), \psi\rangle_{L^2} = \int_0^t \nu\langle u(t'), \triangle \mathbf{P}_n\psi\rangle_{L^2} \le \nu\|\triangle\mathbf{P}_n\psi\|_{\dot{H}^{-s}} \int_0^t \|\mathbf{P}_n u(t')\|_{\dot{H}^s} dt' \\
&\le \nu n^2 \|\psi\|_{\dot{H}^{-s}} \int_0^t \|\mathbf{P}_n u(t')\|_{\dot{H}^s} dt'
\end{aligned}$$

where the integral $\int_0^t \|\mathbf{P}_n u(t')\|_{\dot{H}^s} dt'$ is well defined by $\mathbf{P}_n u \in L^\infty([0,T],\dot{H}^s(\mathbb{R}^d,\mathbb{R}^d))$.
From the above formula

$$\|\mathbf{P}_n u(t)\|_{\dot{H}^s} \le \nu n^2 \int_0^t \|\mathbf{P}_n u(t')\|_{\dot{H}^s} dt'$$

and hence $\|\mathbf{P}_n u(t)\|_{\dot{H}^s} = 0$ by the Gronwall inequality. This implies $u(t) = 0$ for $t \in [0,T]$.

(**Existence**). First of all, there exists a sequence (f_n) in $C^0([0,T],\dot{H}^{s-1}(\mathbb{R}^d,\mathbb{R}^d))$ s.t. $f_n \xrightarrow{n\to+\infty} f$ in $L^2([0,T],\dot{H}^{s-1}(\mathbb{R}^d,\mathbb{R}^d))$. This follows from the density of $C_c^\infty(I,X)$ in $L^p(I,X)$ for $p < \infty$ for I an interval and X a Banach space, see Appendix A.
Applying $\mathbf{P}_n$ to (7.1) and replacing f by f_n we obtain the equation

$$\begin{cases} (u_n)_t - \nu\mathbf{P}_n\triangle u_n = \mathbf{P}_n f_n \\ \qquad u_n(0) = \mathbf{P}_n u_0 \end{cases} \tag{7.11}$$

Notice that $\mathbf{P}_n f_n \in C^0([0,T],\dot{H}^s(\mathbb{R}^d,\mathbb{R}^d))$. Since (7.11) is a standard linear equation it admits a solution $u_n \in C^1([0,T],\dot{H}^s(\mathbb{R}^d,\mathbb{R}^d))$. Notice furthermore that $u_n = \mathbf{P}_n u_n$ and so in particular $u_n \in C^0([0,T],\dot{H}^r(\mathbb{R}^d,\mathbb{R}^d))$ for all $r \ge s$.
Furthermore, applying $\langle\cdot,u_n\rangle_{\dot{H}^s}$ to (7.11) and using

$$\begin{aligned}
&\langle \mathbf{P}_n\triangle u_n, u_n\rangle_{\dot{H}^s} = -\sum_{k=1}^d \int_{B(0,n)} |\xi|^{2s}\xi_k^2 |\widehat{u}_n(t,\xi)|^2 d\xi = -\sum_{k=1}^d \langle \xi_k\widehat{u}_n, \xi_k\widehat{u}_n\rangle_{L^2(B(0,n),|\xi|^{2s}d\xi)} \\
&= \sum_{k=1}^d \langle \xi_k\widehat{u}_n, \xi_k\widehat{u}_n\rangle_{L^2(\mathbb{R}^d,|\xi|^{2s}d\xi)} = \|\nabla u_n\|^2_{\dot{H}^s},
\end{aligned}$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_n\|^2_{\dot{H}^s} + \nu\|\nabla u_n\|^2_{\dot{H}^s} = \langle \mathbf{P}_n f_n, u_n\rangle_{\dot{H}^s}$$

s.t. after integration we obtain

$$\frac{1}{2}\|u_n(t)\|^2_{\dot{H}^s} + \nu \int_0^t \|\nabla u_n(t')\|^2_{\dot{H}^s} dt' = \frac{1}{2}\|\mathbf{P}_n u_0\|^2_{\dot{H}^s} + \int_0^t \langle \mathbf{P}_n f_n(t'), u_n(t')\rangle_{\dot{H}^s} dt'. \tag{7.12}$$

The difference $u_n - u_{n+\ell}$ solves

$$\begin{cases} (u_n - u_{n+\ell})_t - \nu \mathbf{P}_{n+\ell} \triangle (u_n - u_{n+\ell}) = \mathbf{P}_n f_n - \mathbf{P}_{n+\ell} f_{n+\ell} \\ u_n(0) - u_{n+\ell}(0) = (\mathbf{P}_n - \mathbf{P}_{n+\ell}) u_0 \end{cases}$$

Then, like for (7.12) we get

$$\begin{aligned} &\frac{1}{2}\|u_n(t) - u_{n+\ell}(t)\|^2_{\dot{H}^s} + \cancel{2}\frac{\nu}{2} \int_0^t \|\nabla (u_n - u_{n+\ell})(t')\|^2_{\dot{H}^s} dt' = \\ &= \frac{1}{2}\|(\mathbf{P}_n - \mathbf{P}_{n+\ell})u_0\|^2_{\dot{H}^s} + \int_0^t \langle \mathbf{P}_n f_n(t') - \mathbf{P}_{n+\ell} f_{n+\ell}(t'), (u_n - u_{n+\ell})(t')\rangle_{\dot{H}^s} dt' \\ &\le \frac{1}{2}\|(\mathbf{P}_n - \mathbf{P}_{n+\ell})u_0\|^2_{\dot{H}^s} + \int_0^t \|\mathbf{P}_n f_n(t') - \mathbf{P}_{n+\ell} f_{n+\ell}(t')\|_{\dot{H}^{s-1}} \|\nabla (u_n - u_{n+\ell})(t')\|_{\dot{H}^s} dt' \\ &\le \frac{1}{2}\|(\mathbf{P}_n - \mathbf{P}_{n+\ell})u_0\|^2_{\dot{H}^s} + \frac{1}{2\nu} \int_0^t \|\mathbf{P}_n f_n(t') - \mathbf{P}_{n+\ell} f_{n+\ell}(t')\|^2_{\dot{H}^{s-1}} dt' + \cancel{\frac{\nu}{2} \int_0^t \|\nabla (u_n - u_{n+\ell})(t')\|^2_{\dot{H}^s} dt'}. \end{aligned}$$

Hence

$$\begin{aligned} &\|u_n(t) - u_{n+\ell}(t)\|^2_{\dot{H}^s} + \nu \int_0^t \|\nabla (u_n - u_{n+\ell})(s)\|^2_{\dot{H}^s} ds \\ &\le \|(\mathbf{P}_n - \mathbf{P}_{n+\ell})u_0\|^2_{\dot{H}^s} + \frac{1}{\nu} \int_0^t \|\mathbf{P}_n f_n(s) - \mathbf{P}_{n+\ell} f_{n+\ell}(s)\|^2_{\dot{H}^{s-1}} ds. \end{aligned}$$

Since $f_n \xrightarrow{n\to+\infty} f$ in $L^2([0,T], \dot{H}^{s-1}(\mathbb{R}^d, \mathbb{R}^d))$ implies also $\mathbf{P}_n f_n \xrightarrow{n\to+\infty} f$ therein, the last inequality implies that (u_n) is Cauchy in $C([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d))$ and (∇u_n) is Cauchy in $L^2([0,T], \dot{H}^s(\mathbb{R}^d, \mathbb{R}^d))$. Let u be the limit. Notice that u satisfies (7.2) and (7.6), and so obviously also (7.3).
Taking the limit in (7.12) we see that u satisfies the energy equality (7.5).
Next, we check that u is a weak solution of (7.1) in the sense of Def. 7.1. We apply $\langle ., \Psi(t)\rangle_{L^2}$ to (7.11) with $\Psi \in C_c^\infty([0,\infty) \times \mathbb{R}^d, \mathbb{R}^d)$. Then we have

$$\frac{d}{dt}\langle u_n, \Psi\rangle_{L^2} = \nu \langle \triangle u_n, \Psi\rangle_{L^2} + \langle \mathbf{P}_n f_n, \Psi\rangle_{L^2} + \langle u_n, \partial_t \Psi\rangle_{L^2}.$$

Integrating we have

$$\begin{aligned} \langle u_n(t), \Psi(t)\rangle_{L^2} &= \langle \mathbf{P}_n u_0, \Psi(0)\rangle_{L^2} - \nu \int_0^t \langle u_n(t'), \triangle \Psi(t')\rangle_{L^2} dt' \\ &+ \int_0^t \langle \mathbf{P}_n f_n(t'), \Psi(t')\rangle_{L^2} dt' + \int_0^t \langle u_n(t'), \partial_t \Psi(t')\rangle_{L^2} dt'. \end{aligned}$$

Taking the limit for $n \to \infty$ we get

$$\langle u(t), \Psi(t)\rangle_{L^2} = \langle u_0, \Psi(0)\rangle_{L^2} - \nu \int_0^t \langle u(t'), \triangle\Psi(t')\rangle_{L^2} dt' + \int_0^t \langle f(t'), \Psi(t')\rangle_{L^2} dt' + \int_0^t \langle u(t'), \partial_t\Psi(t')\rangle_{L^2} dt'.$$

which yields (7.4). Hence u is a weak solution of (7.1) in the sense of Def. 7.1.
Next, we prove the Duhamel formula (7.7). Applying the Fourier transform to (7.11)

$$\begin{cases} \partial_t \widehat{u}_n(t,\xi) + \nu\chi_{|\xi|\le n}|\xi|^2\widehat{u}_n(t,\xi) = \chi_{|\xi|\le n}\widehat{f}_n(t,\xi) \\ \widehat{u_n}(0,\xi) = \chi_{|\xi|\le n}\widehat{u}_0(\xi) \end{cases} \tag{7.13}$$

Notice that $\mathrm{supp}\widehat{u}_n(t,\cdot) \subseteq \{|\xi| \le n\}$ so that $\chi_{|\xi|\le n}|\xi|^2\widehat{u}_n(t,\xi) = |\xi|^2\widehat{u}_n(t,\xi)$. Then, by the variation of parameters formula

$$\widehat{u}_n(t,\xi) = e^{-t\nu|\xi|^2}\chi_{|\xi|\le n}\widehat{u}_0(\xi) + \int_0^t e^{-(t-t')\nu|\xi|^2}\chi_{|\xi|\le n}\widehat{f}_n(t',\xi)dt'. \tag{7.14}$$

Now we know

$$\begin{aligned} &\widehat{u}_n(t,\xi) \overset{n\to\infty}{\to} \widehat{u}(t,\xi) \text{ in } C([0,T], L^2(\mathbb{R}^d, |\xi|^{2s}d\xi)) \\ &\chi_{|\xi|\le n}\widehat{u}_0(\xi) \overset{n\to\infty}{\to} \widehat{u}_0(\xi) \text{ in } L^2(\mathbb{R}^d, |\xi|^{2s}d\xi), \\ &\chi_{|\xi|\le n}\widehat{f}_n(t',\xi) \overset{n\to\infty}{\to} \widehat{f}(t',\xi) \text{ in } L^2([0,T]\times\mathbb{R}^d, |\xi|^{2(s-1)}dtd\xi) \end{aligned}$$

Notice that

$$\mathbf{T}g(t,\xi) := \int_0^t e^{-(t-t')\nu|\xi|^2} g(t',\xi)dt'$$

is a bounded operator from $L^2([0,T]\times\mathbb{R}^d, |\xi|^{2(s-1)}dtd\xi)$ into $L^\infty([0,T], L^2(\mathbb{R}^d, |\xi|^{2s}d\xi)$. Indeed for $t \in [0,T]$ and fixed $\xi \in \mathbb{R}^d$ and for $g \in C_c([0,T]\times(\mathbb{R}^d\backslash\{0\}))$

$$|\mathbf{T}g(t,\xi)| \le (\int_0^t e^{-2(t-t')\nu|\xi|^2}dt')^{\frac{1}{2}}(\int_0^t |g(t',\xi)|^2dt')^{\frac{1}{2}} \le \frac{1}{\sqrt{2\nu}|\xi|}(\int_0^t |g(t',\xi)|^2dt')^{\frac{1}{2}}$$

and so

$$\int_{\mathbb{R}^d} |\xi|^{2s}|\mathbf{T}g(t,\xi)|^2d\xi \le \frac{1}{2\nu}\int_{[0,T]\times\mathbb{R}^d} |\xi|^{2(s-1)}|g(t',\xi)|^2dt'd\xi.$$

This implies

$$\|\mathbf{T}g\|_{L^\infty([0,T],L^2(\mathbb{R}^d,|\xi|^{2s}d\xi)} \le \sqrt{1/2\nu}\|g\|_{L^2([0,T]\times\mathbb{R}^d,|\xi|^{2(s-1)}dtd\xi)}.$$

Since $C_c([0,T]\times(\mathbb{R}^d\backslash\{0\}))$ is dense in $L^2([0,T]\times\mathbb{R}^d, |\xi|^{2(s-1)}dtd\xi)$ a well defined bounded operator remains defined. Taking the limit for $n \to \infty$ in (7.14) all terms converge in $L^\infty([0,T], L^2(\mathbb{R}^d, |\xi|^{2s}d\xi))$ to the corresponding terms of

$$\widehat{u}(t,\xi) = e^{-t\nu|\xi|^2}\widehat{u}_0(\xi) + \int_0^t e^{-(t-t')\nu|\xi|^2}\widehat{f}(t',\xi)dt'.$$

□

Remark 7.3. Notice that applying the Fourier transform to (7.7) we get

$$u(t) = e^{t\nu\triangle}u_0 + \int_0^t e^{(t-t')\nu\triangle} f(t')dt'. \tag{7.15}$$

The following theorem yields additional estimates.

Theorem 7.4. *Let f be like in Theorem 7.2 and consider the corresponding solution*

$$u \in C([0,T], \dot{H}^s), \quad \nabla u \in L^2([0,T], \dot{H}^s).$$

Then, additionally, we have

$$\|u(t)\|_{\dot{H}^{s+\frac{2}{p}}} \in L^p([0,T], \mathbb{R}) \text{ for any } p \geq 2. \tag{7.16}$$

Moreover we have

$$\begin{aligned} V(t) &:= \left(\int_{\mathbb{R}^d} |\xi|^{2s} \left(\sup_{0\leq t'\leq t} |\widehat{u}(t',\xi)| \right)^2 d\xi \right)^{\frac{1}{2}} \leq \|u_0\|_{\dot{H}^s} + \frac{1}{(2\nu)^{\frac{1}{2}}} \|f\|_{L^2([0,t],\dot{H}^{s-1})} ; \\ \| \|u\|_{\dot{H}^{s+\frac{2}{p}}} \|_{L^p(0,T)} &\leq \nu^{-\frac{1}{p}} \left(\|u_0\|_{\dot{H}^s} + \nu^{-\frac{1}{2}} \|f\|_{L^2([0,T],\dot{H}^{s-1})} \right) . \end{aligned} \tag{7.17}$$

Proof. From the Duhamel formula (7.7) and the previous computation

$$|\widehat{u}(t,\xi)| \leq e^{-t\nu|\xi|^2} |\widehat{u}_0(\xi)| + \frac{1}{\sqrt{2\nu}|\xi|} \|\widehat{f}(\cdot,\xi)\|_{L^2(0,t)}.$$

so that

$$|\xi|^s \sup_{0\leq t'\leq t} |\widehat{u}(t',\xi)| \leq |\xi|^s |\widehat{u}_0(\xi)| + |\xi|^s \frac{1}{\sqrt{2\nu}|\xi|} \|\widehat{f}(\cdot,\xi)\|_{L^2(0,t)}.$$

Taking the $L^2(\mathbb{R}^d, d\xi)$ norm we get

$$V(t) \leq \|u_0(\xi)\|_{L^2(\mathbb{R}^d,|\xi|^{2s}d\xi)} + \frac{1}{\sqrt{2\nu}} \|\widehat{f}\|_{L^2((0,t),L^2(\mathbb{R}^d,|\xi|^{2(s-1)}d\xi))}.$$

and this yields the 1st line in (7.17).
To get the 2nd line in (7.17), from the energy estimate (7.5) we obtain

$$\begin{aligned} &\|u(t)\|^2_{\dot{H}^s} + 2\nu \int_0^t \|\nabla u(t')\|^2_{\dot{H}^s} dt' \leq \|u_0\|^2_{\dot{H}^s} + 2\int_0^t \frac{1}{\sqrt{\nu}} \|f(t')\|_{\dot{H}^{s-1}} \sqrt{\nu} \|\nabla u(t')\|_{\dot{H}^s} dt' \\ &\leq \|u_0\|^2_{\dot{H}^s} + \nu \int_0^t \|\nabla u(t')\|^2_{\dot{H}^s} dt' + \frac{1}{\nu} \int_0^t \|f(t')\|^2_{\dot{H}^{s-1}} dt'. \end{aligned}$$

This yields

$$\|u(t)\|^2_{\dot{H}^s} + \nu \int_0^t \|\nabla u(t')\|^2_{\dot{H}^s} dt' \leq \|u_0\|^2_{\dot{H}^s} + \frac{1}{\nu} \int_0^t \|f(t')\|^2_{\dot{H}^{s-1}} dt'.$$

and hence

$$\|u\|_{L^\infty([0,T],\dot H^s)} \le \|u_0\|_{\dot H^s} + \nu^{-\frac12}\|f\|_{L^2([0,T],\dot H^s)}$$
$$\|\|u\|_{\dot H^{s+1}}\|_{L^2(0,T)} \le \nu^{-\frac12}\left(\|u_0\|_{\dot H^s} + \nu^{-\frac12}\|f\|_{L^2([0,T],\dot H^s)}\right).$$

So by the interpolation of Sobolev norms Lemma 5.1 for $2 < p < \infty$

$$\|\|u\|_{\dot H^{s+\frac2p}}\|_{L^p(0,T)} \le \|\|u\|_{\dot H^s}^{1-\frac2p}\|\nabla u\|_{\dot H^s}^{\frac2p}\|_{L^p(0,T)} \le \|u\|_{L^\infty([0,T],\dot H^s)}^{1-\frac2p}\|\|\nabla u\|_{\dot H^s}^{\frac2p}\|_{L^p(0,T)}$$
$$= \|u\|_{L^\infty([0,T],\dot H^s)}^{1-\frac2p}\|\nabla u\|_{L^2([0,T],\dot H^s)}^{\frac2p} \le \nu^{-\frac1p}\left(\|u_0\|_{\dot H^s} + \nu^{-\frac12}\|f\|_{L^2([0,T],\dot H^s)}\right).$$

□

8 The Navier Stokes equation

We will only deal with the Incompressible Navier Stokes (NS) equation:

$$\begin{cases} u_t + u\cdot\nabla u - \nu\triangle u = -\nabla p \\ \nabla\cdot u = 0 \qquad\qquad (t,x)\in[0,\infty)\times\mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{8.1}$$

where $u : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ with $u = \sum_{j=1}^d u^j e_j$ with e_j the standard basis of $\mathbb{R}^d$,

$$\triangle := \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2},\ \nabla\cdot u = \sum_{j=1}^d \frac{\partial}{\partial x_j}u^j,\ u\cdot\nabla v = \sum_{j=1}^d u_j\frac{\partial}{\partial x_j}v.$$

Here $\nu > 0$ is a fixed constant. We could normalize $\nu = 1$. p is the pressure and its function is simply to absorb the divergence part of the l.h.s. of (8.1).

We can write

$$u\cdot\nabla u = \mathrm{div}(u\otimes u) \text{ for } \mathrm{div}(u\otimes v)^j := \sum_{k=1}^d \partial_k(u^k v^j) \text{ since} \tag{8.2}$$
$$\mathrm{div}(u\otimes u)^j = \sum_{k=1}^d \partial_k(u^k u^j) = \sum_{k=1}^d u^k\partial_k u^j + u^j\underbrace{\mathrm{div} u}_{0} = u\cdot\nabla u^j$$

So we rewrite (8.1) and

$$\begin{cases} u_t + \mathrm{div}(u\otimes u) - \nu\triangle u = -\nabla p \\ \nabla\cdot u = 0 \qquad\qquad (t,x)\in[0,\infty)\times\mathbb{R}^d \\ u(0,x) = u_0(x) \end{cases} \tag{8.3}$$

Definition 8.1 (Weak solutions)**.** Let u_0 be in $L^2(\mathbb{R}^d)$. A vector field $u \in L^2_{loc}([0,\infty)\times\mathbb{R}^d)$ which is weakly continuous as a function from $[0,\infty)$ to $(L^2(\mathbb{R}^d))^d$ and s.t. $\mathrm{div} u(t) = 0$ for every t, is a weak solution of (8.3) if for $\Psi \in C_c^\infty([0,\infty)\times\mathbb{R}^d,\mathbb{R}^d)$ with $\mathrm{div}\Psi = 0$ we have

$$\begin{aligned}\langle u(t),\Psi(t)\rangle_{L^2} = \int_0^t \big(\nu\langle u(t'),\triangle\Psi(t')\rangle_{L^2} + \langle u(t'),\partial_t\Psi(t')\rangle_{L^2} \\ -\langle \mathrm{div}(u\otimes u)(t'),\Psi(t')\rangle_{L^2}\big)\,dt' + \langle u_0,\Psi(0)\rangle_{L^2}.\end{aligned} \tag{8.4}$$

Notice that formally (8.4) is obtained from (8.3) writing

$$\int_0^t\int_{\mathbb{R}^d}(u_t + \mathrm{div}(u\otimes u) - \nu\triangle u)\cdot\Psi = -\int_0^t\int_{\mathbb{R}^d}\nabla p\cdot\Psi = \int_0^t\int_{\mathbb{R}^d} p\underbrace{\nabla\cdot\Psi}_{0}.$$

So integrating by parts (which is formal if u is not sufficiently regular) we have

$$\begin{aligned}0 &= \int_{\mathbb{R}^d} u\cdot\Psi\Big|_0^t - \int_0^t\int_{\mathbb{R}^d} u\cdot\partial_t\Psi + \int_0^t\int_{\mathbb{R}^d}\partial_k(u^ju^k)\Psi^j - \nu\int_0^t\int_{\mathbb{R}^d} u\cdot\triangle\Psi \\ &= \int_{\mathbb{R}^d} u\cdot\Psi\Big|_0^t - \int_0^t\int_{\mathbb{R}^d} u\cdot\partial_t\Psi - \int_0^t\int_{\mathbb{R}^d} u^ju^k\partial_k\Psi^j - \nu\int_0^t\int_{\mathbb{R}^d} u\cdot\triangle\Psi\end{aligned}$$

which gives the desired result. In particular, (8.3) implies (8.4) when u is regular.
But the opposite is also true, and when u is regular (8.4) implies (8.3). Indeed, suppose that u is regular and that it satisfies (8.4) for all the Ψ as in Def. 8.1. Then

$$\begin{aligned}&\int_{\mathbb{R}^d} u(t,x)\cdot\Psi(t,x)dx - \int_{\mathbb{R}^d} u_0(x)\cdot\Psi(0,x)dx = \int_0^t\int_{\mathbb{R}^d}(\nu u\cdot\triangle\Psi + u\otimes u : \nabla\Psi + u\partial_t\Psi)(t',x)dxdt' \\ &= \int_0^t\int_{\mathbb{R}^d}(\nu\triangle u - \mathrm{div}(u\otimes u) - \partial_t u)\cdot\Psi + \int_{\mathbb{R}^d} u(t,x)\cdot\Psi(t,x)dx - \int_{\mathbb{R}^d} u(0,x)\cdot\Psi(0,x)dx.\end{aligned}$$

Hence we get

$$\int_{\mathbb{R}^d} u_0(x)\cdot\Psi(0,x)dx = \int_0^t\int_{\mathbb{R}^d}(\partial_t u - \nu\triangle u + \mathrm{div}(u\otimes u))\cdot\Psi + \int_{\mathbb{R}^d} u(0,x)\cdot\Psi(0,x)dx.$$

Taking $\Psi = \varphi(t)\psi(x)$ with $\varphi \in C_c^\infty((0,T),\mathbb{R})$ and $\psi \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and divergence free, we conclude that

$$\int_0^t dt'\varphi(t)\int_{\mathbb{R}^d}\left[(\partial_t u - \nu\triangle u + \mathrm{div}(u\otimes u))\cdot\psi(x)\right]dx.$$

This implies that for all t

$$\langle \nu\triangle u - \mathrm{div}(u\otimes u) - \partial_t u, \psi\rangle_{L^2(\mathbb{R}^d)} = 0$$

for any t and for any divergence free vector field $\psi \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$. Formally, this implies that the above holds for $\psi = \mathbb{P}\Theta$ for any vector field $\Theta \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$. Then, by $\mathbb{P}^* = \mathbb{P}$, we conclude that

$$\langle \mathbb{P}(\nu\triangle u - \mathrm{div}(u\otimes u) - \partial_t u), \Theta\rangle_{L^2(\mathbb{R}^d)} = 0 \text{ for all } \Theta \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d).$$

This implies

$$\mathbb{P}(\nu\triangle u - \mathrm{div}(u\otimes u) - \partial_t u) = 0 \Rightarrow u_t + u\cdot\nabla u - \nu\triangle u = -\nabla p$$

for some p, see Lemma 3.6.
Then we get

$$\int_{\mathbb{R}^d} u_0(x)\cdot\Psi(0,x)dx = \int_{\mathbb{R}^d} u(0,x)\cdot\Psi(0,x)dx$$

and so $u(0,x) = u_0(x)$.

Let us now formally take the inner product of the first line of (8.1) with u and integrate in $\mathbb{R}^d$

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \langle u\cdot\nabla u, u\rangle_{L^2} - \nu\langle\triangle u, u\rangle_{L^2} = -\langle\nabla p, u\rangle_{L^2}$$

We have, summing on repeated indexes,

$$\langle u\cdot\nabla u, u\rangle_{L^2} = \int_{\mathbb{R}^d} u^j u^k \partial_j u^k dx = 2^{-1}\int_{\mathbb{R}^d} u^j\partial_j (u^k u^k)^2 dx = -2^{-1}\int_{\mathbb{R}^d}|u|^2\mathrm{div}u\, dx = 0 \text{ and}$$

$$\langle\nabla p, u\rangle_{L^2} = \int_{\mathbb{R}^d} u^j\partial_j p dx = -\int_{\mathbb{R}^d} p\mathrm{div}u\, dx = 0.$$

So, formally (rigorously if u is regular and we can integrate by parts), we get

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \nu\|\nabla u\|_{L^2}^2 = 0$$

This in particular yields the following *energy equality*

$$\|u(t)\|_{L^2(\mathbb{R}^d)}^2 + 2\nu\int_0^t \|\nabla u(t')\|_{L^2(\mathbb{R}^d)}^2 dt' = \|u_0\|_{L^2(\mathbb{R}^d)}^2. \tag{8.5}$$

Theorem 8.2 (Leray)**.** *Let $u_0 \in L^2(\mathbb{R}^d)$ for $d = 2,3$ be divergence free. Then* (8.3) *admits a weak solution with $u(t) \in L^\infty(\mathbb{R}_+, L^2)\cap L^2_{loc}(\mathbb{R}_+, H^1)$ such that the following energy inequality holds:*

$$\|u(t)\|_{L^2(\mathbb{R}^d)}^2 + 2\nu\int_0^t \|\nabla u(t')\|_{L^2(\mathbb{R}^d)}^2 dt' \le \|u_0\|_{L^2(\mathbb{R}^d)}^2. \tag{8.6}$$

We will also see the following.

Theorem 8.3 (Case $d = 2$)**.** *When $d = 2$ the solution in Theorem 8.2 is unique, it satisfies* (8.5) *and $u(t) \in C^0([0,\infty), L^2)$.*

Notice that if we apply formally the operator $\mathbb{P}$ to equation (8.3) we obtain formally

$$\begin{cases} u_t - \nu \triangle u = \mathcal{Q}_{NS}(u,u) \\ u(0,x) = u_0(x) \end{cases} (t,x) \in [0,\infty) \times \mathbb{R}^d \tag{8.7}$$

where we set

$$\mathcal{Q}_{NS}(u,v) := -\frac{1}{2}\mathbb{P}(\operatorname{div}(u \otimes v)) - \frac{1}{2}\mathbb{P}(\operatorname{div}(v \otimes u)). \tag{8.8}$$

Here notice that

$$\mathbb{P}(\operatorname{div}(u \otimes v))^j = \sum_{l=1}^{d} \partial_l \left((u^l v^j) - \frac{1}{\triangle} \sum_{k=1}^{d} \partial_j \partial_k (u^l v^k) \right). \tag{8.9}$$

Before starting the proof of Theorem 8.2 we need some preliminary results on 1st order ODE's in Banach spaces.

Definition 8.4. Given a Banach space X a function $F : X \to X$ is locally Lipschitz if for any $M > 0$ $\exists$ $L(M) \in (0, +\infty)$ s.t.

$$\|F(x) - F(y)\| \le L(M)\|x - y\| \text{ for all } x, y \text{ with } \|x\| \le M \text{ and } \|y\| \le M. \tag{8.10}$$

Now consider the system

$$\dot{u} = F(u), \quad u(0) = x \tag{8.11}$$

which we write in integral form as

$$u(t) = x + \int_0^t F(u(s))ds. \tag{8.12}$$

Proposition 8.5. *Let F be as in Definition 8.4. Then for any $M > 0$, for T_M defined by*

$$T_M := \frac{1}{2L(2M + \|F(0)\|) + 2}. \tag{8.13}$$

and for any $x \in X$ with $\|x\| \le M$ there is a unique solution $u \in C^0([0, T_M], X)$ of (8.12).

Proof. Set $K = 2M + \|F(0)\|$ and

$$E = \{u \in C^0([0, T_M], X) : \|u(t)\| \le K \text{ for all } t \in [0, T_M]\}$$

with distance $d_E(u,v) := \sup_{0 \le t \le T_M} \|u(t) - v(t)\|$. (E, d_E) is a complete metric space. Next consider the map $u \in E \to \Phi_u$

$$\Phi_u(t) := x + \int_0^t F(u(s))ds \text{ for all } t \in [0, T_M].$$

By $T_M = \frac{1}{2(L(K)+1)}$ for all $t \in [0, T_M]$ we have

$$\begin{aligned} \|F(u(t))\| &\leq \|F(0)\| + \|F(u(t)) - F(0)\| \leq \|F(0)\| + KL(K) \\ &= \|F(0)\| + (2M + \|F(0)\|)L(K) \leq (M + \|F(0)\|)\ 2\ (L(K)+1) = \frac{M + \|F(0)\|}{T_M} \end{aligned}$$

So for $t \in [0, T_M]$ we have

$$\|\Phi_u(t)\| \leq M + t\frac{M + \|F(0)\|}{T_M} \leq 2M + \|F(0)\| = K$$

and so $\Phi_u \in E$.
For $u, v \in E$ we have

$$\|\Phi_u(t) - \Phi_v(t)\| \leq \int_0^t \|F(u(s)) - F(v(s))\| ds \leq T_M L(K) \|u - v\|_{L^\infty([0,T],X)} = T_M L(K) d_E(u, v).$$

So by $T_M L(K) < T_M(L(K)+1) = 2^{-1}$

$$d_E(\Phi_u, \Phi_v) \leq 2^{-1} d_E(u, v)$$

Hence $u \to \Phi_u$ is a contraction in E and so it has exactly one fixed point.

□

We have the following application of Gronwall's inequality.

Lemma 8.6. *Let $T > 0$, $x \in X$ and let $u, v \in C^0([0,T], X)$ solve* (8.12) *then $u = v$.*

Proof. Let $M = \max_{0 \leq t \leq T}\{\|u(t)\|, \|v(t)\|\}$. Then

$$\|u(t) - v(t)\| \leq \int_0^t \|F(u(s)) - F(v(s))\| ds \leq L(M) \int_0^t \|u(s) - v(s)\| ds$$

and apply Gronwall's inequality. □

It remains defined a function $T : X \to (0, \infty]$ where for any $x \in X$

$$T(x) = \sup\{T > 0 : \exists u \in C^0([0,T), X) \text{ solution of } (8.12)\ \}$$

and the interval $[0, T(x))$ is the largest (positive) half open interval of existence of the (unique, by Lemma 15.6) solution of (8.12).

Theorem 8.7. *We have, for $u(t)$ the corresponding solution in $C([0, T(x)), X)$,*

$$2L(\|F(0)\| + 2\|u(t)\|) \geq \frac{1}{T(x) - t} - 2 \tag{8.14}$$

for all $t \in [0, T(x))$. We have the alternatives

(1) either $T(x) = +\infty$;

(2) or if $T(x) < +\infty$ then $\lim_{t \nearrow T(x)} \|u(t)\| = +\infty$.

Proof. First of all it is obvious that if $T(x) < +\infty$ then by (15.10)

$$\lim_{t \nearrow T(x)} L(\|F(0)\| + 2\|u(t)\|) = +\infty \Rightarrow \lim_{t \nearrow T(x)} \|u(t)\| = +\infty$$

where the implication follows from the fact that $M \to L(M)$ is an increasing function.

We are left with the proof of (15.10), which is clearly true if $T(x) = \infty$. Now suppose that $T(x) < \infty$ and that (15.10) is false. This means that there exists a $t \in [0, T(x))$ with

$$\frac{1}{T_M} - 2 = 2L(\|F(0)\| + 2\|u(t)\|) < \frac{1}{T(x) - t} - 2 \Rightarrow T(x) - t < T_M$$

for $M = \|u(t)\|$, where we recall $T_M := \frac{1}{2L(2M + \|F(0)\|) + 2}$ in (8.13). Consider now $v \in C^0([0, T_M], X)$ the solution of

$$v(s) = u(t) + \int_0^s F(v(s'))ds' \text{ for all } s \in [0, T_M].$$

which exists by the previous Proposition 8.5. Then define

$$w(s) := \begin{cases} u(s) \text{ for } s \in [0, t] \\ v(s - t) \text{ for } s \in [t, t + T_M]. \end{cases}$$

We claim that $w \in C^0([0, t + T_M], X)$ is a solution of (8.12). In $[0, t]$ this is obvious since in $w = u$ in $[0, t]$ and $u \in C^0([0, t], X)$ is a solution of (8.12). Let now $s \in (t, t + T_M]$. We have

$$\begin{aligned} w(s) = v(s - t) &= u(t) + \int_0^{s-t} F(v(s'))ds' \\ &= x + \int_0^t F(u(s'))ds' + \int_0^{s-t} F(v(s'))ds' \\ &= x + \int_0^t F(\underbrace{u(s')}_{w(s')})ds' + \int_t^s F(\underbrace{v(s' - t)}_{w(s')})ds' \\ &= x + \int_0^s F(w(s'))ds. \end{aligned}$$

□

8.1 Proof of Theorem 8.2

We will need the following elementary lemma.

Lemma 8.8. *Let $d = 2, 3$. Then the trilinear form*

$$(u, v, \varphi) \in (C_c^\infty(\mathbb{R}^d))^d \times (C_c^\infty(\mathbb{R}^d))^d \times (C_c^\infty(\mathbb{R}^d))^d \to \langle div(u \otimes v), \varphi \rangle_{L^2} \in \mathbb{R} \tag{8.15}$$

extends into a unique bounded trilinear form $(H^1(\mathbb{R}^d))^d \times (H^1(\mathbb{R}^d))^d \times (H^1(\mathbb{R}^d))^d$ *which satisfies for a fixed* C

$$\langle div(u \otimes v), \varphi \rangle_{L^2} \le C \|\nabla u\|_{L^2}^{\frac{d}{4}} \|\nabla v\|_{L^2}^{\frac{d}{4}} \|u\|_{L^2}^{1-\frac{d}{4}} \|v\|_{L^2}^{1-\frac{d}{4}} \|\nabla \varphi\|_{L^2} \tag{8.16}$$

If furthermore $div\ u = 0$ *then*

$$\langle div(u \otimes v), v \rangle_{L^2} = 0. \tag{8.17}$$

Proof. Recall that from (8.2) we have $\mathrm{div}(u\otimes v)^j := \sum_{k=1}^d \partial_k(u^k v^j)$. Then for fields like in (8.15) we have

$$\langle \mathrm{div}(u \otimes v), \varphi \rangle_{L^2} = \sum_{j=1}^d \langle \mathrm{div}(u \otimes v)^j, \varphi^j \rangle_{L^2} = \sum_{j=1}^d \langle \sum_{k=1}^d \partial_k(u^k v^j), \varphi^j \rangle_{L^2} = -\sum_{j=1}^d \sum_{k=1}^d \langle u^k v^j, \partial_k \varphi^j \rangle_{L^2}.$$

Now the r.h.s. can be bounded by

$$|\langle u^k v^j, \partial_k \varphi^j \rangle_{L^2}| \le \|u^k v^j\|_{L^2} \|\nabla \varphi\|_{L^2} \le \|u^k\|_{L^4} \|v^j\|_{L^4} \|\nabla \varphi\|_{L^2}.$$

Finally, we apply Gagliardo-Nirenberg inequality writing

$$\|u^k\|_{L^4} \le C \|\nabla u^k\|_{L^2}^{\frac{d}{4}} \|u^k\|_{L^2}^{1-\frac{d}{4}}.$$

The same equality holds for v^j. Then we obtain (8.16), obviously with a different C. This implies that that the form in (8.15) is continuous, and by density of $C_c^\infty(\mathbb{R}^d)$ in $H^1(\mathbb{R}^d)$ it extends in a unique way.
Next, we write for $\varphi = v$

$$\begin{aligned} \langle \mathrm{div}(u \otimes v), v \rangle_{L^2} &= -\sum_{j=1}^d \sum_{k=1}^d \langle u^k v^j, \partial_k v^j \rangle_{L^2} \\ &= -2^{-1} \sum_{j=1}^d \sum_{k=1}^d \langle u^k, \partial_k (v^j)^2 \rangle_{L^2} = 2^{-1} \sum_{j=1}^d \langle (\mathrm{div} u) v^j, v^j \rangle_{L^2} = 0. \end{aligned}$$

Notice that this formal computation (the Leibnitz rule used for the 2nd equality requires some explaining) is certainly rigorous for $v \in (C_c^\infty(\mathbb{R}^d))^d$. On the other hand inequality (8.16) yields (8.17) by a density argument also for $v \in (H^1(\mathbb{R}^d))^d$. □

Remark 8.9. Notice that $u, v \in (H^1(\mathbb{R}^d))^d$ implies $\mathrm{div}(u \otimes v) \in (L^1(\mathbb{R}^d))^d$. Indeed we have

$$\mathrm{div}(u \otimes v)^j = \sum_{k=1}^d \partial_k(u^k v^j) = \sum_{k=1}^d (v^j \partial_k u^k + u^k \partial_k v^j) \tag{8.18}$$

where the above product rule can be proved by taking sequences $(C_c^\infty(\mathbb{R}^d))^d \ni u_n \overset{n\to\infty}{\to} u$ in $(H^1(\mathbb{R}^d))^d$ and $(C_c^\infty(\mathbb{R}^d))^d \ni v_n \overset{n\to\infty}{\to} v$ in $(H^1(\mathbb{R}^d))^d$. Then clearly for $\psi \in (\mathcal{S}^\infty(\mathbb{R}^d))^d$ summing on double indexes

$$\begin{aligned}\langle \partial_k(u^k v^j), \psi^j\rangle &= -\langle u^k v^j, \partial_k \psi^j\rangle = -\lim_{n\to\infty}\langle u_n^k v_n^j, \partial_k\psi^j\rangle \\ &= \lim_{n\to\infty}\Big(\langle v_n^j\partial_k u_n^k, \psi^j + u_n^k\partial_k v_n^j, \psi^j\rangle\Big) = \langle v^j\partial_k u^k + u^k\partial_k v^j, \psi^j\rangle\end{aligned}$$

and this yields (8.18).
Hence $\mathfrak{F} := \mathcal{F}(\mathrm{div}(u\otimes v)) \in (L^\infty(\mathbb{R}^d))^d \subset (L^1_{loc}(\mathbb{R}^d))^d$. Furthermore, (8.16) implies that $\mathfrak{F} \in (L^2(\mathbb{R}^d, |\xi|^{-2}d\xi))^d$. Indeed the bilinear map

$$\langle\cdot,\cdot\rangle_{L^2(\mathbb{R}^d,d\xi)} : L^2(\mathbb{R}^d, |\xi|^{-2}d\xi)\times L^2(\mathbb{R}^d,|\xi|^2 d\xi)\to\mathbb{R}$$

can be used to define an embedding

$$L^2(\mathbb{R}^d,|\xi|^{-2}d\xi)\hookrightarrow (L^2(\mathbb{R}^d,|\xi|^2d\xi))'$$

by $f\to\langle f,\cdot\rangle_{L^2(\mathbb{R}^d,d\xi)}$. Furthermore we have the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^d,|\xi|^{-2}d\xi) & \overset{f\to\langle f,\cdot\rangle_{L^2(\mathbb{R}^d,d\xi)}}{\to} & (L^2(\mathbb{R}^d,|\xi|^2d\xi))' \\ f\to|\xi|^{-1}f\downarrow & & \uparrow \\ L^2(\mathbb{R}^d,d\xi) & \overset{h\to\langle h,\cdot\rangle_{L^2(\mathbb{R}^d,d\xi)}}{\to} & (L^2(\mathbb{R}^d,d\xi))' \end{array} \tag{8.19}$$

where the $\uparrow$ is the map $(L^2(\mathbb{R}^d,d\xi))'\to(L^2(\mathbb{R}^d,|\xi|^2d\xi))'$ given by $\langle g,\cdot\rangle_{L^2(\mathbb{R}^d,d\xi)}\to\langle|\xi|^{-1}g,\cdot\rangle_{L^2(\mathbb{R}^d,|\xi|^2d\xi)}$ where the latter map is an isomorphism since it closes the diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^d,d\xi) & \overset{f\to|\xi|^{-1}f}{\to} & L^2(\mathbb{R}^d,|\xi|^2d\xi) \\ f\to\langle f,\cdot\rangle_{L^2(\mathbb{R}^d,d\xi)}\downarrow & & \downarrow f\to\langle f,\cdot\rangle_{L^2(\mathbb{R}^d,|\xi|^2d\xi)} \\ (L^2(\mathbb{R}^d,d\xi))' \dashrightarrow & & (L^2(\mathbb{R}^d,|\xi|^2d\xi))' \end{array}$$

Since the other maps in (8.19) are isomorphisms, also the 1st line in (8.19) is an isomorphism. Hence we conclude that $\mathfrak{F}\in(L^2(\mathbb{R}^d,|\xi|^{-2}d\xi))^d$ since $\langle\mathfrak{F},\cdot\rangle_{L^2(\mathbb{R}^d,d\xi)}\in(L^2(\mathbb{R}^d,|\xi|^2d\xi))'$ by (8.16).
So we conclude $\mathrm{div}(u\otimes v)\in(\dot{H}^{-1}(\mathbb{R}^d))^d$. Now applying Lemma 3.6 we have in $(\dot{H}^{-1}(\mathbb{R}^d))^d$

$$\mathrm{div}(u\otimes u) = \mathbb{P}\mathrm{div}(u\otimes u) - \nabla p$$

for a function $p\in L^2(\mathbb{R}^d)$ which is what we get in the r.h.s. in (8.1).

We consider now the following truncation of the NS equation.

$$\begin{cases} (u_n)_t + \mathbf{P}_n\mathbb{P}\mathrm{div}(\mathbf{P}_n u_n\otimes\mathbf{P}_n u_n) - \nu(\mathbf{P}_n\triangle)u_n = 0 \\ u_n(0,x) = \mathbf{P}_n u_0(x). \end{cases} \quad (t,x)\in[0,\infty)\times\mathbb{R}^d \tag{8.20}$$

Lemma 8.10. *For any n the system* (8.3) *admits exactly one solution*

$$u_n \in C^\infty([0,\infty),(H^N(\mathbb{R}^d))^d) \text{ for any } N \in \mathbb{N}\cup\{0\}.$$

Furthermore we have $\mathbb{P}u_n = u_n$ and $\mathbf{P}_n u_n = u_n$.

Proof. First of all, we consider for any n local existence. Set

$$F_n(v) := \nu(\mathbf{P}_n\triangle)v - \mathbf{P}_n\mathbb{P}\mathrm{div}(\mathbf{P}_n v \otimes \mathbf{P}_n v).$$

Then we have

$$\|F_n(v)\|_{(H^N(\mathbb{R}^d))^d} \le \|\nu(\mathbf{P}_n\triangle)v\|_{(H^N(\mathbb{R}^d))^d} + \|\mathbf{P}_n\mathbb{P}\mathrm{div}(\mathbf{P}_n v\otimes \mathbf{P}_n v)\|_{(H^N(\mathbb{R}^d))^d}$$

with

$$\|\nu(\mathbf{P}_n\triangle)v\|_{(H^N(\mathbb{R}^d))^d} \le \nu n^{2+N}\|v\|_{(L^2(\mathbb{R}^d))^d}$$

and

$$\begin{aligned}\|\mathbf{P}_n\mathbb{P}\mathrm{div}(\mathbf{P}_n v\otimes \mathbf{P}_n v)\|_{(H^N(\mathbb{R}^d))^d} &\lesssim n^{N+1}\|\mathbf{P}_n v\otimes\mathbf{P}_n v\|_{L^2} \lesssim n^{N+1}\|\mathbf{P}_n v\|_{L^4}^2\\ &\lesssim n^{N+1}\|\nabla\mathbf{P}_n v\|_{L^2}^{\frac{d}{4}}\|\nabla\mathbf{P}_n v\|_{L^2}^{\frac{d}{4}}\|\mathbf{P}_n v\|_{L^2}^{1-\frac{d}{4}}\|\mathbf{P}_n v\|_{L^2}^{1-\frac{d}{4}}\\ &\lesssim n^{N+1+\frac{d}{2}}\|v\|_{L^2}^2 \quad .\end{aligned}$$

So for some constant $C_{n,N}$ we have

$$\|F_n(v)\|_{(H^N(\mathbb{R}^d))^d} \le C_{n,N}(\|v\|_{(L^2(\mathbb{R}^d))^d} + \|v\|_{(L^2(\mathbb{R}^d))^d}^2).$$

Furthermore, as a sum of a bounded linear operator and a bounded quadratic form each F_n is a locally Lipchitz function. Then for any n and N we know that (8.3) admits a solution $u_n \in C^1([0,T_{n,N}),(H^N(\mathbb{R}^d))^d))$ for some maximal $T_{n,N} > 0$. Furthermore we must have

$$\lim_{t\nearrow T_{n,N}} \|u_n(t)\|_{(H^N(\mathbb{R}^d))^d} = +\infty \text{ if } T_{n,N} < \infty. \tag{8.21}$$

Next we have $u_n = \mathbb{P}u_n$ since applying $1-\mathbb{P}$ to (8.20)

$$\begin{cases}((1-\mathbb{P})u_n)_t - \nu(\mathbf{P}_n\triangle)(1-\mathbb{P})u_n = 0\\ (1-\mathbb{P})u_n(0,x) = 0\end{cases} \Rightarrow (1-\mathbb{P})u_n = 0,$$

and $u_n = \mathbf{P}_n u_n$ since applying $1-\mathbf{P}_n$ to (8.20)

$$\begin{cases}((1-\mathbf{P}_n)u_n)_t = 0\\ (1-\mathbf{P}_n)u_n(0,x) = 0\end{cases} \Rightarrow (1-\mathbf{P}_n)u_n = 0$$

Now we show that the *finite time blow up* in (8.21) cannot occur for any (n,N) (in fact, the following argument proves that also *infinite time blow up*, that is (8.21) but with $T_{n,N}=\infty$, cannot occur).

Let us consider (8.21) first in the case $N = 0$. When we apply $\langle \cdot, u_n \rangle_{L^2}$ to the 1st line in (8.3) and get

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{L^2}^2 + \langle \mathbf{P}_n \mathbb{P}\mathrm{div}(u_n \otimes u_n), u_n \rangle_{L^2} - \nu \langle \triangle u_n, u_n \rangle_{L^2} = 0 \, .$$

Notice that summing on repeated indexes $\langle \triangle u_n, \varphi \rangle_{L^2} = -\langle \partial_j u_n, \partial_j \varphi \rangle_{L^2}$ for all $\varphi \in (C_0^\infty(\mathbb{R}^d))^d$ and since this is dense in $(H^1(\mathbb{R}^d))^d$ and both sides define bounded functionals in $(H^1(\mathbb{R}^d))^d$, we conclude

$$\nu \langle \triangle u_n, u_n \rangle_{L^2} = -\nu \|\nabla u_n\|_{L^2}^2 .$$

Next, using $\mathbb{P}^* = \mathbb{P}$, $\mathbf{P}_n^* = \mathbf{P}_n$ and (8.17), we have

$$\langle \mathbf{P}_n \mathbb{P}\mathrm{div}(u_n \otimes u_n), u_n \rangle_{L^2} = \langle \mathrm{div}(u_n \otimes u_n), u_n \rangle_{L^2} = 0.$$

Hence we conclude

$$\frac{1}{2}\frac{d}{dt}\|u_n\|_{(L^2(\mathbb{R}^d))^d}^2 + \nu \|\nabla u_n\|_{(L^2(\mathbb{R}^d))^{d^2}}^2 = 0$$

and we obtain

$$\|u_n(t)\|_{(L^2(\mathbb{R}^d))^d}^2 + 2\nu \int_0^t \|\nabla u_n(t')\|_{(L^2(\mathbb{R}^d))^{d^2}}^2 dt' = \|\mathbf{P}_n u_0\|_{(L^2(\mathbb{R}^d))^d}^2 . \tag{8.22}$$

In particular this yields the bound $\|u_n(t)\|_{L^2} \le \|\mathbf{P}_n u_0\|_{L^2}$ for all $t \in [0, T_{n,0})$ and by (8.21) we conclude that the lifespan is $T_{n,0} = \infty$ for all $n \in \mathbb{N}$. This proves the case $N = 0$ in Lemma 8.10.

Consider now the case $N \in \mathbb{N}$. If $u_n \in C^1([0, T_{n,N}), (H^N(\mathbb{R}^d))^d)$ with $T_{n,N} < \infty$ is a maximal solution, obviously it is the restriction in $[0, T_{n,N})$ of a solution $u_n \in C^1([0,\infty), (L^2(\mathbb{R}^d))^d)$. On the other hand, the blow up (8.21) is impossible because otherwise we would have

$$\infty = \lim_{t \nearrow T_{n,N}} \|u_n(t)\|_{(H^N(\mathbb{R}^d))^d} \le n^N \lim_{t \nearrow T_{n,N}} \|u_n(t)\|_{(L^2(\mathbb{R}^d))^d} \le n^N \|\mathbf{P}_n u_0\|_{L^2} < \infty$$

which is absurd. Hence the lifespan is $T_{n,N} = \infty$ for all $n \in \mathbb{N}$ and $N \in \mathbb{N} \cup \{0\}$.

□

8.1.1 Compactness properties of $\{u_n\}_{n \in \mathbb{N}}$

Now we consider the sequence of solutions $\{u_n\}_{n \in \mathbb{N}}$ of solutions of (8.3). We will prove the following result.

Proposition 8.11. *There exists a* $u \in L^\infty(\mathbb{R}_+, (L^2(\mathbb{R}^d))^d) \cap L^2_{loc}(\mathbb{R}_+, H^1(\mathbb{R}^d))^d)$ *with* $\mathrm{div}\, u = 0$ *and a subsequence of* $\{u_n\}_{n \in \mathbb{N}}$ *such that for any* $T > 0$ *and any compact subset* $K \subset \mathbb{R}^d$ *we have (after extracting this subsequence)*

$$\lim_{n \to \infty} \int_{[0,T] \times K} |u_n(t,x) - u(t,x)|^2 dt dx = 0. \tag{8.23}$$

Moreover, for all vector fields $\Psi \in L^2([0,T],(H^1(\mathbb{R}^d))^d)$ and all $\Phi \in L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ we have

$$\lim_{n\to\infty}\int_{[0,T]\times\mathbb{R}^d}(u_n(t,x)-u(t,x))\cdot\Phi(t,x)dtdx=0, \tag{8.24}$$

$$\lim_{n\to\infty}\sum_{j,k=1}^{d}\int_{[0,T]\times\mathbb{R}^d}\partial_k(u_n^j(t,x)-u^j(t,x))\partial_k\Psi^j(t,x)dtdx=0. \tag{8.25}$$

Finally, for any $\psi \in C^0([0,\infty),(H^1(\mathbb{R}^d))^d)$ we have $\langle u_n,\psi\rangle_{(L^2(\mathbb{R}^d))^d} \to \langle u,\psi\rangle_{(L^2(\mathbb{R}^d))^d}$ in $L^\infty_{loc}([0,\infty))$, that is

$$\lim_{n\to\infty}\|\langle u_n(t)-u(t),\psi(t)\rangle\|_{L^\infty([0,T])}=0 \text{ for any } T. \tag{8.26}$$

Proof. Fix an arbitrary $T>0$ and an arbitrary compact subset K of $\mathbb{R}^d$.

Claim 8.12. The set formed by the elements of the sequence $\{u_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T]\times K,\mathbb{R}^d)$.

Proof of Claim 8.12. Notice that (8.22) implies that $u_n \in L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ for all n. We will show the following statement, which is equivalent to Claim 8.12.

Claim 8.13. For any $\varepsilon>0$ there exists a finite family of balls of the space $L^2([0,T]\times K,\mathbb{R}^d)$ which have radius ε and whose union covers the set $\{u_n\}_{n\in\mathbb{N}}$.

Proof of Claim 8.13. First of all, if we want to approximate $\{u_n\}_{n\in\mathbb{N}}$ with $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ for a fixed n_0, we can use the fact that for any n_0 and any n we have

$$\|u_n-\mathbf{P}_{n_0}u_n\|^2_{L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)}=\int_0^T\|u_n-\mathbf{P}_{n_0}u_n\|^2_{(L^2(\mathbb{R}^d))^d}dt$$
$$\le n_0^{-2}\int_0^T\|\nabla u_n-\nabla\mathbf{P}_{n_0}u_n\|^2_{(L^2(\mathbb{R}^d))^{d^2}}dt\le n_0^{-2}\int_0^T\|\nabla u_n\|^2_{(L^2(\mathbb{R}^d))^{d^2}}dt\le n_0^{-2}\|u_0\|^2_{(L^2(\mathbb{R}^d))^d}.$$

Hence we can choose n_0 large enough s.t.

$$\|u_n-\mathbf{P}_{n_0}u_n\|_{L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)}<\frac{\varepsilon}{2} \text{ for all } n\in\mathbb{N}. \tag{8.27}$$

Now consider $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$. Then Claim 8.13 is a consequence of

Claim 8.14. $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2([0,T]\times K,\mathbb{R}^d)$.

Indeed Claim 8.14 implies that for any $\varepsilon>0$ there is a finite number of balls $B_{L^2([0,T]\times K,\mathbb{R}^d)}(f_j,\frac{\varepsilon}{2})$ which cover $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$. Hence by (8.27) we conclude that for any $\varepsilon>0$ the balls $B_{L^2([0,T]\times K,\mathbb{R}^d)}(f_j,\varepsilon)$ cover $\{u_n\}_{n\in\mathbb{N}}$ and so we get Claim 8.13.

Proof of Claim 8.14. It will be a consequence of the following stronger claim.

Claim 8.15. $\{\mathbf{P}_{n_0}u_n\}_{n\in\mathbb{N}}$ is relatively compact in $C^0([0,T],(L^2(K))^d)\subset L^\infty([0,T],(L^2(K))^d)$.

Proof of Claim 8.15. To get this result we want to apply the Ascoli–Arzela Theorem (for which a sufficient condition for a sequence of continuous functions $f_n : K \to X$, with K compact and separable metric space and X a complete metric space, to admit a subsequence that converges uniformly to a continuous function $f : K \to X$ is that it is equicontinuous and $\{f_n(k)\}_n$ is relatively compact for any $k \in K$ [1]). So it is enough to show that $\{\mathbf{P}_{n_0} u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions in $C^0([0,T],(L^2(K))^d)$ and that for any $t \in [0,T]$ the sequence $\{\mathbf{P}_{n_0} u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in $(L^2(K))^d$.
First of all we want to show that $\{\mathbf{P}_{n_0} u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions in $C^0([0,T],(L^2(K))^d)$. This will follow from Hölder inequality (since $\frac{4}{d} > 1$ if $d = 2,3$) and from the following claim.

Claim 8.16. There exists a fixed constant C s.t.

$$\|(\mathbf{P}_{n_0} u_n)_t\|_{L^{\frac{4}{d}}([0,T],(L^2(\mathbb{R}^d))^d)} \leq C \text{ for all } n.$$

Proof of Claim 8.16. We apply $\mathbf{P}_{n_0}$ to (8.3) and we obtain

$$(\mathbf{P}_{n_0} u_n)_t = -\mathbf{P}_{n_0}\mathbf{P}_n\mathbb{P}\mathrm{div}(u_n \otimes u_n) + \nu\mathbf{P}_{n_0}\triangle u_n.$$

We have

$$\|\nu\mathbf{P}_{n_0}\triangle u_n\|_{(L^2(\mathbb{R}^d))^d} \leq \nu n_0^2 \|u_n\|_{(L^2(\mathbb{R}^d))^d} \leq \nu n_0^2 \|u_0\|_{(L^2(\mathbb{R}^d))^d}$$

and, by the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
&\|\mathbf{P}_{n_0}\mathbf{P}_n\mathbb{P}\mathrm{div}(u_n \otimes u_n)\|_{(L^2(\mathbb{R}^d))^d} \leq \|\mathbf{P}_{n_0}\mathrm{div}(u_n \otimes u_n)\|_{(L^2(\mathbb{R}^d))^d} \\
&= \sum_{j=1}^{d} \|\mathbf{P}_{n_0} \sum_{k=1}^{d} \partial_k(u_n^k u_n^j)\|_{L^2(\mathbb{R}^d)} \leq n_0 \sum_{j,k=1}^{d} \|u_n^k u_n^j\|_{L^2(\mathbb{R}^d)} \\
&\leq C n_0 \|u_n\|^2_{(L^4(\mathbb{R}^d))^d} \leq C' n_0 \left(\|\nabla u_n\|_{L^2}^{\frac{d}{4}} \|u_n\|_{L^2}^{1-\frac{d}{4}} \right)^2 .
\end{aligned}$$

Then we have

$$\begin{aligned}
\|(\mathbf{P}_{n_0} u_n)_t\|_{L^{\frac{4}{d}}([0,T],(L^2(\mathbb{R}^d))^d)} &\leq \nu n_0^2 T^{\frac{d}{4}} \|u_0\|_{(L^2(\mathbb{R}^d))^d} \\
&\quad + C' n_0 \|u_n\|^{2\left(1-\frac{d}{4}\right)}_{L^\infty([0,T],(L^2(\mathbb{R}^d))^d)} \|\nabla u_n\|^{\frac{d}{2}}_{L^2([0,T],L^2)} \leq C
\end{aligned}$$

for some constant C independent of n by the energy equality (8.22) and the fact that $\|\mathbf{P}_n u_0\|_{(L^2(\mathbb{R}^d))^d} \leq \|u_0\|_{(L^2(\mathbb{R}^d))^d}$ for all n.
Hence we have concluded the proof that $\{\mathbf{P}_{n_0} u_n\}_{n\in\mathbb{N}}$ is a sequence of equicontinuous functions in $C^0([0,T],(L^2(\mathbb{R}^d))^d)$.

[1] The proof goes as follows. One first considers a dense countable subset $\mathcal{N}$ of K. Then by a diagonal argument, one considers a subsequence $\{f_{n_m}\}$ s.t. $\{f_{n_m}(k)\}$ converges for any $k \in \mathcal{N}$ to a limit that we denote by $f(k)$. Using equicontinuity and the completeness of X it is easy to see that $\{f_{n_m}(k)\}$ converges for any $k \in K$. We denote again by $f(k)$ the limit. Finally, using equicontinuity we conclude that $f : K \to X$ is continuous

To complete the proof of Claim 8.15 we need to show that for any $t \in [0,T]$ the sequence $\{\mathbf{P}_{n_0} u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in $(L^2(K))^d$. It is here that we will exploit the fact that K is a compact subspace of $\mathbb{R}^d$.
We know that $\{\mathbf{P}_{n_0} u_n(t)\}_{n\in\mathbb{N}}$ is a bounded sequence in $(H^1(\mathbb{R}^d))^d$ for any $t \in [0,T]$. This follows immediately from $\|\mathbf{P}_{n_0} u_n(t)\|_{H^1} \le n_0 \|u_n(t)\|_{L^2} \le n_0 \|u_0\|_{L^2}$, which follows from the energy inequality (8.22) which guarantees $\|u_n(t)\|_{L^2} \le \|u_0\|_{L^2}$. We recall now that

Claim 8.17. The restriction map $H^1(\mathbb{R}^d) \to L^2(K)$ is compact for any compact K .

Sketch of proof Indeed this is equivalent at showing that

$$\mathcal{T} f := \chi_K \mathcal{F}^* \left(\frac{f}{\langle \xi \rangle} \right) \text{ is compact as } L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d).$$

We have $\mathcal{T} f = \int \mathcal{K}(x,\xi) f(\xi) d\xi$ with integral *kernel* $\mathcal{K}(x,\xi) := \chi_K(x) \langle \xi \rangle^{-1} e^{-\mathrm{i}x\cdot\xi}$. It is easy to see that $\mathcal{T}_n \overset{n\to\infty}{\to} \mathcal{T}$ in the operator norm where the $\mathcal{T}_n$ has kernel $\mathcal{K}_n(x,\xi) := \chi_K(x)\langle \xi \rangle^{-1} e^{-\mathrm{i}x\cdot\xi} \chi_{B(0,n)}(\xi)$. Since $\mathcal{K}_n \in L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$, it follows that $\mathcal{T}_n$ is a Hilbert–Schmidt operator, with $\|\mathcal{T}_n\|_{HS} := \|\mathcal{K}_n\|_{L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)}$. It is easy to show that $\|\mathcal{T}_n\|_{L^2 \to L^2} \le \|\mathcal{T}_n\|_{HS}$. $\mathcal{K}_n$ is the limit in $L^2(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$ of elements in $L^2(\mathbb{R}^d_x) \otimes L^2(\mathbb{R}^d_\xi)$. The latter ones are integral kernels of finite rank operators and their operators converge in the Hilbert–Schmidt norm, and so also in the $\|\cdot\|_{L^2\to L^2}$ norm, to $\mathcal{T}_n$. We conclude that there is a sequence of finite rank operators which converges in the operator norm to $\mathcal{T}$, which then is compact. $\square$
It follows that $\{\mathbf{P}_{n_0} u_n(t)\}_{n\in\mathbb{N}}$ is relatively compact in $(L^2(K))^d$ for any $t \in [0,T]$.
Hence the hypotheses of the Ascoli–Arzela Theorem have been checked and we can conclude that Claim 8.15, that is the claim that $\{\mathbf{P}_{n_0} u_n\}_{n\in\mathbb{N}}$ is relatively compact in $C^0([0,T],(L^2(K))^d)$, is true.
Hence there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ (and it is not restrictive to assume this is true for the whole sequence) which converges to an $u \in L^2([0,T] \times K, \mathbb{R}^d)$. By a diagonal argument, we can assume that this is true for any compact $K \subset \mathbb{R}^d$ and any $T > 0$. This yields (8.23). Notice that this implies

$$u_n \to u \text{ in } \mathcal{D}'((0,T) \times \mathbb{R}^d, \mathbb{R}^d). \tag{8.28}$$

We claim now that $u \in L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ and that

$$u_n \rightharpoonup u \text{ in } L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d) \tag{8.29}$$

(convergence in the weak topology). Indeed, since from (8.22) we have that $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$, it follows that up to a subsequence we have $u_n \rightharpoonup v$ for some $v \in L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$. Then (8.28) implies that $v = u$ as distributions in $\mathcal{D}'((0,T) \times \mathbb{R}^d, \mathbb{R}^d)$. This implies that $u \in L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d)$ with $u = v$.
In particular this implies

$$\lim_{n\to\infty} \int_{[0,T]\times\mathbb{R}^d} (u_n(t,x) - u(t,x)) \cdot \Phi(t,x) dt dx = 0 \text{ for all } \Phi \in L^2([0,T] \times \mathbb{R}^d, \mathbb{R}^d),$$

that is (8.24). Notice that, for $\Phi(t,x)=\chi(t)\nabla\psi(x)$ we have from the above limit

$$\int_{[0,T]} dt\chi(t)\int_{\mathbb{R}^d}\mathrm{div}_x u(t,x)\psi(x)dx=0 \text{ for all } \psi\in C_c^\infty(\mathbb{R}^d,\mathbb{R}) \text{ and any } \chi\in C^\infty([0,T],\mathbb{R}),$$

This implies that

$$\int_{\mathbb{R}^d}\mathrm{div}_x u(t,x)\psi(x)dx=0 \text{ for a.e. } t.$$

In fact, for the argument below, which proves (8.26) and is independent of what we are discussing right here, the integral on the l.h.s. is continuous in t. This integral equals 0 for all t, and not just for a.a. t. Since this is true for all t and for all $\psi\in C_c^\infty(\mathbb{R}^d,\mathbb{R})$, it follows that $\mathrm{div}_x u(t,x)=0$ for all t.

We now turn to the proof of (8.25).
By (8.22) we know that $\{\nabla u_n\}_{n\in\mathbb{N}}$ is bounded in $L^2((0,T)\times\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)$. This implies that up to a subsequence there exists $V\in L^2((0,T)\times\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)$ s.t. $\nabla u_n\rightharpoonup V$. On the other hand (8.29) implies $u_n\to u$ in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$. This in turn implies $\partial_j u_n\to\partial_j u$ in $\mathcal{D}'((0,T)\times\mathbb{R}^d,\mathbb{R}^d)$ for any $j=1,...,d$. Hence $\nabla u=V$ in $\mathcal{D}'((0,T)\times\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)$, $\nabla u\in L^2((0,T)\times\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)$ and $\nabla u=V$ in $L^2((0,T)\times\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)$. This proves (8.25).
Notice also that, up to a subsequence, $u_n(t,x)\xrightarrow{n\to+\infty}u(t,x)$ for almost any (t,x) and $\nabla u_n\rightharpoonup\nabla u$ as $n\to+\infty$ in $L^2((0,T)\times\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)$. In particular, this implies that, up to a subsequence, for almost any t we have the above limits for a.e. x. Then the energy inequalities (8.22) imply by Fathou

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)}+2\nu\int_0^t\|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)}dt'\le\|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{8.6}$$

We turn now to the proof of (8.26).
Fix a function $\psi\in C^0([0,\infty),H^1(\mathbb{R}^d,\mathbb{R}^d))$. For a given n_0 consider

$$g_n(t):=\langle u_n(t),\psi(t)\rangle_{(L^2(\mathbb{R}^d))^d} \text{ and } g_n^{(n_0)}(t):=\langle\mathbf{P}_{n_0}u_n(t),\psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}.$$

Then for any $\epsilon>0$ and any fixed $T>0$ there exists n_0 s.t.

$$\|(\mathbf{P}_{n_0}-1)\psi(t)\|_{L^\infty([0,T],(L^2(\mathbb{R}^d))^d)}<\epsilon.$$

This and $\|u_n(t)\|_{L^\infty([0,T],(L^2(\mathbb{R}^d))^d)}\le\|u_0\|_{(L^2(\mathbb{R}^d))^d}$ imply

$$\|g_n-g_n^{(n_0)}\|_{L^\infty([0,T]}\le\|u_0\|_{(L^2(\mathbb{R}^d))^d}\epsilon.$$

Furthermore, for any fixed $T>0$ there exists a compact K s.t.

$$\|\psi(t)\|_{L^\infty([0,T],(L^2(\mathbb{R}^d\backslash K))^d)}<\epsilon.$$

Then, if we set $g_n^{(n_0,K)}(t):=\langle\mathbf{P}_{n_0}u_n(t),\psi(t)\rangle_{(L^2(K)))^d}$ we have

$$\|g_n^{(n_0,K)}-g_n^{(n_0)}\|_{L^\infty([0,T]}\le\|u_0\|_{(L^2(\mathbb{R}^d))^d}\epsilon.$$

We claim that

$$\mathbf{P}_{n_0} u_n \to \mathbf{P}_{n_0} u \text{ in } C^0([0,T],(L^2(K))^d). \tag{8.30}$$

Indeed, by Claim 8.15, and by a diagonal argument, we know that there exists a v s.t. $\mathbf{P}_{n_0} u_n \to v$ in $C^0([0,T],(L^2(K))^d)$ for any T and K. It is easy to conclude that $v \in L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ and that $\mathbf{P}_{n_0} u_n \rightharpoonup v$ therein. On the other hand, we know that $u_n \to u$ in $L^2([0,T]\times K,\mathbb{R}^d)$, and that $u_n \rightharpoonup u$ in $L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$. In turn. this implies $\mathbf{P}_{n_0} u_n \rightharpoonup \mathbf{P}_{n_0} u$ in $L^2([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$. But then this implies $v = \mathbf{P}_{n_0} u$, and so we get (8.30).

In turn, (8.30) implies

$$\{g_n^{(n_0,K)}\}_n = \langle \mathbf{P}_{n_0} u_n(t), \psi(t)\rangle_{(L^2(K)))^d} \to \langle \mathbf{P}_{n_0} u(t), \psi(t)\rangle_{(L^2(K)))^d} \text{ in } C^0([0,T]).$$

But then also

$$\begin{aligned}
&\|\langle u_n(t),\psi(t)\rangle_{(L^2(\mathbb{R}^d))^d} - \langle u(t),\psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}\|_{L^\infty([0,T]} \\
&\le \|\langle \mathbf{P}_{n_0} u_n(t),\psi(t)\rangle_{(L^2(K))^d} - \langle \mathbf{P}_{n_0} u(t),\psi(t)\rangle_{(L^2(K))^d}\|_{L^\infty([0,T]} + 2\|u_0\|_{(L^2(\mathbb{R}^d))^d}\epsilon \\
&+ \langle u(t),(1-\mathbf{P}_{n_0})\psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}\|_{L^\infty([0,T]} + \langle u(t),(1-\chi_K)\psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}\|_{L^\infty([0,T]} \le \\
&\le \|\langle \mathbf{P}_{n_0} u_n(t),\psi(t)\rangle_{(L^2(K))^d} - \langle \mathbf{P}_{n_0} u(t),\psi(t)\rangle_{(L^2(K))^d}\|_{L^\infty([0,T]} + 4\|u_0\|_{(L^2(\mathbb{R}^d))^d}\epsilon.
\end{aligned}$$

Since ϵ is arbitrarily small, it follows that we obtain that g_n converges to $\langle u(t),\psi(t)\rangle_{(L^2(\mathbb{R}^d))^d}$ in $L^\infty([0,T])$, and hence in $C^0([0,T])$. In particular we have shown that $u \in C^0([0,\infty), L^2_w(\mathbb{R}^d,\mathbb{R}^d))$. The proof of Proposition 8.11 is completed. □

8.1.2 End of the proof of Leray's Theorem 8.2

Proposition 8.11 has provided us with a function

$$u \in L^\infty([0,\infty), L^2(\mathbb{R}^d,\mathbb{R}^d)) \cap L^2_{loc}([0,\infty), H^1(\mathbb{R}^d,\mathbb{R}^d)) \cap C^0([0,\infty), L^2_w(\mathbb{R}^d,\mathbb{R}^d))$$

which satisfies the energy inequality

$$\|u(t)\|^2_{L^2(\mathbb{R}^d)} + 2\nu\int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^d)} dt' \le \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{8.6}$$

Our aim in this section is to prove that u is a weak solution in the sense of Definition 8.1. Let us consider $\Psi \in C^1([0,\infty), \mathbb{P}(H^1(\mathbb{R}^d))^d)$ and let us apply to (8.3) the inner product $\langle\cdot,\Psi\rangle_{L^2}$. Then we get

$$\langle (u_n)_t,\Psi\rangle_{(L^2(\mathbb{R}^d))^d} + \langle \mathbf{P}_n \mathbb{P}\mathrm{div}(u_n\otimes u_n),\Psi\rangle_{(L^2(\mathbb{R}^d))^d} - \nu\langle \triangle u_n,\Psi\rangle_{(L^2(\mathbb{R}^d))^d} = 0.$$

Hence

$$\frac{d}{dt}\langle u_n,\Psi\rangle_{(L^2(\mathbb{R}^d))^d} - \langle u_n,\Psi_t\rangle_{(L^2(\mathbb{R}^d))^d} + \langle \mathrm{div}(u_n\otimes u_n),\mathbf{P}_n\Psi\rangle_{(L^2(\mathbb{R}^d))^d} + \nu\langle \triangle u_n,\Psi\rangle_{(L^2(\mathbb{R}^d))^d} = 0.$$

So, integrating in t we get

$$\int_{\mathbb{R}^d} u_n(t,x)\cdot\Psi(t,x)dx = \int_{\mathbb{R}^d} \mathbf{P}_n u_0(x)\cdot\Psi(0,x)dx - \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x) : \nabla \mathbf{P}_n \Psi(s,x)dx$$
$$+ \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\cdot\Psi_t(s,x)dx - \nu \sum_{j,k}\int_0^t ds \int_{\mathbb{R}^d} \partial_k u_n^j(s,x)\partial_k \Psi^j(s,x)dx. \qquad (8.31)$$

By (8.26) for any t

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} u_n(t,x)\cdot\Psi(t,x)dx = \int_{\mathbb{R}^d} u(t,x)\cdot\Psi(t,x)dx. \qquad (8.32)$$

By the definition of $\mathbf{P}_n$ we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \mathbf{P}_n u_0(x)\cdot\Psi(0,x)dx = \int_{\mathbb{R}^d} u_0(x)\cdot\Psi(0,x)dx. \qquad (8.33)$$

By (8.24) we have

$$\lim_{n\to\infty} \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\cdot\Psi_t(s,x)dx = \int_0^t ds \int_{\mathbb{R}^d} u(s,x)\cdot\Psi_t(s,x)dx. \qquad (8.34)$$

By (8.25) we have

$$\lim_{n\to\infty} \nu \int_0^t ds \int_{\mathbb{R}^d} \partial_k u_n^j(s,x)\partial_k \Psi^j(s,x)dx = \nu \int_0^t ds \int_{\mathbb{R}^d} \partial_k u^j(s,x)\partial_k \Psi^j(s,x)dx. \qquad (8.35)$$

The above limits (8.32)–(8.35) are straightforward consequences of Proposition 8.11. By taking the limit in (8.31), Leray's Theorem will be a consequence of the following claim, which is the delicate point of this part of the proof.

Claim 8.18. We have

$$\lim_{n\to\infty} \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x) : \nabla \mathbf{P}_n \Psi(s,x)dx = \int_0^t ds \int_{\mathbb{R}^d} u(s,x)\otimes u(s,x) : \nabla \Psi(s,x)dx. \qquad (8.36)$$

Proof of Claim 8.18. The 1st step, algebraic, is to write

$$\int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x) : \nabla \mathbf{P}_n \Psi(s,x)dx = \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x) : \nabla \Psi(s,x)dx$$
$$+ \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x) : \nabla (\mathbf{P}_n \Psi(s,x) - \Psi(s,x))dx.$$

Claim 8.18 will be a consequence of

$$\lim_{n\to\infty} \int_0^t ds \int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x) : \nabla \Psi(s,x)dx = \int_0^t ds \int_{\mathbb{R}^d} u(s,x)\otimes u(s,x) : \nabla \Psi(s,x)dx. \qquad (8.37)$$

and of

$$\lim_{n\to\infty}\int_0^t ds\int_{\mathbb{R}^d} u_n(s,x)\otimes u_n(s,x):\nabla(\mathbf{P}_n\Psi(s,x)-\Psi(s,x))dx=0. \tag{8.38}$$

In order to prove (8.37)–(8.38) we observe that since $\Psi\in C^1([0,\infty),(H^1(\mathbb{R}^d))^d)$ for any $\varepsilon>0$ there is a compact set $K\subset\mathbb{R}^d$ s.t.

$$\sup_{s\in[0,T]}\|\nabla\Psi(s,\cdot)\|_{L^2(\mathbb{R}^d\setminus K)}<\varepsilon. \tag{8.39}$$

(8.39) is elementary to prove and it is assumed in the sequel. Now we show (8.37). By Hölder, (8.39), Gagliardo–Nirenberg and the energy equality (8.22) we have

$$\begin{aligned}
&|\int_0^t ds\int_{\mathbb{R}^d\setminus K} u_n(s,x)\otimes u_n(s,x):\nabla\Psi(s,x)dx|\le\int_0^T ds\|u_n\otimes u_n\|_{L^2(\mathbb{R}^d)}\|\nabla\Psi(s)\|_{L^2(\mathbb{R}^d\setminus K)}\\
&\le T^{\frac{4-d}{4}}\|u_n\otimes u_n\|_{L^{\frac{4}{d}}([0,T],L^2(\mathbb{R}^d))}\|\nabla\Psi\|_{L^\infty([0,T],L^2(\mathbb{R}^d\setminus K))}\\
&\le T^{\frac{4-d}{4}}\|\|u_n\|^2_{L^4(\mathbb{R}^d)}\|_{L^{\frac{4}{d}}(0,T)}\varepsilon\lesssim\varepsilon T^{\frac{4-d}{4}}\|\|u_n\|^{2(1-d/4)}_{L^2(\mathbb{R}^d)}\|\nabla u_n\|^{d/2}_{L^2(\mathbb{R}^d)}\|_{L^{\frac{4}{d}}(0,T)}\\
&\lesssim\varepsilon T^{\frac{4-d}{4}}\|u_n\|^{2\left(1-\frac{d}{4}\right)}_{L^\infty([0,T],L^2(\mathbb{R}^d))}\|\nabla u_n\|^{\frac{d}{2}}_{L^2([0,T],L^2(\mathbb{R}^d))}\le\varepsilon T^{\frac{4-d}{4}}\|u_0\|^2_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Hence, to prove (8.37) it is enough to show for any compact set $K\subset\mathbb{R}^d$

$$\lim_{n\to\infty}\int_0^t ds\int_K u_n(s,x)\otimes u_n(s,x):\nabla\Psi(s,x)dx=\int_0^t ds\int_K u(s,x)\otimes u(s,x):\nabla\Psi(s,x)dx. \tag{8.40}$$

The limit (8.40) is a consequence of

$$\lim_{n\to\infty}u_n\otimes u_n=u\otimes u \text{ in } L^1([0,T],L^2(K))$$

which in turn is a consequence of

$$\lim_{n\to\infty}u_n=u \text{ in } L^2([0,T],L^4(K)). \tag{8.41}$$

Let us consider $\chi\in C_c^\infty(\mathbb{R}^d,[0,1])$ s.t. $\chi=1$ in K, $\Omega:=\text{supp}\chi$ and with $\|\nabla\chi\|_{L^\infty(\mathbb{R}^d)}\le 1$. Then by Gagliardo Nirenberg we have

$$\|f\|_{L^4(K)}\le C\|f\|^{1-d/4}_{L^2(\Omega)}(\|\chi\nabla f\|_{L^2(\mathbb{R}^d)}+\|f\nabla\chi\|_{L^2(\mathbb{R}^d)})^{d/4}\le C\|f\|^{1-d/4}_{L^2(\Omega)}\|f\|^{d/4}_{H^1(\mathbb{R}^d)}.$$

Using this inequality and Hölder (using $\frac{1}{2}=\frac{4-d}{8}+\frac{d}{8}$):

$$\begin{aligned}
&\|u-u_n\|_{L^2([0,T],L^4(K))}\lesssim\|\|u-u_n\|^{1-\frac{d}{4}}_{L^2(\Omega)}\|u-u_n\|^{\frac{d}{4}}_{H^1(\mathbb{R}^d)}\|_{L^2(0,T)}\\
&\le\|\|u-u_n\|^{1-\frac{d}{4}}_{L^2(\Omega)}\|_{L^{\frac{8}{4-d}}(0,T)}\|\|u-u_n\|^{\frac{d}{4}}_{H^1(\mathbb{R}^d)}\|_{L^{\frac{8}{d}}(0,T)}\\
&=\|u-u_n\|^{1-\frac{d}{4}}_{L^2([0,T],L^2(\Omega))}\|u-u_n\|^{\frac{d}{4}}_{L^2([0,T],H^1(\mathbb{R}^d))}\\
&\le(2(1+\sqrt{T})\|u_0\|_{(L^2(\mathbb{R}^d))^d})^{\frac{d}{4}}\|u-u_n\|^{1-\frac{d}{4}}_{L^2([0,T],L^2(\Omega))}\xrightarrow{n\to+\infty}0
\end{aligned}$$

where the limit holds because $u_n \xrightarrow{n\to+\infty} u$ in $L^2([0,T],(L^2(\Omega))^d)$, by Proposition 8.11. This yields (8.41) and so also (8.40).
The proof of (8.38) will follow from the fact that for any $\varepsilon > 0$ there is N s.t. $n \geq N$ implies

$$\sup_{s\in[0,T]} \|\nabla(\mathbf{P}_n\Psi(s) - \Psi(s))\|_{L^2(\mathbb{R}^d)} < \varepsilon$$

In turn this, like (8.39), is a simple consequence of the fact that $\Psi \in C^1([0,\infty),(H^1(\mathbb{R}^d))^d)$. To prove (8.38) observe that

$$\begin{aligned}
&|\text{r.h.s. of (8.38)}| \leq \|u_n \otimes u_n\|_{L^1([0,T],(L^2(\mathbb{R}^d))^{d^2})} \|\nabla(\mathbf{P}_n\Psi - \Psi)\|_{L^2([0,T],(L^2(\mathbb{R}^d))^d)} \\
&\leq \varepsilon \|u_n\|^2_{L^2([0,T],(L^4(\mathbb{R}^d))^d)} = \varepsilon \| \|u_n\|_{L^4(\mathbb{R}^d)} \|^2_{L^2(0,T)} \lesssim \varepsilon \| \|u_n\|^{1-d/4}_{L^2(\mathbb{R}^d)} \|\nabla u_n\|^{d/4}_{L^2(\mathbb{R}^d)} \|^2_{L^2(0,T)} \\
&\lesssim \varepsilon \|u_n\|^{2(1-\frac{d}{4})}_{L^\infty([0,T],L^2(\mathbb{R}^d))} \|\nabla u_n\|^{\frac{d}{2}}_{L^2([0,T],L^2(\mathbb{R}^d))} \leq T^{1-\frac{d}{4}} \varepsilon \|u_0\|^2_{(L^2(\mathbb{R}^d))^d} \xrightarrow{\varepsilon\to 0} 0
\end{aligned}$$

This completes the proof of Leray's Theorem 8.2.

□

In the next section we prove the 2nd Leray's theorem, that is Theorem 8.3. Uniqueness in this case will follow from the fact that we will frame the problem as a fixed point argument using a contraction.

9 Well posedness in Sobolev spaces

For this section see [1].

Consider the equation (8.7). If $\mathcal{Q}_{NS}(u,u)$ is a force like the f in (7.1), we can interpret the solutions of (8.7) as solutions of a linear heat equation (7.1). We denote by $B(u,v)$ the weak solution of

$$\begin{cases} \partial_t B(u,v) - \nu \triangle B(u,v) = \mathcal{Q}_{NS}(u,v) \\ B(u,v)|_{t=0} = 0. \end{cases} \tag{9.1}$$

Then, when we are within the scope of the theory of Sect. 7, the solutions of (8.7) can be rewritten as

$$u = e^{\nu t \triangle} u_0 + B(u,u). \tag{9.2}$$

In the sequel we will use repeatedly the following abstract lemma.

Lemma 9.1. *Let X be a Banach space and $B : X^2 \to X$ a continuous bilinear map. Let $\alpha < \frac{1}{4\|B\|}$ where $\|B\| = \sup_{\|x\|=\|y\|=1} \|B(x,y)\|$. Then for any $x_0 \in X$ in $D_X(0,\alpha)$ (the open ball of center 0 and radius α in X) there exists a unique $x \in \overline{D}_X(0,2\alpha)$ s.t. $x = x_0 + B(x,x)$.*

Proof. We consider the map

$$x \to x_0 + B(x,x). \tag{9.3}$$

We will frame this as a fixed point problem in $\overline{D}_X(0,2\alpha)$.

First of all, we claim that the map (9.3) leaves $\overline{D}_X(0,2\alpha)$ invariant. Indeed

$$\|x_0 + B(x,x)\| \leq \|x_0\| + \|B(x,x)\| \leq \|x_0\| + \|B\|\|x\|^2 \leq \alpha \overbrace{(1 + \underbrace{4\|B\|\alpha}_{<1})}^{\leq 2} < 2\alpha.$$

Next, we check that the map (9.3) is a contraction. Indeed

$$\|B(x,x) - B(y,y)\| \leq \|B(x-y,x)\| + \|B(y,x-y)\| \leq 4\alpha\|B\|\|x-y\|$$

where $4\alpha\|B\| < 1$. So the map (9.3) has a unique fixed point in $\overline{D}_X(0,2\alpha)$.

□

Using the above lemma we will prove the following well posedness result.

Theorem 9.2. *For any $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)$ there exists a T and a solution of* (9.2) *with $u \in L^4([0,T],\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))$. This solution is unique. Furthermore we have*

$$u \in C([0,T],\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)),\ \nabla u \in L^2([0,T],\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d \times \mathbb{R}^d)). \tag{9.4}$$

Let T_{u_0} be the lifespan of the solution. Then:

(1) there exists a c s.t.

$$\|u_0\|_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)} \leq c\nu \Rightarrow T_{u_0} = \infty;$$

(2) if $T_{u_0} < \infty$ then

$$\int_0^{T_{u_0}} \|u(t)\|^4_{\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)} dt = \infty. \tag{9.5}$$

(3) if $T_{u_0} < \infty$ then

$$\int_0^{T_{u_0}} \|\nabla u(t)\|^2_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)} dt = \infty. \tag{9.6}$$

Moreover, if u and v are solutions, then

$$\begin{aligned} &\|u(t)-v(t)\|^2_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)} + \nu \int_0^t \|\nabla(u-v)(s)\|^2_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d\times\mathbb{R}^d)} ds \\ &\leq \|u_0 - v_0\|^2_{\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)} e^{C\nu^{-3}\int_0^t \left(\|u(t')\|^4_{\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)} + \|v(t')\|^4_{\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)}\right) dt'} \end{aligned} \tag{9.7}$$

where C is a fixed constant.

Remark 9.3. Notice that the following transformation preserves the solutions of the Navier Stokes equation:

$$u\,(t,x) \mapsto u_\lambda\,(t,x) := \lambda u\left(\lambda^2 t, \lambda x\right), \tag{9.8}$$

Furthermore, notice that the norms of u in the spaces in (9.4) coincide with the analogous norms of u_λ in the interval $[0, T/\lambda^2]$. Notice also that the norm of $u_0(x)$ in $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d, \mathbb{R}^d)$ coincides with the norm of $u_0(x/\lambda)$ in the same space. So the space $\dot{H}^{\frac{d}{2}-1}$ is an example of space *critical* for the Navier Stokes equation. One obvious consequence of this is the following: there exists no function $T(\cdot) : [0, +\infty) \to (0, +\infty]$ s.t. $T_{u_0} \geq T(\|u_0\|_{\dot{H}^{\frac{d}{2}-1}})$ for all $u_0 \in \dot{H}^{\frac{d}{2}-1}$. This is different from what we will see in a later chapter, where we will treat energy subcritical semilinear Schrödinger equations.

Remark 9.4. While for $d = 2$ the solutions provided by Theorem 9.2 are exactly Leray's solutions, for $d = 3$ we could have $u_0 \in (\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))^3$ with $u_0 \notin (L^2(\mathbb{R}^3))^3$. The corresponding solutions of the Navier Stokes equations provided by Theorem 9.2 are not Leray's solutions.

Remark 9.5. Notice that the finite lifespan (9.5) is relevant only for $d = 3$. Furthermore, if $T_{u_0} < \infty$, it has been shown that

$$\|u\|_{L^\infty([0,T_{u_0}],(\dot{H}^1(\mathbb{R}^3))^3)} = \infty,$$

but the proof is a much harder.
There is no blow up at $T = \infty$, at least when $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^3)$. Indeed, we will see in Sect. 10.1 that for such u_0 if $T_{u_0} = \infty$ we have $\lim_{t\to+\infty} \|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} = 0$.

We will assume for the moment Theorem 9.2 and prove the following.

10 Proof of Theorem 9.2

This section is devoted to the proof of this theorem. First we have the following lemma.

Lemma 10.1. *Let $d = 2, 3$. There exists a constant $C > 0$ s.t.*

$$\|\mathcal{Q}_{NS}(u, v)\|_{\dot{H}^{\frac{d}{2}-2}(\mathbb{R}^d,\mathbb{R}^d)} \leq C\|u\|_{\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)}\|v\|_{\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)}. \tag{10.1}$$

Proof. If $d = 2$ we have

$$\begin{aligned}
\|\mathcal{Q}_{NS}(u, v)\|_{\dot{H}^{-1}} &\leq \sum_{j,k=1}^{2} \left(\|\partial_k(u^k v^j)\|_{\dot{H}^{-1}} + \|\partial_k(v^k u^j)\|_{\dot{H}^{-1}}\right) \\
&\leq 2\sum_{j,k}^{2} \|u^k v^j\|_{L^2} \leq C\|u\|_{L^4}\|v\|_{L^4} \leq C\|u\|_{\dot{H}^{\frac{1}{2}}}\|v\|_{\dot{H}^{\frac{1}{2}}}
\end{aligned}$$

by the Sobolev embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^2) \subset L^4(\mathbb{R}^2)$, since $\frac{1}{4} = \frac{1}{2} - \frac{\frac{1}{2}}{2}$. This yields (10.1) for $d = 2$.
For $d = 3$

$$\begin{aligned}
\|\mathcal{Q}_{NS}(u, v)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} &\leq \sum_{j,k}^{2} \left(\|\partial_k(u^k v^j)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} + \|\partial_k(v^k u^j)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}\right) \\
&\lesssim \|(\nabla u)v\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} + \|u\nabla v\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \lesssim \|(\nabla u)v\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} + \|u\nabla v\|_{L^{\frac{3}{2}}(\mathbb{R}^3)}
\end{aligned}$$

where we are using the Sobolev embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$ (since $\frac{1}{3} = \frac{1}{2} - \frac{\frac{1}{2}}{3}$) which in turn by duality implies $L^{\frac{3}{2}}(\mathbb{R}^3) \subset \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$.
Hence, by $\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$ and Hölder,

$$\|\mathcal{Q}_{NS}(u,v)\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \lesssim \|\nabla u\|_{L^2(\mathbb{R}^3)}\|v\|_{L^6(\mathbb{R}^3)} + \|u\|_{L^6(\mathbb{R}^3)}\|\nabla v\|_{L^2(\mathbb{R}^3)} \le 2\|u\|_{\dot{H}^1(\mathbb{R}^3)}\|v\|_{\dot{H}^1(\mathbb{R}^3)}.$$

This yields (10.1) for $d = 3$. □

A straightforward consequence of Lemma 10.1 is the following for C the constant in Lemma 10.1.

Lemma 10.2. *Let $d = 2, 3$. Then for $u, v \in L^4([0,T], (\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d, \mathbb{R}^d))$ we have*

$$\|\mathcal{Q}_{NS}(u,v)\|_{L^2([0,T],\dot{H}^{\frac{d}{2}-2}(\mathbb{R}^d,\mathbb{R}^d))} \le C\|u\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))}\|v\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d))} \quad (10.2)$$

□

Proof of Theorem 9.2. By Theorem 7.4 we have for $s = \frac{d}{2} - 1$ and $p = 4$

$$\begin{aligned} \|B(u,v)\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} &= \|\|B(u,v)\|_{\dot{H}^{s+\frac{2}{p}}}\|_{L^p(0,T)} \lesssim \frac{1}{\nu^{\frac{1}{p}+\frac{1}{2}}}\|\mathcal{Q}_{NS}(u,v)\|_{L^2([0,T],\dot{H}^{s-1})} \\ &= \nu^{-\frac{3}{4}}\|\mathcal{Q}_{NS}(u,v)\|_{L^2([0,T],\dot{H}^{\frac{d}{2}-2})} \le C\nu^{-\frac{3}{4}}\|u\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})}\|v\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})}. \end{aligned} \quad (10.3)$$

So in the Banach space $X = L^4([0,T], \dot{H}^{\frac{d-1}{2}})$ we have $\|B\| \le C\nu^{-\frac{3}{4}}$. Obviously this is the same as $\frac{\nu^{\frac{3}{4}}}{4C} \le \frac{1}{4\|B\|}$. Our strategy is to prove

$$\|e^{\nu t\triangle}u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} < \frac{\nu^{\frac{3}{4}}}{4C} \le \frac{1}{4\|B\|} \quad (10.4)$$

where $e^{\nu t\triangle}u_0$ plays the role of x_0 in the abstract Lemma 9.1.
If (10.4) happens, that is if the l.h.s.of (10.4) is less than an $\alpha < \frac{1}{4\|B\|}$, then by Lemma 9.1 we can conclude that problem (9.2) admits a unique solution in $L^4([0,T], \dot{H}^{\frac{d-1}{2}})$ with norm less than $2\alpha < \frac{\nu^{\frac{3}{4}}}{2C}$.
We consider two distinct proofs of (10.4). The 1st, simpler, is valid only if $\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}$ is sufficiently small and shows that (10.4) holds for all T. In the second proof, which is general, we drop the assumption that $\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}$ is small, and we prove (10.4) for T sufficiently small.
Step 1: small initial data. By Theorem 7.4 we have for $s = \frac{d}{2} - 1$ and $p = 4$

$$\|e^{\nu t\triangle}u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} = \|\|e^{\nu t\triangle}u_0\|_{\dot{H}^{s+\frac{2}{p}}}\|_{L^p(0,T)} \le \nu^{-\frac{1}{p}}\|u_0\|_{\dot{H}^s} = \nu^{-\frac{1}{4}}\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}. \quad (10.5)$$

So, if $\|u_0\|_{\dot{H}^{\frac{d}{2}-1}} < \frac{\nu}{4C}$ then (10.4) is true for any $T > 0$. In particular $T_{u_0} = \infty$ and we have just proved (1) in Theorem 9.2.

Step 2: possibly large initial data. Now we consider the case when $u_0 \in \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$ is possibly large. We consider a low–high energy decomposition: $u_0 = \mathbf{P}_\rho u_0 + \chi_{\sqrt{-\triangle}\geq\rho} u_0$ where we pick $\rho = \rho_{u_0}$ large enough so that

$$\|\chi_{\sqrt{-\triangle}\geq\rho} u_0\|_{\dot{H}^{\frac{d}{2}-1}} < \frac{\nu}{8C}.$$

Then by (10.5) we get

$$\begin{aligned} &\|e^{\nu t\triangle} u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} \leq \|e^{\nu t\triangle}\chi_{\sqrt{-\triangle}\geq\rho} u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} + \|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} \\ &< \frac{\nu^{\frac{3}{4}}}{8C} + \|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} \end{aligned} \tag{10.6}$$

where we made the high energy contribution small by the choice of ρ large.
We now exploit the fact that we have the freedom to choose T small, in order to make the contribution to (10.6) small too. Indeed we have

$$\begin{aligned} &\|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} = \|e^{\nu t\triangle}\chi_{[0,\rho]}(\sqrt{-\triangle})u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} \\ &= \|e^{\nu t\triangle}\chi_{[0,\rho]}(\sqrt{-\triangle})\sqrt{\rho}\frac{(-\triangle)^{\frac{1}{4}}}{\sqrt{\rho}}u_0\|_{L^4([0,T],\dot{H}^{\frac{d}{2}-1})} \\ &\leq \sqrt{\rho}\|e^{\nu t\triangle}\chi_{[0,\rho]}(\sqrt{-\triangle})u_0\|_{L^4([0,T],\dot{H}^{\frac{d}{2}-1})} = \sqrt{\rho}\|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^4([0,T],\dot{H}^{\frac{d}{2}-1})} \\ &\leq (\rho^2 T)^{\frac{1}{4}}\|e^{\nu t\triangle}\mathbf{P}_\rho u_0\|_{L^\infty([0,T],\dot{H}^{\frac{d}{2}-1})} \leq (\rho^2 T)^{\frac{1}{4}}\|\mathbf{P}_\rho u_0\|_{\dot{H}^{\frac{d}{2}-1}} \leq (\rho^2 T)^{\frac{1}{4}}\|u_0\|_{\dot{H}^{\frac{d}{2}-1}} \leq \frac{\nu^{\frac{3}{4}}}{8C} \end{aligned}$$

if we choose T small enough so that the last inequality holds, that is if we choose T such that

$$T \leq \left(\frac{\nu^{\frac{3}{4}}}{8\rho^{\frac{1}{2}}C\|u_0\|_{\dot{H}^{\frac{d}{2}-1}}}\right)^4, \tag{10.7}$$

then all terms in the r.h.s. of (10.6) have been made small enough s.t.

$$\|e^{\nu t\triangle}u_0\|_{L^4([0,T],\dot{H}^{\frac{d-1}{2}})} < \frac{\nu^{\frac{3}{4}}}{4C} \leq \frac{1}{4\|B\|},$$

that is we obtained (10.4).

We have proved the 1st sentence in the statement of Theorem 9.2.
Now we turn to the proof that a solution $u \in L^4([0,T],\dot{H}^{\frac{d-1}{2}})$ satisfies (9.4).
By (10.1) we have $\mathcal{Q}_{NS}(u,u) \in L^2([0,T],\dot{H}^{\frac{d}{2}-2})$. Then it must be remarked that by its definition $B(u,u)$ is a solution in the sense of Definition 7.1 of the Heat Equation written above (9.2). Similarly, by Theorem 7.2 also $e^{\nu t\triangle}u_0$ is a solution of the homogeneous Heat Equation with initial value u_0. Hence, since u satisfies (9.2), then u is the solution of the Heat Equation (8.7), where the latter can be framed in terms of the theory in Sect. 7 for

$s = \frac{d}{2} - 1$. Then by Theorem 7.2 we have $u \in C^0([0,T], \dot{H}^{\frac{d}{2}-1})$ and $\nabla u \in L^2([0,T], \dot{H}^{\frac{d}{2}})$. This yields (9.4).
We turn now to the proof of (9.7). We consider two solutions u and v, and set $w = u - v$. Then

$$\begin{cases} w_t - \nu \triangle w = \mathcal{Q}_{NS}(w, u+v) \\ \quad\quad w(0) = u_0 - v_0 \end{cases}$$

where we used the symmetry $\mathcal{Q}_{NS}(u,v) = \mathcal{Q}_{NS}(v,u)$ and

$$\mathcal{Q}_{NS}(u-v, u+v) = \mathcal{Q}_{NS}(u,u) - \mathcal{Q}_{NS}(v,v) + \underbrace{\mathcal{Q}_{NS}(u,v) - \mathcal{Q}_{NS}(v,u)}_{0}.$$

By the energy estimate (7.5) for $s = \frac{d}{2} - 1$ we have

$$\Delta_w := \|w(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\nu \int_0^t \|\nabla w(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' = \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t \langle \mathcal{Q}_{NS}(w, u+v), w\rangle_{\dot{H}^{\frac{d}{2}-1}}(t')dt'.$$

Claim 10.3. We have

$$\langle \mathcal{Q}_{NS}(a,b), c\rangle_{\dot{H}^{\frac{d}{2}-1}} \leq C\|a\|_{\dot{H}^{\frac{d-1}{2}}}\|b\|_{\dot{H}^{\frac{d-1}{2}}}\|c\|_{\dot{H}^{\frac{d}{2}}}. \tag{10.8}$$

Proof. Indeed, trading derivatives we have

$$\langle \mathcal{Q}_{NS}(a,b), c\rangle_{\dot{H}^{\frac{d}{2}-1}} \leq \|\mathcal{Q}_{NS}(a,b)\|_{\dot{H}^{\frac{d}{2}-2}}\|c\|_{\dot{H}^{\frac{d}{2}}}$$

and by (10.1) we have

$$\|\mathcal{Q}_{NS}(a,b)\|_{\dot{H}^{\frac{d}{2}-2}} \leq C\|a\|_{\dot{H}^{\frac{d-1}{2}}}\|b\|_{\dot{H}^{\frac{d-1}{2}}}.$$

This proves Claim 10.3.

Now for $N(t) := \|u(t)\|_{\dot{H}^{\frac{d-1}{2}}} + \|v(t)\|_{\dot{H}^{\frac{d-1}{2}}}$ by Claim 10.3 we have

$$\Delta_w \leq \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t \|w(t')\|_{\dot{H}^{\frac{d-1}{2}}} N(t')\|\nabla w(t')\|_{\dot{H}^{\frac{d}{2}-1}} dt'.$$

By the interpolation estimate in Lemma 5.1 we have

$$\|w(t')\|_{\dot{H}^{\frac{d-1}{2}}} \leq \|w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}}\|\nabla w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}}.$$

This implies

$$\Delta_w \leq \|w_0\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\int_0^t \|w(t')\|^{\frac{1}{2}}_{\dot{H}^{\frac{d}{2}-1}} N(t')\|\nabla w(t')\|^{\frac{3}{2}}_{\dot{H}^{\frac{d}{2}-1}} dt'.$$

Using the inequality $ab \leq \frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}$, which follows by

$$\log(ab) = \frac{1}{4}\log(a^4) + \frac{3}{4}\log(b^{\frac{4}{3}}) \leq \log\left(\frac{1}{4}a^4 + \frac{3}{4}b^{\frac{4}{3}}\right),$$

we get

$$\begin{aligned}\text{the integrand} &= \left(\|w(t')\|^{\frac{1}{2}}_{\dot H^{\frac{d}{2}-1}} N(t')\nu^{-\frac{3}{4}} \left(\frac{3}{4}\right)^{\frac{3}{4}} \right) \left(\frac{4}{3}\nu \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} \right)^{\frac{3}{4}} \\ &\le \frac{3^3}{4^4\nu^3} \|w(t')\|^2_{\dot H^{\frac{d}{2}-1}} N^4(t') + \nu \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}}.\end{aligned}$$

Then

$$\Delta_w \le \|w_0\|^2_{\dot H^{\frac{d}{2}-1}} + \frac{3^3}{4^4\nu^3} \int_0^t \|w(t')\|^2_{\dot H^{\frac{d}{2}-1}} N^4(t') dt' + \nu \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt'.$$

In other words, by the definition of Δ_w

$$\begin{aligned}&\|w(t)\|^2_{\dot H^{\frac{d}{2}-1}} + \cancel{2}\nu \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' \\ &\le \|w_0\|^2_{\dot H^{\frac{d}{2}-1}} + \frac{3^3}{4^4\nu^3} \int_0^t \|w(t')\|^2_{\dot H^{\frac{d}{2}-1}} N^4(t') dt' + \cancel{\nu \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt'}\end{aligned}$$

so that, if we set

$$X(t) := \|w(t)\|^2_{\dot H^{\frac{d}{2}-1}} + \nu \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt'$$

we have

$$\begin{aligned}X(t) &\le \|w_0\|^2_{\dot H^{\frac{d}{2}-1}} + \frac{3^3}{4^4\nu^3} \int_0^t \|w(t')\|^2_{\dot H^{\frac{d}{2}-1}} N^4(t') dt' \\ &\le \|w_0\|^2_{\dot H^{\frac{d}{2}-1}} + \frac{3^3}{4^4\nu^3} \int_0^t X(t') N^4(t') dt'.\end{aligned}$$

So by Gronwall's inequality

$$\|w(t)\|^2_{\dot H^{\frac{d}{2}-1}} + \nu \int_0^t \|\nabla w(t')\|^2_{\dot H^{\frac{d}{2}-1}} dt' \le \|w_0\|^2_{\dot H^{\frac{d}{2}-1}} \exp\left(\frac{3^3}{4^4\nu^3} \int_0^t N^4(t') dt' \right).$$

This proves the stability inequality (9.7)

We now consider the blow up criterion (9.5). Suppose that $u(t)$ is a solution in $[0,T)$ with

$$\int_0^T \|u(t)\|^4_{\dot H^{\frac{d-1}{2}}} dt < \infty.$$

Notice that then $u \in L^4([0,T], \dot H^{\frac{d-1}{2}})$ and

$$\|\mathcal{Q}_{NS}(u,u)\|_{L^2([0,T],\dot H^{\frac{d}{2}-2})} \le C \|u\|^2_{L^4([0,T],\dot H^{\frac{d-1}{2}})}. \tag{10.9}$$

We claim that we can extend $u(t)$ beyond T.

Claim 10.4. There exists a $\tau > 0$ s.t. u extends in a solution in $L^4([0, T+\tau), \dot H^{\frac{d-1}{2}}(\mathbb{R}^d, \mathbb{R}^d))$.

First of all we set

$$g(\xi) := \sup_{0 \le t' \le T} |\widehat{u}(t', \xi)|.$$

Claim 10.5. We have $|\xi|^{\frac{d}{2}-1} g \in L^2(\mathbb{R}^d)$.

Proof of Claim 10.5. By (7.17) for $s = \frac{d}{2} - 1$ and by (10.1) we have

$$\begin{aligned}
\| |\xi|^{\frac{d}{2}-1} g \|_{L^2} &= \left(\int_{\mathbb{R}^d} |\xi|^{d-2} \left(\sup_{0 \le t' \le t} |\widehat{u}(t', \xi)| \right)^2 d\xi \right)^{\frac{1}{2}} \\
&\le \|u_0\|_{\dot{H}^{\frac{d}{2}-1}} + \frac{1}{(2\nu)^{\frac{1}{2}}} \|\mathcal{Q}_{NS}\|_{L^2([0,T], \dot{H}^{\frac{d}{2}-2})} \\
&\le \|u_0\|_{\dot{H}^{\frac{d}{2}-1}} + \frac{C}{(2\nu)^{\frac{1}{2}}} \|u\|^2_{L^4([0,T], \dot{H}^{\frac{d-1}{2}})} < \infty.
\end{aligned}$$

This proves Claim 10.5.

Proof of Claim 10.4. Claim 10.5 implies

$$\int_{|\xi| \ge \rho} |\xi|^{d-2} |g(\xi)|^2 d\xi \xrightarrow{\rho \to +\infty} 0.$$

Thus there exists $\rho > 0$ s.t for any preassigned $c > 0$

$$\int_{|\xi| \ge \rho} |\xi|^{d-2} |\widehat{u}(t, \xi)|^2 d\xi < (c\nu)^2 \text{ for all } t \in [0, T).$$

Now, recalling the splitting in high and low energies in the proof of the 1st sentence in the statement of Theorem 9.2, there exists a fixed $\tau > 0$ s.t. the lifespan of the solution with initial datum $u(t)$ is bounded below by τ independently of $t \in [0, T)$. Indeed there exists a $c_1 > 0$ independent from $t \in [0, T)$ s.t.

$$\left(\frac{\nu^{\frac{3}{4}}}{8 \rho^{\frac{1}{2}} C \|u(t)\|_{\dot{H}^{\frac{d}{2}-1}}} \right)^4 > c_1 > 0.$$

This follows from the fact that

$$\|u(t)\|_{\dot{H}^{\frac{d}{2}-1}} \le \| |\xi|^{\frac{d}{2}-1} g \|_{L^2} < \infty$$

So we can take $\tau = c_1$. Then $T_{u_0} \ge T + \tau$ and this yields Claim 10.4.

Let us now discuss the blow up criterion (9.6). Suppose that $T_{u_0} < \infty$ and that

$$C_{L2} := \int_0^{T_{u_0}} \|\nabla u(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} dt < \infty. \tag{10.10}$$

Since we have (9.5) and

$$L^4([0,T],\dot{H}^{\frac{d-1}{2}}(\mathbb{R}^d,\mathbb{R}^d)) \subseteq L^\infty([0,T],\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d,\mathbb{R}^d)) \cap L^2([0,T],\dot{H}^{\frac{d}{2}}(\mathbb{R}^d,\mathbb{R}^d))$$

it follows that since we must have (9.5), then (10.10) implies that

$$\lim_{T\to T_{u_0}} \|u(t)\|_{L^\infty([0,T],\dot{H}^{\frac{d}{2}-1})} = \infty \tag{10.11}$$

For $0 \leq t \leq T < T_{u_0}$ we have, by (10.8) and interpolation,

$$\begin{aligned}
&\|u(t)\|^2_{\dot{H}^{\frac{d}{2}-1}} + 2\nu \int_0^t \|\nabla u(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt' = \|u(t_1)\|^2_{\dot{H}^{\sigma_s}} + 2\int_0^t \langle Q(u(t'),u(t')),u(t')\rangle_{\dot{H}^{\frac{d}{2}-1}} dt' \\
&\leqslant \|u(0)\|^2_{\dot{H}^{\frac{d}{2}-1}} + C'_d \int_0^t \|u(t')\|^2_{\dot{H}^{\frac{d-1}{2}}} \|\nabla u(t')\|_{\dot{H}^{\frac{d}{2}-1}} dt' \\
&\leqslant \|u(0)\|^2_{\dot{H}^{\frac{d}{2}-1}} + C_d \int_0^t \|u(t')\|_{\dot{H}^{\frac{d}{2}-1}} \|\nabla u(t')\|^2_{\dot{H}^{\frac{d}{2}-1}} dt'
\end{aligned} \tag{10.12}$$

and so

$$\|u\|^2_{L^\infty([0,T],\dot{H}^{\frac{d}{2}-1})} \leqslant \|u(0)\|^2_{\dot{H}^{\frac{d}{2}-1}} + C_d C_{L2} \|u\|_{L^\infty([0,T],\dot{H}^{\frac{d}{2}-1})}.$$

But this means that

$$\|u\|_{L^\infty([0,T],\dot{H}^{\frac{d}{2}-1})} \leq \frac{1}{2} C_d C_{L2} + \frac{1}{2}\sqrt{C_d^2 C_{L2}^2 + 4\|u(0)\|^2_{\dot{H}^{\frac{d}{2}-1}}} < \infty,$$

contradicting (10.11). This contradiction proves the blow up criterion (9.6).
The proof of Theorem 9.2 is completed. □

Corollary 10.6. *In the case $d = 2$, Theorem 9.2 implies Leray's Theorem 8.3 for $d = 2$*

Proof. By the Leray's Theorem 8.2 we know that given a divergence free $u_0 \in L^2(\mathbb{R}^2,\mathbb{R}^2)$ there are weak solutions in the sense of Leray with $u \in L^\infty([0,\infty), L^2(\mathbb{R}^2,\mathbb{R}^2))$ and $\nabla u \in L^2([0,\infty), L^2(\mathbb{R}^2,\mathbb{R}^4))$. Interpolating, for each such a solution we have

$$\|\|u\|_{\dot{H}^{\frac{1}{2}}}\|_{L^4_t} \leq \|\|u\|^{\frac{1}{2}}_{L^2}\|\nabla u\|^{\frac{1}{2}}_{L^2}\|_{L^4_t} \leq \|u\|^{\frac{1}{2}}_{L^\infty_t L^2}\|\nabla u\|^{\frac{1}{2}}_{L^2_t L^2}$$

and so we obtain also $u \in L^4([0,\infty),\dot{H}^{\frac{1}{2}}(\mathbb{R}^2,\mathbb{R}^2))$.
By Lemma 10.2 we know that this implies

$$\mathcal{Q}_{NS}(u,u) \in L^2([0,\infty),\dot{H}^{-1}(\mathbb{R}^2,\mathbb{R}^2)).$$

Notice that the right hand side of (8.7) satisfies the hypothesis of the force term in the linear heat equation (7.1). As a weak solution of the Navier Stokes equation in the sense of Definition 8.1, u is then also a solution of the linear heat equation (7.1) in the sense of Definition 7.1. This means that it is also a solution of (9.2). Since by Theorem 9.2 such solution is a unique, we conclude that the solution of Leray's Theorem 8.2 in the case $d = 2$ is unique. Furthermore by Theorem 9.2 we know also that $u \in C^0([0, \infty), L^2(\mathbb{R}^2, \mathbb{R}^2))$. We now turn to the energy identity. By Leray's Theorem 8.2 we know that

$$\|u(t)\|^2_{L^2(\mathbb{R}^2)} + 2\nu \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^2)} dt' \le \|u_0\|^2_{L^2(\mathbb{R}^2)}.$$

We want now to prove that $\le$ can be replaced by $=$ in this formula. As we have mentioned above, u solves in the sense of Definition 7.1 the problem

$$\partial_t u - \nu \triangle u = \mathcal{Q}_{NS}(u, u) \text{ with } \mathcal{Q}_{NS}(u, u) \in L^2(\mathbb{R}_+, \dot{H}^{-1}(\mathbb{R}^2, \mathbb{R}^2)),$$

Then, by Theorem 7.2 for $s = 0$ the identity (7.5) yields

$$\|u(t)\|^2_{L^2} + 2\nu \int_0^t \|\nabla u(t')\|^2_{L^2} dt' = \|u_0\|^2_{L^2} + 2 \int_0^t \langle \mathcal{Q}_{NS}(u(t'), u(t')), u(t')\rangle_{L^2} dt'.$$

By Lemma 8.8 we have the cancelation

$$\langle \mathcal{Q}_{NS}(u, u), u\rangle = \langle \mathbb{P}(\operatorname{div}(u \otimes u), u\rangle = \langle \operatorname{div}(u \otimes u), u\rangle = 0.$$

This completes the proof, by giving the energy identity. □

10.1 Global solutions.

Proposition 10.7. *Let $d = 3$ and let $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^3)$ be s.t. $T_{u_0} = \infty$. Then*

$$\lim_{t \to +\infty} \|u(t)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^3))} = 0. \tag{10.13}$$

Proof. Since $u_0 \in H^{\frac{1}{2}}(\mathbb{R}^3, \mathbb{R}^3))$ we have also $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3))$, and u is also a weak solution in the sense of Leray. Hence it satisfies the energy inequality

$$\|u(t)\|^2_{L^2(\mathbb{R}^3)} + 2\nu \int_0^t \|\nabla u(t')\|^2_{L^2(\mathbb{R}^3)} dt' \le \|u_0\|^2_{L^2(\mathbb{R}^3)},$$

which implies in particular

$$\|\nabla u\|_{L^2(\mathbb{R}_+, L^2(\mathbb{R}^3))} \le \frac{1}{\sqrt{2\nu}} \|u_0\|_{L^2(\mathbb{R}^3)} \text{ and}$$
$$\|u\|_{L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^3))} \le \|u_0\|_{L^2(\mathbb{R}^3)}.$$

So by Hölder inequality and the interpolation of Lemma 5.1, we have

$$\|u\|_{L^4(\mathbb{R}_+, \dot{H}^{\frac{1}{2}}(\mathbb{R}^3))} \le \frac{1}{\sqrt[4]{2\nu}} \|u_0\|_{L^2(\mathbb{R}^3)}.$$

This implies that for $1 \gg \epsilon > 0$ arbitrarily small, there exists $t_\epsilon > 0$ s.t. $\|u(t_\epsilon)\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \le \epsilon$. Then, by the part of the proof for small initial data in Theorem 9.2, we know that

$\|u(t_\epsilon)\|_{L^4(t_\epsilon,+\infty)\dot H^1(\mathbb{R}^3))} \le C_3\epsilon$ for a fixed constant $C_3 > 0$. In turn, from inequality (9.7), for $t > t_\epsilon$ we get

$$\|u(t)\|^2_{\dot H^{\frac12}} + \nu \int_{t_\epsilon}^t \|\nabla u(s)\|^2_{\dot H^{\frac12}} ds \le \|u(t_\epsilon)\|^2_{\dot H^{\frac12}} e^{C\nu^{-3}\int_{t_\epsilon}^t \|u(t')\|^4_{\dot H^1} dt'}.$$

So, in the half-line $[t_\epsilon, \infty)$ we get $\|u(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)} \le 2\|u(t_\epsilon)\|_{\dot H^{\frac12}} \le 2\epsilon$ and, since $\epsilon > 0$ is arbitrary, we have the limit in (10.13). □

Notice that in the previous proposition, in fact $\|u(t)\|_{\dot H^{\frac12}(\mathbb{R}^3)}$ from a certain moment on is decreasing. In fact, we have the following result.

Lemma 10.8. *There exists $\varepsilon_1 > 0$ s.t. for $\|u_0\|_{\dot H^{\frac d2 -1}} \le \varepsilon_1$ the function $t \to \|u(t)\|_{\dot H^{\frac d2 -1}}$ is decreasing.*

Proof. From Theorem 9.2 we know that for $\varepsilon_1 \in (0, \varepsilon_0]$ then we have, arguing like in Proposition 10.7, $\|u(t)\|_{\dot H^{\frac d2 -1}} \lesssim \|u_0\|_{\dot H^{\frac d2 -1}} \le \varepsilon_1$ for all t. Now, given any pair $0 \le t_1 < t_2$ we have like in (10.12)

$$\|u(t_2)\|^2_{\dot H^{\frac d2 -1}} + 2\nu \int_{t_1}^{t_2} \|\nabla u(t')\|^2_{\dot H^{\frac d2 -1}} dt' \leqslant \|u(t_1)\|^2_{\dot H^{\frac d2 -1}} + C\int_{t_1}^{t_2} \|u(t')\|_{\dot H^{\frac d2 -1}} \|\nabla u(t')\|^2_{\dot H^{\frac d2 -1}} dt'$$
$$\leqslant \|u(t_1)\|^2_{\dot H^{\frac d2 -1}} + C\varepsilon_1 \int_{t_1}^{t_2} \|\nabla u(t')\|^2_{\dot H^{\frac d2 -1}} dt',$$

where C is a fixed constant. Choosing ε_1 s.t. $C\varepsilon_1 < \nu$, it follows

$$\|u(t_2)\|^2_{\dot H^{\frac d2 -1}} + \nu \int_{t_1}^{t_2} \|\nabla u(t')\|^2_{\dot H^{\frac d2 -1}} dt' \leqslant \|u(t_1)\|^2_{\dot H^{\frac d2 -1}}. \tag{10.14}$$

Hence $t \to \|u(t)\|_{\dot H^{\frac d2 -1}}$ is decreasing. □

11 The case of initial data in $L^3(\mathbb{R}^3)$

It is possible to prove the following theorem.

Theorem 11.1. *For any divergence free $u_0 \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ there is a $T > 0$ and a unique solution $u \in C^0([0,T), L^3(\mathbb{R}^3, \mathbb{R}^3))$ of*

$$u = e^{\nu t \triangle} u_0 + B(u,u). \tag{9.2}$$

Furthermore there exists a $\varepsilon_{3,\nu} > 0$ s.t. for $\|u_0\|_{L^3} < \varepsilon_{3,\nu}$ we have $T = \infty$. Furthermore, if $u_0 \in \dot H^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$, the life span is the same of Theorem 9.2.

Exercise 11.2. Prove that the mapping $\dot H^{1/2}(\mathbb{R}^3, \mathbb{R}^3) \to L^3(\mathbb{R}^3, \mathbb{R}^3)$ is not surjective.

Exercise 11.3. Prove that the subspace of divergence free vector fields in $\dot H^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ is closed in $\dot H^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$. Prove the same for with $\dot H^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ replaced by $L^3(\mathbb{R}^3, \mathbb{R}^3)$.

Exercise 11.4. Prove that the Sobolev embedding from the subspace of divergence free vector fields in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ to the subspace of divergence free vector fields in $L^3(\mathbb{R}^3, \mathbb{R}^3)$ is not surjective.

Exercise 11.5. Pick a divergence free u_0 belonging to $L^3(\mathbb{R}^3, \mathbb{R}^3)$ but not to $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$. Show that there exists a sequence of divergence free vector fields $\{u_0^{(n)}\}$ in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ with $u_0^{(n)} \to u_0$ in $L^3(\mathbb{R}^3, \mathbb{R}^3)$. Show also that $\|u_0^{(n)}\|_{\dot{H}^{1/2}} \to \infty$.

Exercise 11.6. Show that it is possible to define divergence free sequences $\{v_0^{(n)}\}$ in $\dot{H}^{1/2}(\mathbb{R}^3, \mathbb{R}^3)$ with $\|v_0^{(n)}\|_{\dot{H}^{1/2}} \to \infty$ and $\|v_0^{(n)}\|_{L^3} \to 0$.

Remark 11.7. For a sequence such as in Exercise 11.6, for $n \gg 1$ the corresponding solutions of the NS equation are globally defined in time by Theorem 11.14, while Theorem 9.2 is able to guarantee only on short intervals of time.

To prove Theorem 11.14 we will apply the abstract Lemma 9.1 in an appropriate Banach space X. The striking fact though, is that the space X will not be of the form $C^0([0,T], L^3(\mathbb{R}^3, \mathbb{R}^3))$. This because if X where this space, then the bilinear form B defined by (9.1) is known not to be continuous. It turns out that to get the right Banach space X, has required a certain degree of imagination and insight.

Definition 11.8. For $p \in [3, \infty]$ and $T \in (0, \infty)$ we set

$$K_p(T) = \{u \in C^0((0,T], L^p(\mathbb{R}^3, \mathbb{R}^3)) : \|u\|_{K_p(T)} := \sup_{t \in (0,T]} (\nu t)^{\frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right)} \|u(t)\|_{L^p} < \infty\} \tag{11.1}$$

and for $p \in [1, 3)$

$$K_p(T) = \{u \in C^0([0,T], L^p(\mathbb{R}^3, \mathbb{R}^3)) : \|u\|_{K_p(T)} := \sup_{t \in (0,T]} (\nu t)^{\frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right)} \|u(t)\|_{L^p} < \infty\}. \tag{11.2}$$

We denote by $K_p(\infty)$ the spaces defined as above, with $(0,T]$ replaced by $(0, \infty)$.

We recall that the solution of the heat equation $u_t - \nu \triangle u = 0$ is $e^{t\nu\triangle} f = K_t * f$ where $K_t(x) := (4\pi\nu t)^{-\frac{3}{2}} e^{-\frac{|x|^2}{4t\nu}}$. Notice that $K_t(x) = (\nu t)^{-\frac{3}{2}} K((\nu t)^{-\frac{1}{2}} x)$, where $K(x) := (4\pi)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4}}$ and where $\widehat{K}(\xi) = e^{-|\xi|^2}$.

Notice that for $u_0 \in L^3(\mathbb{R}^3)$ and $p \geq 3$ we have from (1.15),

$$\|e^{t\nu\Delta} u_0\|_{L^p(\mathbb{R}^3)} \leqslant (4\pi\nu t)^{\frac{3}{2}\left(\frac{1}{p} - \frac{1}{3}\right)} \|u_0\|_{L^3(\mathbb{R}^3)} \text{ for all } p \geq 3, \tag{11.3}$$

it can be proved that $e^{t\nu\Delta} u_0 \in C(\mathbb{R}_+, L^p)$, and so $e^{t\nu\Delta} u_0 \in K_p(\infty)$.

Lemma 11.9. *Let $u_0 \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ and $p > 3$. Then*

$$\lim_{T \to 0} \|e^{t\nu\Delta} u_0\|_{K_p(T)} = 0. \tag{11.4}$$

Proof. For any $\epsilon > 0$ there exists $\phi \in L^3(\mathbb{R}^3, \mathbb{R}^3) \cap L^p(\mathbb{R}^3, \mathbb{R}^3)$ s.t. $\|u - \phi\|_{L^3} < \epsilon$. Then by (11.3) we have

$$\|u - \phi\|_{K_p(T)} \leq (4\pi\nu T)^{\frac{3}{2}\left(\frac{1}{p} - \frac{1}{3}\right)} \epsilon.$$

Since $\|e^{t\nu\Delta}\phi\|_{L^p} \leq \|\phi\|_{L^p}$, it follows

$$\|e^{t\nu\Delta}\phi\|_{K_p(T)} = \sup_{t\in(0,T]} (\nu t)^{\frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right)} \|e^{t\nu\Delta}\phi\|_{L^p} \leq (\nu T)^{\frac{3}{2}\left(\frac{1}{3} - \frac{1}{p}\right)} \|\phi\|_{L^p} \xrightarrow{T\to 0} 0.$$

□

Lemma 11.10. *Let p, q and r satisfy*

$$\begin{aligned} 0 < \frac{1}{p} + \frac{1}{q} \leqslant 1 \\ \frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{q} < \frac{1}{3} + \frac{1}{r} \end{aligned} \tag{11.5}$$

Then the bilinear map B defined in (9.1) *maps $K_p(T) \times K_q(T) \to K_r(T)$ and there is a constant C independent from T s.t.*

$$\|B(u, v)\|_{K_r(T)} \leqslant C\|u\|_{K_p(T)}\|v\|_{K_q(T)}. \tag{11.6}$$

To prove Lemma 11.10 we consider for any $m = 1, 2, 3$ the problem

$$\begin{cases} (L_m f)_t - \nu\triangle L_m f = \mathbb{P}\partial_m f \\ L_m f(0, x) = 0 \end{cases} \tag{11.7}$$

($L_m f$ is by definition the solution of the above heat equation). Then by (7.7) and (8.9) for appropriate constants c_{jk} we have

$$\widehat{L_m f}(t, \xi) = \sum_{j,k=1}^{3} c_{jk} \int_0^t e^{-(t-t')\nu|\xi|^2} \xi_j\xi_k\xi_m|\xi|^{-2}\widehat{f}(t', \xi)dt'. \tag{11.8}$$

This means, for $\Gamma_{jkm}(t, x)$ the inverse Fourier transform of $e^{-t\nu|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}$,

$$L_m f(t) = \sum_{j,k=1}^{3} c_{jk} \int_0^t \Gamma_{jkm}(t - t') * \widehat{f}(t')dt'. \tag{11.9}$$

We claim the following.

Claim 11.11. We have for a fixed $C > 0$

$$|\Gamma_{jkm}(t, x)| \leqslant C(\sqrt{\nu t} + |x|)^{-4}. \tag{11.10}$$

Proof. It is elementary that $\Gamma_{jkm}(t,x) = (\nu t)^{-2}\Gamma_{jkm}((\nu t)^{-1/2}x)$ with $\widehat{\Gamma}_{jkm}(x) = e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}$. Then (11.10) is a consequence of

$$|\Gamma_{jkm}(x)| \leqslant C(1+|x|)^{-4}. \tag{11.11}$$

It is straightforward that $\Gamma_{jkm} \in C^\infty(\mathbb{R}^4) \cap L^\infty(\mathbb{R}^4)$, because of the rapid decay to 0 at infinity of $e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}$. Hence, to prove (11.11) it suffices to consider $|x| \gg 1$. For χ_0 a smooth cutoff of compact support equal to 1 near 0 and with $\chi_1 := 1 - \chi_0$, we set

$$\begin{aligned}\Gamma_{jkm}(x) = (2\pi)^{-\frac{3}{2}} &\int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x}\chi_0\left(|x|\xi\right) e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}d\xi \\ + (2\pi)^{-\frac{3}{2}} &\int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x}\chi_1\left(|x|\xi\right) e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}d\xi\end{aligned}$$

The 1st term in the r.h.s. is

$$\lesssim \int_{|\xi|\le|x|^{-1}} |\xi|\ d\xi \sim |x|^{-4}.$$

We next consider the other term, which we split as

$$(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x}\chi_1\left(|x|\xi\right)\chi_0\left(\xi\right) e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}d\xi \tag{11.12}$$

$$+ (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x}\chi_1\left(\xi\right) e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}d\xi. \tag{11.13}$$

Let us consider the term in (11.12). Set $L := \mathrm{i}\frac{x}{|x|^2}\cdot\nabla_\xi$ and notice that $Le^{-\mathrm{i}\xi\cdot x} = e^{-\mathrm{i}\xi\cdot x}$. Then, the the term in (11.12) is

$$(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x}L^6\left(\chi_1\left(|x|\xi\right) e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}\right)d\xi.$$

The absolute value of the integrand is for fixed C

$$\left|L^6\left(\cdots\right)\right| \le C|x|^{-6}|\xi|^{-5}.$$

Here we used that in the support of $\nabla_\xi\left(\chi_1\left(|x|\xi\right)\right)$ we have $|x| \sim |\xi|^{-1}$. So the last integral is bounded

$$\lesssim |x|^{-6}\int_{1\ge|\xi|\ge|x|^{-1}} |\xi|^{-5}d\xi \sim |x|^{-6}|x|^2 = |x|^{-4}$$

where the 2nd term is $\sim |x|^{-6} \ll |x|^{-4}$ and the 1st term is $\sim |x|^{-6}|x|^2 = |x|^{-4}$.

Finally, for (11.13) we consider

$$(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-\mathrm{i}\xi\cdot x}L^N\left(\chi_1\left(\xi\right) e^{-|\xi|^2}\xi_j\xi_k\xi_m|\xi|^{-2}\right)d\xi,$$

where we can use

$$\left|L^N(\cdots)\right| \le C|x|^{-N}e^{-|\xi|^2}|\xi|^N.$$

So the last integral is bounded

$$|x|^{-N}\int_{|\xi|\ge 1} e^{-|\xi|^2}|\xi|^N d\xi \sim |x|^{-N},$$

which, taking N arbitrarily large, is shown to be much smaller than $|x|^{-4}$. This completes the proof of Claim 11.11.

□

Completion of proof of Lemma 11.10. By (11.10) we have by Young's inequality for convolutions and Hölder's inequality for the tensor product of u and v the bound (here $\frac{1}{a} = 1 + \frac{1}{r} - \frac{1}{\beta}$ and $\frac{1}{\beta} = \frac{1}{p} + \frac{1}{q}$)

$$\begin{aligned}\|B(u,v)\|_{L^r} &\leqslant C_1 \sum_{j,m,k} \int_0^t \left\|\Gamma_{j,m,k}(t-t')\right\|_{L^a} \left\|u(t')\otimes v(t')\right\|_{L^\beta} dt' \\ &\leqslant C_1 \sum_{j,m,k} \int_0^t \left\|\Gamma_{j,m,k}(t-t')\right\|_{L^a} \left\|u(t')\right\|_{L^p} \left\|v(t')\right\|_{L^q} dt' \\ &\lesssim \int_0^t (t-t')^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)}(t')^{-\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}-\frac{1}{q}\right)}dt' \,\|u\|_{K_p(t)}\|v\|_{K_q(t)} \end{aligned} \tag{11.14}$$

where in the 3rd line we used

$$\begin{aligned}\left\|\Gamma_{j,m,k}(t-t')\right\|_{L^a(\mathbb{R}^3)} &\lesssim \left\|(\sqrt{t-t'}+|x|)^{-4}\right\|_{L^a(\mathbb{R}^3)} = (t-t')^{-2}\left\|\left(1+\frac{|x|}{\sqrt{t-t'}}\right)^{-4}\right\|_{L^a(\mathbb{R}^3)} \\ &= (t-t')^{-2}(t-t')^{\frac{3}{2a}}\left\|(1+|x|)^{-4}\right\|_{L^a(\mathbb{R}^3)} \sim (t-t')^{-2+\frac{3}{2}\left(1+\frac{1}{r}-\frac{1}{p}-\frac{1}{q}\right)} \\ &= (t-t')^{-\frac{1}{2}-\frac{3}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)}.\end{aligned}$$

We then conclude

$$\|B(u,v)\|_{L^r} \leqslant Ct^{-\frac{3}{2}\left(\frac{1}{3}-\frac{1}{r}\right)}\|u\|_{K_p(t)}\|v\|_{K_q(t)} \tag{11.15}$$

where we used the fact that $\forall\ \alpha,\beta \in (-\infty,1)$ we have

$$\int_0^t (t-t')^{-\alpha}(t')^{-\beta}dt' = C(\alpha,\beta)t^{1-\alpha-\beta} \text{ for all } t>0 \text{ and for } C(\alpha,\beta) := \int_0^1 (1-t')^{-\alpha}(t')^{-\beta}dt'. \tag{11.16}$$

and

$$\begin{aligned}&\frac{1}{2}+\frac{3}{2}\left(\frac{1}{p}+\frac{1}{q}-\frac{1}{r}\right)+\frac{3}{2}\left(\frac{2}{3}-\frac{1}{p}-\frac{1}{q}\right) = \frac{1}{2}+\frac{3}{2}\left(\frac{2}{3}-\frac{1}{r}\right) = \frac{1}{2}+1-\frac{3}{2r} \\ &= 2-\frac{1}{2}-\frac{3}{2r} = 1+1-\frac{3}{2r} = 1+\frac{3}{2}\left(\frac{1}{3}-\frac{1}{r}\right).\end{aligned}$$

Notice that in the inequalities in (11.5) we need:

- $\frac{1}{\beta} := \frac{1}{p} + \frac{1}{q} \leqslant 1$ in order for $u \otimes v$ to belong to the Lebesgue space $L^\beta(\mathbb{R}^3)$;
- $0 < \frac{1}{p} + \frac{1}{q}$ is needed because otherwise in (11.15) we get $(t')^{-1}$ and the integral is undefined;
- $\frac{1}{r} \leqslant \frac{1}{p} + \frac{1}{q}$ is needed for $a \geq 1$;
- $\frac{1}{p} + \frac{1}{q} < \frac{1}{3} + \frac{1}{r}$ is needed to get $-\frac{1}{2} - \frac{3}{2}\left(\frac{1}{p} + \frac{1}{q} - \frac{1}{r}\right) > -1$ in the exponent of $(t-t')$ in (11.14).

□

Exercise 11.12. Prove (11.16). Hint, split the integral into sum of integrals in $[0, t/2]$ and $[t/2, t]$.

We have the following fact.

Proposition 11.13. *For any $p \in (3, \infty]$ there exists a constant $\varepsilon_{p\nu} > 0$ s.t. if*

$$\|e^{t\triangle} u_0\|_{K_p(T)} < \varepsilon_{p\nu} \tag{11.17}$$

then there exists and is unique u in the ball of center 0 and radius $2\varepsilon_{p\nu}$ in $K_p(T)$ which satisfies (9.2).

Proof. Setting $r = q = p$, we see that for $p > 3$ we have $B : K_p(T) \times K_p(T) \to K_p(T)$ is bounded and with norm that admits a finite upper bound independent from T. The proof follows then from the abstract Lemma 9.1. □

Theorem 11.14. *For any $u_0 \in L^3(\mathbb{R}^3, \mathbb{R}^3)$ there is a $T > 0$ and solution $u \in C^0([0, T), L^3(\mathbb{R}^3, \mathbb{R}^3))$ of* (9.2) *which is unique. Furthermore there exists a $\varepsilon_3 > 0$ s.t. for $\|u_0\|_{L^3} < \varepsilon_{3\nu}$ we have $T = \infty$.*

Proof. We have $e^{t\triangle} u_0 \in K_p(T)$ for any $p > 3$, see (11.3). Furthermore, $\|e^{t\triangle} u_0\|_{K_p(T)} \xrightarrow{T \to 0} 0$ for $p > 3$ by Lemma 11.9. Then we can apply Proposition 11.13 concluding that there exists a solution u of (9.2) in $K_6(T)$ for $T > 0$ small enough. Applying Lemma 11.10 for $p = q = 6$ and $r = 3$ we get $B(u, u) \in C^0([0, T], L^3)$, and so $u \in C^0([0, T], L^3)$.
We assume now that there are two solutions u_1 and u_2. Setting $u_{21} = u_2 - u_1$ and $w_j = B(u_j, u_j)$ we have

$$\begin{cases} \partial_t u_{21} - \nu \triangle u_{21} = f_{21} \\ \qquad u_{21}(0) = 0 \end{cases} \text{with}$$

$$f_{21} = 2Q(e^{\nu t\triangle} u_0, u_{21}) + Q(w_2, u_{21}) + Q(w_1, u_{21}).$$

By $L^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$, which is the dual of Sobolev's Embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$, we have

$$\|Q(u,v)\|_{\dot{H}^{-\frac{3}{2}}(\mathbb{R}^3)} \le \|u\otimes v\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \lesssim \|u\otimes v\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \le \|u\|_{L^3}\|v\|_{L^3}.$$

Then, by (7.5) and entering the definition of f_{21}

$$\begin{aligned}
&\|u_{21}(t)\|^2_{\dot{H}^{-\frac{1}{2}}} + 2\nu\int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}}dt' \le 4\int_0^t \langle f(t'),u(t')\rangle_{\dot{H}^{-\frac{1}{2}}}dt' \\
&\le 2\int_0^t \|Q(e^{\nu t'\triangle}u_0,u_{21})\|_{\dot{H}^{-\frac{3}{2}}}\|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}dt' \\
&+2\int_0^t \|Q(w_2,u_{21})+Q(w_1,u_{21})\|_{\dot{H}^{-\frac{3}{2}}}\|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}dt'.
\end{aligned} \tag{11.18}$$

We bound the last line with, for $j=1,2$,

$$\begin{aligned}
&2\int_0^t \|Q(w_j,u_{21})\|_{\dot{H}^{-\frac{3}{2}}}\|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}dt' \lesssim \|w_j\|_{K_3(t)}\int_0^t \|u_{21}(t')\|_{L^3}\|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}dt' \\
&\lesssim \|w_j\|_{K_3(t)}\int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}}dt',
\end{aligned} \tag{11.19}$$

where in the last line we used Sobolev's Embedding $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$.
So, the last line of (11.18) is

$$\lesssim \left(\|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)}\right)\int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}}dt'. \tag{11.20}$$

We split now

$$u_0 = u_0^{(1)} + u_0^{(2)} \text{ with } \|u_0^{(1)}\|_{L^3} < \epsilon \text{ and } u_0^{(2)} \in L^6\cap L^3$$

and we bound similarly to (11.19)

$$\int_0^t \|Q(e^{\nu t'\triangle}u_0^{(1)},u_{21})\|_{\dot{H}^{-\frac{3}{2}}}\|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}dt' \lesssim \|u_0^{(1)}\|_{L^3}\int_0^t \|\nabla u_{21}(t')\|^2_{\dot{H}^{-\frac{1}{2}}}dt'.$$

Finally, we bound

$$\begin{aligned}
&\int_0^t \|Q(e^{\nu t'\triangle}u_0^{(2)},u_{21})\|_{\dot{H}^{-\frac{3}{2}}}\|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}dt' \\
&\le \int_0^t \|e^{\nu t'\triangle}u_0^{(2)}\otimes u_{21}\|_{\dot{H}^{-\frac{1}{2}}}\|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}dt' \lesssim \int_0^t \|e^{\nu t'\triangle}u_0^{(2)}\otimes u_{21}\|_{L^{\frac{3}{2}}}\|\nabla u_{21}(t')\|_{\dot{H}^{-\frac{1}{2}}}dt' \\
&\le \int_0^t \|e^{\nu t'\triangle}u_0^{(2)}\|_{L^6}\ \|u_{21}\|_{L^2}\|\nabla u_{21}\|_{H^{\frac{1}{2}}}dt' \le \|u_0^{(2)}\|_{L^6}\int_0^t \|u_{21}\|^{\frac{1}{2}}_{H^{-\frac{1}{2}}}\|\nabla u_{21}\|^{\frac{3}{2}}_{H^{\frac{1}{2}}}dt'.
\end{aligned}$$

So we get

$$\|u_{21}(t)\|^2_{\dot H^{-\frac12}} + 2\nu \int_0^t \|\nabla u_{21}(t')\|^2_{\dot H^{-\frac12}} dt' \lesssim \Big(\|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)} + \|u_0^{(1)}\|_{L^3} \Big) \int_0^t \|\nabla u_{21}(t')\|^2_{\dot H^{-\frac12}} dt$$
$$+ \frac{3}{4\mathbf{C}^{\frac43}} \int_0^t \|\nabla u_{21}(t')\|^2_{\dot H^{-\frac12}} dt' + \frac{\mathbf{C}^4}{4} \|u_0^{(2)}\|^4_{L^6} \int_0^t \|u_{21}\|^2_{H^{-\frac12}} dt'.$$

Taking $\mathbf{C}$ large, and t small, so that $\|w_1\|_{K_3(t)} + \|w_2\|_{K_3(t)} + \|u_0^{(1)}\|_{L^3} < 3\epsilon$ with ϵ sufficiently small, we obtain

$$\|u_{21}(t)\|^2_{\dot H^{-\frac12}} + \nu \int_0^t \|\nabla u_{21}(t')\|^2_{\dot H^{-\frac12}} dt' \lesssim \frac{\mathbf{C}^4}{4} \|u_0^{(2)}\|^4_{L^6} \int_0^t \|u_{21}\|^2_{H^{-\frac12}} dt'.$$

Gronwall's Inequality implies that $u_{21}(t') = 0$ for $t' \in [0, t]$ with $t > 0$ sufficiently small. The above argument shows that the set

$$\{t \in [0, T) : u_{21} \equiv 0 \text{ in } [0, t]\}$$

is open (and, obviously, non empty) in $[0, T)$. On the other hand, since $u_{21} \in C^0([0, T), L^3(\mathbb{R}^3, \mathbb{R}^3))$, it is also closed in $[0, T)$. Hence it coincides with $[0, T)$.

□

Remark 11.15. Let $u_0 \in \dot H^{\frac12}(\mathbb{R}^3, \mathbb{R}^3)$. Then it can be proved that if $T_3 > 0$ is the lifespan of the corresponding solution $u \in C^0([0, T_3), L^3(\mathbb{R}^3, \mathbb{R}^3))$ provided by Theorem 11.14 and if $T_{u_0} > 0$ is the lifespan of the solution provided by Theorem 9.2, we have $T_3 = T_{u_0}$. We will prove the simpler result in Proposition 11.16.

Proposition 11.16. *Let $u_0 \in \dot H^{\frac12}(\mathbb{R}^3, \mathbb{R}^3)$. There there exists $\epsilon_{3\nu} > 0$ s.t. for $\|u_0\|_{L^3(\mathbb{R}^3)} < \epsilon_{3\nu}$ and if $T_{u_0} > 0$ is the lifespan of the solution provided by Theorem 9.2, we have $T_{u_0} = \infty$.*

Proof. Taking $\epsilon_{3\nu} > 0$ sufficiently small we can assume by Theorem 11.14 that $u \in C^0([0, \infty), L^3)$. In fact, if it is sufficiently small we can prove $\|u\|_{L^\infty([0,\infty), L^3)} < C_\nu \|u_0\|_{L^3}$ for a fixed $C_\nu > 0$. Suppose that $T_{u_0} < \infty$. Then by Theorem 9.2 we have the blow up

$$\lim_{T \nearrow T_{u_0}} \int_0^T \|\nabla u(t)\|^2_{\dot H^{\frac12}} dt = \infty. \tag{11.21}$$

By Theorem 9.2 and by (7.5), for $0 < t \le T < T_{u_0}$ we have

$$\|u(t)\|^2_{\dot H^{\frac12}} + 2\nu \int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' = \|u_0\|^2_{\dot H^{\frac12}} + 2 \int_0^t \langle u(t') \cdot \nabla u(t'), u(t') \rangle_{\dot H^{\frac12}} dt'. \tag{11.22}$$

By Sobolev's Embedding $\dot H^{\frac12}(\mathbb{R}^3, \mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3, \mathbb{R}^3)$ we obtain

$$|\langle u \cdot \nabla u, u \rangle_{\dot H^{\frac12}}| = |\langle u \cdot \nabla u, \nabla u \rangle_{L^2}| \le \|u\|_{L^3} \|\nabla u\|^2_{L^3} \le C \|u\|_{L^3} \|\nabla u\|^2_{\dot H^{\frac12}}.$$

Then

$$\|u(t)\|^2_{\dot H^{\frac12}} + 2\nu\int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' \le \|u_0\|^2_{\dot H^{\frac12}} + C\|u\|_{L^\infty(\mathbb{R}_+,L^3)} \int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt'$$
$$\le \|u_0\|^2_{\dot H^{\frac12}} + C_\nu C\|u_0\|_{L^3}\int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt'.$$

So, for $C_\nu C\|u_0\|_{L^3} < \nu$, we get

$$\|u(t)\|^2_{\dot H^{\frac12}} + \nu\int_0^t \|\nabla u(t')\|^2_{\dot H^{\frac12}} dt' \le \|u_0\|^2_{\dot H^{\frac12}},$$

which contradicts (11.21).

□

12 Schrödinger equations

For $u_0 \in \mathcal{S}'(\mathbb{R}^d,\mathbb{C})$ the linear homogeneous Schrödinger equation is

$$\mathrm{i}u_t + \triangle u = 0\,, \quad u(0,x) = u_0(x).$$

By applying $\mathcal{F}$ we transform the above problem into

$$\widehat{u}_t + \mathrm{i}|\xi|^2\widehat{u} = 0\,, \quad \widehat{u}(0,\xi) = \widehat{u}_0(\xi).$$

This yields $\widehat{u}(t,\xi) = e^{-\mathrm{i}t|\xi|^2}\widehat{u}_0(\xi)$. We have $e^{-\mathrm{i}t|\xi|^2} = \widehat{G}(t,\xi)$ with $G(t,x) = (2t\mathrm{i})^{-\frac{d}{2}}e^{\frac{\mathrm{i}|x|^2}{4t}}$. This follows from the following generalization of (1.2) for $\operatorname{Re} z > 0$

$$e^{-z\frac{|\xi|^2}{2}} = (2\pi z)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x}e^{-\frac{|x|^2}{2z}}dx.$$

This formula follows from the fact that both sides are holomorphic in $\operatorname{Re} z > 0$ and coincide for $z \in \mathbb{R}_+$. Then taking the limit $z \to 2\mathrm{i}$ for $\operatorname{Re} z > 0$ and using the continuity of $\mathcal{F}$ in $\mathcal{S}'(\mathbb{R}^d,\mathbb{C})$ we get

$$e^{-\mathrm{i}|\xi|^2} = (4\pi\mathrm{i})^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\mathrm{i}\xi\cdot x}e^{\frac{\mathrm{i}|x|^2}{4}}dx.$$

Then $u(t,x) = (2\pi)^{-\frac{d}{2}}G(t,\cdot) * u_0(x)$. In particular, for $u_0 \in L^p(\mathbb{R}^d,\mathbb{C})$ for $p \in [1,2]$ and by Reisz's interpolation defines for any $t > 0$ an operator which we denote by

$$e^{\mathrm{i}\triangle t}u_0(x) = (4\pi\mathrm{i}t)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{\frac{\mathrm{i}|x-y|^2}{4t}}u_0(y)dy \tag{12.1}$$

which is s.t. $e^{\mathrm{i}\triangle t} : L^p(\mathbb{R}^d,\mathbb{C}) \to L^{p'}(\mathbb{R}^d,\mathbb{C})$ for $p \in [1,2]$ and $p' = \frac{p}{p-1}$ with $\|e^{\mathrm{i}\triangle t}u_0\|_{L^{p'}} \le (4\pi t)^{-d(\frac12 - \frac{1}{p'})}\|u_0\|_{L^p}$ by Riesz interpolation.

Remark 12.1. Notice that for no $p \neq 2$ and $t > 0$ we have that $e^{\mathrm{i}\triangle t}$ defines a bounded operator $L^p(\mathbb{R}^d, \mathbb{C}) \to L^p(\mathbb{R}^d, \mathbb{C})$, see [9].

Remark 12.2. Notice that $e^{\triangle t} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ is a bounded operator for all $1 \leq p \leq q \leq \infty$.

In the sequel, given $v, w \in L^2(\mathbb{R}^d, \mathbb{C})$ we will use the notation

$$\langle v, w \rangle = \mathrm{Re} \int_{\mathbb{R}^d} v(x)\overline{w}(x)dx. \tag{12.2}$$

In the sequel we will reinterpret the equation

$$\mathrm{i}u_t + \triangle u = f\,, \quad u(0) = u_0 \in H^1(\mathbb{R}^d) \tag{12.3}$$

in the integral form

$$u(t) = e^{-\mathrm{i}t\triangle}u_0 - \mathrm{i}\int_0^t e^{-\mathrm{i}(t-t')\triangle} f(t')dt'. \tag{12.4}$$

To understand this formula we will need Strichartz's inequalities.

We say that a pair (q, r) is *admissible* when

$$\begin{aligned} &\frac{2}{q} + \frac{d}{r} = \frac{d}{2} \\ &2 \leq r \leq \frac{2d}{d-2} \ (2 \leq r \leq \infty \text{ if } d = 1,\ 2 \leq r < \infty \text{ if } d = 2). \end{aligned} \tag{12.5}$$

The pair $(\infty, 2)$ is always admissible. The *endpoint* $(2, \frac{2d}{d-2})$ is admissible for $d \geq 3$. We have the following important result.

Theorem 12.3 (Strichartz's estimates)**.** *The following facts hold.*

(1) For every $u_0 \in L^2(\mathbb{R}^d)$ we have $e^{\mathrm{i}\triangle t}u_0 \in L^q(\mathbb{R}, L^r(\mathbb{R}^d)) \cap C^0(\mathbb{R}, L^2(\mathbb{R}^d))$ for every admissible (q, r). Furthermore, there exists a C s.t.

$$\|e^{\mathrm{i}\triangle t}u_0\|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C\|u_0\|_{L^2}. \tag{12.6}$$

(2) Let I be an interval and let $t_0 \in \overline{I}$. If (γ, ρ) is an admissible pair and $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))$ then for any admissible pair (q, r) the function

$$\mathcal{T}f(t) = \int_{t_0}^t e^{\mathrm{i}\triangle(t-s)} f(s)ds \tag{12.7}$$

belongs to $L^q(I, L^r(\mathbb{R}^d)) \cap C^0(\overline{I}, L^2(\mathbb{R}^d))$ and there exists a constant C independent of I and f s.t.

$$\|\mathcal{T}f\|_{L^q(I, L^r(\mathbb{R}^d))} \leq C\|f\|_{L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^d))}. \tag{12.8}$$

13 Keel and Tao's proof of Strichartz estimates

We will follow the argument by Keel and Tao [8]. We will assume that (X, dx) is a measurable space and that H is a Hilbert space. We consider a family of operators $U(t) : H \to L^2(X)$. We assume the following two hypotheses.

(1) There exists a $C > 0$ s.t.

$$\|U(t)f\|_{L^2} \le C\|f\|_H \text{ for all } f \in H;$$

(2) there exist a $\sigma > 0$ and a $C > 0$ s.t. for all $t \neq s$ and all $g \in L^1(X)$ we have

$$\|U(t)(U(s))^* g\|_{L^\infty} \le C|t-s|^{-\sigma}\|g\|_{L^1}.$$

We say that a pair (q, r) is σ*–admissible* when

$$\begin{aligned} &\frac{2}{q} + \frac{2\sigma}{r} = \sigma \\ &r, q \ge 2 \text{ and } (q, r, \sigma) \neq (2, \infty, 1). \end{aligned} \tag{13.1}$$

Particularly important, for $\sigma > 1$, is the point $P = \left(2, \dfrac{2\sigma}{\sigma - 1}\right)$.

Notice that (1) implies $\|U^*(t)F\|_{L^2} \le C\|F\|_{L^2}$ by duality.

Theorem 13.1 (Keel and Tao's Strichartz estimates)**.** *If $U(t)$ satisfies (1) and (2), and if furthermore there exists an appropriate scaling operator in X and H, then we have*

(3)

$$\|U(t)u_0\|_{L^q(\mathbb{R}, L^r(X))} \le C_{q,r}\|u_0\|_H.$$

(4)

$$\| \int_{\mathbb{R}} (U(s))^* F(s) ds \|_H \le C\|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))}.$$

(5)

$$\| \int_{t>s} U(t)(U(s))^* F(s) ds \|_{L^q(\mathbb{R}, L^r(X))} \le C_{q,r,\widetilde{q},\widetilde{r}} \|F\|_{L^{\widetilde{q}'}(\mathbb{R}, L^{\widetilde{r}'}(X))}.$$

for all admissible pairs (q, r) and $(\widetilde{q}, \widetilde{r})$.

(3) is called the homogeneous estimate and (5) the non–homogeneous estimate or also the retarded estimate. (3) and (4) are equivalent by duality. The scaling operators are used only in Sect. 13.2.

13.1 Proof of the nonendpoint homogeneous estimate

We consider the case $(q,r) \neq P$. The proof of this case predates the paper by Keel and Tao.
It is immediate that (4) is equivalent to

$$\left| \int_{\mathbb{R}^2} \langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H \, dtds \right| \leq C \|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \|G\|_{L^{q'}(\mathbb{R}, L^{r'}(X))}.$$

So we have to prove the above estimate. Furthermore, it is enough to prove the above bound for

$$T(F,G) := \int_{t>s} \langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H \, dtds. \tag{13.2}$$

By (1) we know that (3) holds for $q = \infty$ and $r = 2$. So pointwise

$$\begin{aligned} &|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H| = |\langle U(t)(U(s))^* F(s), G(t) \rangle_{L^2}| \\ &\leq \|U(t)(U(s))^* F(s)\|_{L^2(X)} \|G(t)\|_{L^2(X)} \leq C^2 \|F(s)\|_{L^2(X)} \|G(t)\|_{L^2(X)}. \end{aligned}$$

Furthermore

$$\begin{aligned} &|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H| = |\langle U(t)(U(s))^* F(s), G(t) \rangle_H| \leq \|U(t)(U(s))^* F(s)\|_{L^\infty(X)} \|G(t)\|_{L^1(X)} \\ &\leq C|t-s|^{-\sigma} \|F(s)\|_{L^1(X)} \|G(t)\|_{L^1(X)}. \end{aligned}$$

From the Riesz–Thorin Interpolation Theorem, see Theorem 1.6, we have (omitting the constant) for any $r \in 2, \infty]$

$$\|U(t)(U(s))^* F(s)\|_{L^r(X)} \lesssim |t-s|^{-\sigma\left(1-\frac{2}{r}\right)} \|F(s)\|_{L^{r'}(X)} = |t-s|^{-1-\beta(r,r)} \|F(s)\|_{L^{r'}(X)}$$
$$\text{where } \beta(r,\widetilde{r}) := \sigma - 1 - \frac{\sigma}{r} - \frac{\sigma}{\widetilde{r}}.$$

Then we conclude

$$|\langle (U(s))^* F(s), (U(t))^* G(t) \rangle_H| \lesssim |t-s|^{-1-\beta(r,r)} \|F(s)\|_{L^{r'}(X)} \|G(t)\|_{L^{r'}(X)}.$$

Then for $\frac{1}{q'} - \frac{1}{q} = -\beta(r,r)$, using the Hardy,Littlewood Sobolev inequality, see Theorem 4.4 , which requires $q > q'$,

$$|T(F,G)| \lesssim \| \int_{\mathbb{R}} |t-s|^{-1-\beta(r,r)} \|F(s)\|_{L^{r'}(X)} ds \|_{L^q(\mathbb{R})} \|G\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \lesssim \|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \|G\|_{L^{q'}(\mathbb{R}, L^{r'}(X))}.$$

Notice that $\frac{1}{q'} - \frac{1}{q} = -\beta(r,r)$ means

$$1 - \frac{2}{q} = -\sigma + 1 + 2\frac{\sigma}{r} \Leftrightarrow \frac{2}{q} + \frac{2\sigma}{r} = \sigma$$

and $-\beta(r,r) > 0$ means

$$r < \frac{2\sigma}{\sigma - 1}.$$

13.2 Proof of the endpoint homogeneous estimate

Here we consider the endpoint case $(q, r) = P = (2, \frac{2\sigma}{\sigma-1})$, when $\sigma > 1$.

The introduction of a scaling operator will simplify considerably the discussion. We will denote it by D_λ for $\lambda > 0$. We assume the following:

1. there exist operators $D_\lambda : H \to H$ s.t. $\langle D_\lambda f, D_\lambda g\rangle_H = \lambda^{-\sigma} \langle f, g\rangle_H$
2. there exist operators $D_\lambda : L^r(X) \to L^r(X)$ s.t $\|D_\lambda F\|_{L^r(X)} = \lambda^{-\frac{\sigma}{r}} \|F\|_{L^r(X)}$
3. in all cases $D_\lambda^{-1} = D_{\lambda^{-1}}$.

Notice that for $\sigma = \frac{d}{2}$, $H = L^2(\mathbb{R}^d)$ and $X = \mathbb{R}^d$ with $L^r(X)$ the standard Lebesgue spaces, then $D_\lambda f(x) := f(\lambda^{\frac{1}{2}} x)$ satisfies the desired requirements.

Lemma 13.2. *Let the function $t \to U(t)$ satisfy (1) and (2) in Sect. 13. Then $t \to D_\lambda U(\lambda t) D_{\lambda^{-1}}$ satisfies (1) and (2) in Sect. 13 with exactly the same constants C.*

Proof. Indeed

$$\|D_\lambda U(\lambda t) D_{\lambda^{-1}} f\|_{L^2} = \lambda^{-\frac{\sigma}{2}} \|U(\lambda t) D_{\lambda^{-1}} f\|_{L^2} \le C\lambda^{-\frac{\sigma}{2}} \|D_{\lambda^{-1}} f\|_H = C\|f\|_H$$

and from $(D_\lambda U(\lambda s) D_{\lambda^{-1}})^* = D_\lambda (U(\lambda s))^* D_{\lambda^{-1}}$,

$$\begin{aligned} &\|D_\lambda U(\lambda t) D_{\lambda^{-1}} (D_\lambda U(\lambda s) D_{\lambda^{-1}})^* f\|_{L^\infty} \|D_\lambda U(\lambda t)(U(\lambda s))^* D_{\lambda^{-1}} f\|_{L^\infty} \\ &= \|U(\lambda t)(U(\lambda s))^* D_{\lambda^{-1}} f\|_{L^\infty} \le C\lambda^{-\sigma} |t-s|^{-\sigma} \|D_{\lambda^{-1}} f\|_{L^1} = C|t-s|^{-\sigma}\|f\|_{L^1}. \end{aligned}$$

□

After the above preliminary on scaling operators, expand

$$T(F, G) = \sum_{j\in\mathbb{Z}} T_j(F, G) \text{ where } T_j(F, G) := \int_{t-2^j > s > t-2^{j+1}} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, dt ds. \tag{13.3}$$

We will prove

$$\sum_{j\in\mathbb{Z}} |T_j(F, G)| \lesssim \|F\|_{L^2 L^{r'}} \|G\|_{L^2 L^{r'}}. \tag{13.4}$$

We will prove the following.

Lemma 13.3. *For a fixed constant C dependent only on the constants in (1) –(2) Sect. 13. we have*

$$|T_j(F, G)| \le C 2^{-j\beta(a,b)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}. \tag{13.5}$$

with $(1/a, 1/b)$ in a sufficiently small, but fixed neighborhood of $(1/r, 1/r)$, dependent only on σ.

Proof. Notice that

$$
\begin{aligned}
&T_j(F,G) = \int_{t-2^j>s>t-2^{j+1}} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, dtds \\
&= 2^{2j} 2^{j\sigma} \int_{t-1>s>t-2} \left\langle D_{2^j}(U(2^j s))^* D_{2^{-j}} D_{2^j} F(2^j s), D_{2^j}(U(2^j t))^* D_{2^{-j}} D_{2^j} G(2^j t)\right\rangle_H dtds.
\end{aligned}
$$

Suppose now that we have (13.4) in the particular case $j = 0$. But then we have

$$
\begin{aligned}
&|T_j(F,G)| \le C 2^{2j} 2^{j\sigma} \|D_{2^j} F(2^j s)\|_{L^2 L^{a'}} \|D_{2^j} G(2^j t)\|_{L^2 L^{b'}} = C 2^{2j} 2^{j\sigma} 2^{-j\left(1+\frac{\sigma}{a'}+\frac{\sigma}{b'}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} \\
&= C 2^{j\left(2+\sigma-1-2\sigma+\frac{\sigma}{a}+\frac{\sigma}{b}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} = C 2^{j\left(1-\sigma+\frac{\sigma}{a}+\frac{\sigma}{b}\right)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}} = C 2^{-j\beta(a,b)} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}
\end{aligned}
$$

where we recall $\beta(a,b) = \sigma - 1 - \frac{\sigma}{a} - \frac{\sigma}{b}$.

So we have reduced to the case $j = 0$. Next we do another reduction. We claim that to prove the case $j = 0$ it is enough to assume that F and G are supported in time intervals of length 1. Indeed, assuming this case, then we have

$$
\begin{aligned}
&|T_0(F,G)| \le \sum_{n\in\mathbb{Z}} \left| \int_{n+1>t>n} dt \int_{t-1>s>t-2} \langle (U(s))^* F(s), (U(t))^* G(t)\rangle_H \, ds \right| \\
&\le C \sum_{n\in\mathbb{Z}} \|F\|_{L^2((n,n+1),L^{a'})} \|G\|_{L^2((n-2,n),L^{b'})} \le C \left(\sum_{n\in\mathbb{Z}} \|F\|^2_{L^2((n,n+1),L^{a'})} \right)^{\frac12} \left(\sum_{n\in\mathbb{Z}} \|G\|^2_{L^2((n-2,n),L^{a'})} \right)^{\frac12} \\
&= C\sqrt{2} \left(\sum_{n\in\mathbb{Z}} \|F\|^2_{L^2((n,n+1),L^{a'})} \right)^{\frac12} \left(\sum_{n\in\mathbb{Z}} \|G\|^2_{L^2((n-1,n),L^{b'})} \right)^{\frac12} = C\sqrt{2} \|F\|_{L^2 L^{a'}} \|G\|_{L^2 L^{b'}}.
\end{aligned}
$$

Hence, in the rest of the proof we will assume that F and G are supported in time intervals of length 1. To prove (13.5) for $j = 0$ we consider three cases:

(i) $a = b = \infty$;

(ii) $2 \le a < r$ and $b = 2$;

(iii) $a = 2$ and $2 \le b < r$.

Then the desired result follows by interpolation.

Let us start with (i). The proof is elementary and straightforward, because we have

$$
\begin{aligned}
&|T_0(F,G)| \le \int dt \int_{t-1>s>t-2} |\langle U(t)(U(s))^* F(s), G(t)\rangle_H| ds \\
&\le C \int dt \int_{t-1>s>t-2} |t-s|^{-\sigma} \|F(s)\|_{L^1} \|G(t)\|_{L^1} \le C \int dt \int_{t-1>s>t-2} \|F(s)\|_{L^1} \|G(t)\|_{L^1} \\
&\le C \|F\|_{L^1 L^1} \|G\|_{L^1 L^1} \le C \|F\|_{L^2 L^1} \|G\|_{L^2 L^1}.
\end{aligned}
$$

Let us now consider (ii). Here we will use the Strichartz estimates in Sect. 13.1. We have

$$
\begin{aligned}
|T_0(F,G)| &\le \int |\left\langle \int_{t-1>s>t-2} (U(s))^* F(s)ds, (U(t))^* G(t) \right\rangle_H |dt \\
&\le \int \left\| \int_{t-1>s>t-2} (U(s))^* F(s)ds \right\|_H \|(U(t))^* G(t)\|_H dt \\
&\le \sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s)ds \right\|_H \int \|(U(t))^* G(t)\|_H dt \\
&\le C\|G\|_{L^1L^2} \sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s)ds \right\|_H ,
\end{aligned}
$$

where we used (1) in Sect. 13. Now, using the Strichartz estimates in Sect. 13.1 we have, for $(q(a),a)$ admissible,

$$
\sup_t \left\| \int_{t-1>s>t-2} (U(s))^* F(s)ds \right\|_H \le C\|F\|_{L^{q(a)'}L^{a'}} \le C\|F\|_{L^2L^{a'}}.
$$

This proves (ii) and by symmetry yields also (iii). □

Now we need to show that (13.5) implies (13.4). Obviously, we cannot just take $a = b = r$ and sum up, since $\beta(r,r) = 0$. To get the idea on how to overcome this problem, Keel and Tao consider functions of the form

$$
F(t) = 2^{-\frac{k}{r'}} f(t)\chi_{E(t)}(x) \text{ and } G(s) = 2^{-\frac{\widetilde{k}}{r'}} g(s)\chi_{\widetilde{E}(s)}(x), \tag{13.6}
$$

with scalar functions $f(t), g(s)$ and $E(t)$ resp. $\widetilde{E}(s)$ sets of size 2^k resp. $2^{\widetilde{k}}$. Applying (13.5) we obtain

$$
\begin{aligned}
|T_j(F,G)| &\le C2^{-j\left(\sigma-1-\frac{\sigma}{a}-\frac{\sigma}{b}\right)} 2^{-\frac{k}{r'}} 2^{\frac{k}{a'}} 2^{-\frac{\widetilde{k}}{r'}} 2^{\frac{\widetilde{k}}{b'}} \|f\|_{L^2}\|g\|_{L^2} \\
&= C2^{-j\left(\frac{2\sigma}{r}-\frac{\sigma}{a}-\frac{\sigma}{b}\right)} 2^{-(k+\widetilde{k})\left(\cancel{1}-\frac{1}{r}\right)+\cancel{(k+\widetilde{k})}-\frac{k}{a}-\frac{\widetilde{k}}{b}} \|f\|_{L^2}\|g\|_{L^2} \\
&= C2^{-j\left(\frac{2\sigma}{r}-\frac{\sigma}{a}-\frac{\sigma}{b}\right)+k\left(\frac{1}{r}-\frac{1}{a}\right)+\widetilde{k}\left(\frac{1}{r}-\frac{1}{b}\right)} \|f\|_{L^2}\|g\|_{L^2} \\
&= C2^{(k-j\sigma)\left(\frac{1}{r}-\frac{1}{a}\right)+\left(\widetilde{k}-j\sigma\right)\left(\frac{1}{r}-\frac{1}{b}\right)} \|f\|_{L^2}\|g\|_{L^2}.
\end{aligned} \tag{13.7}
$$

Notice now that we can adjust (a,b) s.t. for a fixed small $\varepsilon > 0$ the last term equals

$$
C2^{-\varepsilon|k-j\sigma|-\varepsilon\left|\widetilde{k}-j\sigma\right|} \|f\|_{L^2}\|g\|_{L^2} \tag{13.8}
$$

whose sum for $j \in \mathbb{Z}$ is finite.

To convert the above intuition in a proof we consider the following preliminary lemma.

Lemma 13.4. *Let $p \in (0,\infty)$. Then any $f \in L^p_x$ can be written as*

$$
f = \sum_{k\in\mathbb{Z}} c_k \chi_k
$$

where $meas(supp\chi_k) \le 2\ 2^k$, $|\chi_k| \le 2^{-\frac{k}{p}}$ and $\|c_k\|_{\ell^p} \le 2^{\frac{1}{p}}\|f\|_{L^p}$.

Proof. Consider the distribution function $\lambda(\alpha) = \text{meas}(\{|f(x)| > \alpha\})$. Then for each k consider

$$\alpha_k := \inf_{\lambda(\alpha)<2^k} \alpha\,, \quad c_k := 2^{\frac{k}{p}}\alpha_k\,, \quad \chi_k := \frac{1}{c_k}\chi_{(\alpha_{k+1},\alpha_k]}(|f|)f.$$

We show the desired properties. We have

$$\text{supp}\chi_k \subseteq \{x : \alpha_{k+1} < |f(x)| \le \alpha_k\} \subseteq \{x : |f(x)| > \alpha_{k+1}\}.$$

Then we get the 1st inequality:

$$\text{meas}(\text{supp}\chi_k) \le \text{meas}(\{x : |f(x)| > \alpha_{k+1}\}) = \lim_{\alpha\to\alpha_{k+1}^+} \lambda(\alpha) = \sup\{\lambda(\alpha) : \alpha > \alpha_{k+1}\}) \le 2^{k+1}.$$

Next, by $|f(x)| \le \alpha_k$ in $\text{supp}\chi_k$, we have

$$|\chi_k(x)| \le 2^{-\frac{k}{p}}\frac{|f(x)|}{\alpha_k} \le 2^{-\frac{k}{p}}.$$

Let now $\lim\limits_{k\to+\infty} \alpha_k = \inf\limits_{k\in\mathbb{Z}} \alpha_k = \underline{\alpha}$ and $\lim\limits_{k\to-\infty} \alpha_k = \sup\limits_{k\in\mathbb{Z}} \alpha_k = \overline{\alpha}$. Then we claim that $\underline{\alpha} = 0$ and that $|f(x)| \le \overline{\alpha}$ a.e. Indeed, suppose that $|f(x)| > \overline{\alpha}$ on a set of positive measure. There there is $\alpha > \overline{\alpha}$ with $\lambda(\alpha) > 2^k$ for some $k \in \mathbb{Z}$. Then $\alpha_k \ge \alpha > \overline{\alpha}$, which is a contradiction. On the other hand, suppose we have $0 < \alpha < \underline{\alpha}$. Then $\lambda(\alpha) = \infty$, since otherwise $\lambda(\alpha) < 2^k$ for a k, and then $\alpha \ge \alpha_k \ge \underline{\alpha} > \alpha$, getting a contradiction. But by Chebyshev's inequality,

$$\infty > \|f\|_{L^p}^p \ge \alpha^p\lambda(\alpha),$$

hence getting a contradiction. The above claim and the obvious fact that for any x we have $|f(x)| \in (\alpha_{k+1}, \alpha_k]$ for at most one k, prove $f = \sum\limits_{k\in\mathbb{Z}} c_k\chi_k$ (the claim guarantees the existence of one such k).

We have $\|f\|_{L^p} \le 2^{\frac{1}{p}}\|c_k\|_{\ell^p}$ by

$$\|f\|_{L^p}^p = \int |f|^p dx = \int \sum_{k\in\mathbb{Z}} |c_k|^p|\chi_k|^p dx = \sum_{k\in\mathbb{Z}} |c_k|^p \int |\chi_k|^p dx \le \sum_{k\in\mathbb{Z}} |c_k|^p 2^{-k}\text{meas}(\text{supp}\chi_k)$$
$$\le 2\sum_{k\in\mathbb{Z}} |c_k|^p$$

Next we have

$$\sum_{k\in\mathbb{Z}} |c_k|^p = \sum_{k\in\mathbb{Z}} 2^k\alpha_k^p = \int_{\mathbb{R}_+} \alpha^p \left(\sum 2^k\delta(\alpha - \alpha_k)\right) d\alpha = \int_{\mathbb{R}_+} \alpha^p(-F'(\alpha))d\alpha$$

where

$$F(\alpha) := \sum_{k\in\mathbb{Z}} 2^k H(\alpha - \alpha_k) = \sum_{\alpha_k>\alpha} 2^k \le \sum_{2^k\le\lambda(\alpha)} 2^k \le 2\lambda(\alpha).$$

Then, integrating by parts and using (4.12),

$$\sum_{k\in\mathbb{Z}} |c_k|^p = p\int_{\mathbb{R}_+} \alpha^{p-1}F(\alpha)d\alpha \le 2p\int_{\mathbb{R}_+} \alpha^{p-1}\lambda(\alpha)d\alpha = 2\|f\|^p_{L^p},$$

so that $\|c_k\|_{\ell^p} \le 2^{\frac{1}{p}}\|f\|_{L^p}$.

□

Furthermore we have the following.

Lemma 13.5. *Let $1 \le q, r \le \infty$ and let $f \in L^q(I, L^r_x)$ with I an interval. Then we can write the expansion of Lemma 13.4*

$$f = \sum_{k\in\mathbb{Z}} c_k(t)\chi_k(t) \tag{13.9}$$

with $t \to \{c_k(t)\}$ a map in $L^q(I, \ell^r)$.

Proof. Formally this follows immediately from

$$\|\{c_k(t)\}\|_{L^q(I,\ell^r)} = \|\|\{c_k(t)\}\|_{\ell^r}\|_{L^q(I)} \le 2^{\frac{1}{p}}\|\|f\|_{L^r_x}\|_{L^q(I)}.$$

However one needs to argue that the function $t \to \{c_k(t)\}$ is measurable. By a density argument it is enough to consider the case of simple functions $f = \sum\limits_{j=1,\dots,n} \chi_{E_j}(t)g_j(x)$ with E_j mutually disjoints sets. Then $\lambda(t,\alpha) = \text{meas}(\{|f(t,x)| > \alpha\}) = \sum\limits_{j=1,\dots,n} \chi_{E_j}(t)\lambda_j(\alpha)$ with λ_j the distribution function of g_j. Then $\alpha_k(t) = \sum_{j=1,\dots,n} \chi_{E_j}(t)\alpha_k^{(j)}$ with $\alpha_k^{(j)}$ defined like in Lemma 13.4 for each g_j. Then

$$\{c_k(t)\} = \sum_{j=1,\dots,n} \chi_{E_j}(t)\{c_k^{(j)}\} \text{ for } c_k^{(j)} = 2^{\frac{k}{p}}\alpha_k^{(j)}.$$

This is measurable in t.

□

Consider now the

$$F(t) = \sum_{k\in\mathbb{Z}} f_k(t)\chi_k(t)\,, \quad G(s) = \sum_{k\in\mathbb{Z}} g_k(s)\widetilde{\chi}_k(s). \tag{13.10}$$

By (13.6)–(13.8) e have

$$\sum_j |T_j(F,G)| \le \sum_{j,k,\widetilde{k}} |T_j(f_k\chi_k, g_{\widetilde{k}}\widetilde{\chi}_{\widetilde{k}})| \le C\sum_{j,k,\widetilde{k}} 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|}\|f_k\|_{L^2_t}\|g_{\widetilde{k}}\|_{L^2_t}$$

$$= C\sum_{k,\widetilde{k}} \left(\sum_j 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|}\right)\|f_k\|_{L^2_t}\|g_{\widetilde{k}}\|_{L^2_t}.$$

We claim that for a fixed $C = C(\sigma, \varepsilon)$

$$\sum_j 2^{-\varepsilon|k-j\sigma|-\varepsilon|\widetilde{k}-j\sigma|} \le C 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|). \tag{13.11}$$

To prove this inequality, it is not restrictive to assume $k \le \widetilde{k}$. Then the summation on the left can be rewritten as

$$\sum_{j\sigma\le k} 2^{2\varepsilon j\sigma-\varepsilon(k+\widetilde{k})} + \sum_{k<j\sigma\le\widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} + \sum_{\widetilde{k}<j\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon j\sigma}.$$

Then (here $[t] \in \mathbb{Z}$ is the integer part of $t \in \mathbb{R}$, defined by $[t] \le t < [t]+1$)

$$\sum_{j\sigma\le k} 2^{2\varepsilon j\sigma-\varepsilon(k+\widetilde{k})} = 2^{-\varepsilon(k+\widetilde{k})} \sum_{j\le\left[\frac{k}{\sigma}\right]} 2^{2\varepsilon j\sigma} = 2^{-\varepsilon(k+\widetilde{k})} \sum_{j=0}^{\infty} 2^{2\varepsilon\sigma\left(\left[\frac{k}{\sigma}\right]-j\right)} = C_{\varepsilon\sigma}\ 2^{-\varepsilon(k+\widetilde{k})+2\varepsilon\sigma\left[\frac{k}{\sigma}\right]}$$

$$\le C_{\varepsilon\sigma} 2^{-\varepsilon(k+\widetilde{k})+2\varepsilon\sigma\frac{k}{\sigma}} = C_{\varepsilon\sigma} 2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon\sigma} 2^{-\varepsilon|k-\widetilde{k}|} \text{ where } C_{\varepsilon\sigma} = \frac{1}{1-2^{-2\varepsilon\sigma}}.$$

We have

$$\sum_{\widetilde{k}<j\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon j\sigma} \le 2^{\varepsilon(k+\widetilde{k})} \sum_{j\ge\left[\frac{\widetilde{k}}{\sigma}\right]+1} 2^{-2\varepsilon j\sigma} = 2^{\varepsilon(k+\widetilde{k})} \sum_{j=0}^{\infty} 2^{-2\varepsilon\sigma\left(\left[\frac{\widetilde{k}}{\sigma}\right]+1+j\right)} = C_{\varepsilon\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon\sigma\left(\left[\frac{\widetilde{k}}{\sigma}\right]+1\right)}$$

$$\le C_{\varepsilon\sigma} 2^{\varepsilon(k+\widetilde{k})-2\varepsilon\sigma\frac{\widetilde{k}}{\sigma}} = C_{\varepsilon\sigma} 2^{-\varepsilon(\widetilde{k}-k)} = C_{\varepsilon\sigma} 2^{-\varepsilon|k-\widetilde{k}|}.$$

Finally

$$\sum_{k<j\sigma\le\widetilde{k}} 2^{-\varepsilon(\widetilde{k}-k)} = 2^{-\varepsilon(\widetilde{k}-k)} \sum_{\left[\frac{k}{\sigma}\right]+1\le j\sigma\le\left[\frac{\widetilde{k}}{\sigma}\right]} 1 = 2^{-\varepsilon(\widetilde{k}-k)} \left(\left[\frac{\widetilde{k}}{\sigma}\right] - \left[\frac{k}{\sigma}\right] - 1 \right) \le \sigma^{-1} 2^{-\varepsilon(\widetilde{k}-k)}(\widetilde{k}-k)$$

Hence (13.11) is proved. From this we conclude that for a fixed C

$$\begin{aligned}
\sum_j |T_j(F,G)| &\le C \sum_{k,\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|)\|f_k\|_{L_t^2}\|g_{\widetilde{k}}\|_{L_t^2} \\
&\le C\|\{\|f_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})} \left\| \{\sum_{\widetilde{k}} 2^{-\varepsilon|k-\widetilde{k}|}(1+|k-\widetilde{k}|)\|g_{\widetilde{k}}\|_{L_t^2}\} \right\|_{\ell^2(\mathbb{Z})} \\
&\le C \left(\sum_k 2^{-\varepsilon|k|}(1+|k|) \right) \|\{\|f_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})} \|\{\|g_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})}
\end{aligned}$$

where we used Lemma 1.9. So, using $r' \le 2$,

$$\sum_j |T_j(F,G)| \le C'\|\{\|f_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})}\|\{\|g_k\|_{L_t^2}\}\|_{\ell^2(\mathbb{Z})} = C'\|\|\{f_k\}\|_{\ell^2(\mathbb{Z})}\|_{L_t^2}\|\|\{g_k\}\|_{\ell^2(\mathbb{Z})}\|_{L_t^2}$$

$$\le C''\|\|\{f_k\}\|_{\ell^{r'}(\mathbb{Z})}\|_{L_t^2}\|\|\{g_k\}\|_{\ell^{r'}(\mathbb{Z})}\|_{L_t^2} \le C'''\|\|F\|_{L_x^{r'}}\|_{L_t^2}\|\|G\|_{L_x^{r'}}\|_{L_t^2}$$

which completes the proof of (13.4).

13.3 Proof of the non homogeneous estimate

We need to prove that for all admissible pairs (q, r) and $(\widetilde{q}, \widetilde{r})$ we have

$$|T(F, G)| \le C_{q,r,\widetilde{q},\widetilde{r}} \|F\|_{L^{q'}(\mathbb{R}, L^{r'}(X))} \|G\|_{L^{\widetilde{q}'}(\mathbb{R}, L^{\widetilde{r}'}(X))}. \tag{13.12}$$

We have already proved that this is true for $(q, r) = (\widetilde{q}, \widetilde{r})$. Furthermore, proceeding like in Lemma 13.3

$$\begin{aligned}
|T(F,G)| &\le \int \left| \left\langle \int_{t>s} (U(s))^* F(s) ds, (U(t))^* G(t) \right\rangle_H \right| dt \\
&\le \int \| \int_{t>s} (U(s))^* F(s) ds \|_H \| (U(t))^* G(t) \|_H dt \le \sup_t \| \int_{t>s} (U(s))^* F(s) ds \|_H \int \| (U(t))^* G(t) \|_H dt \\
&\le C \|G\|_{L^1 L^2} \sup_t \| \int_{t>s} (U(s))^* F(s) ds \|_H,
\end{aligned}$$

Then, by (4) in Theorem 13.1 (that is the dual homogenous estimates, which are already proved) for any admissible pair (q, r)

$$\sup_t \| \int_{t>s} (U(s))^* F(s) ds \|_H = \sup_t \| \int_{\mathbb{R}} (U(s))^* F(s) \chi_{(-\infty,t)}(s) ds \|_H \le C \|F \chi_{(-\infty,t)}\|_{L^{q'}(\mathbb{R}, L^{r'})} \le C \|F\|_{L^{q'}(\mathbb{R}, L^{r'})}.$$

So (13.12) holds for $(\widetilde{q}, \widetilde{r}) = (\infty, 2)$ and any admissible pair (q, r). Obviously, symmetrically (13.12) holds for $(q, r) = (\infty, 2)$ and any admissible pair $(\widetilde{q}, \widetilde{r})$. Finally, let us consider (q, r) and $(\widetilde{q}, \widetilde{r})$ not in one of the cases already covered. Then it is not restrictive to assume that $(\widetilde{q}, \widetilde{r}) = (a_{t_0}, b_{t_0})$ for $t_0 \in (0, 1)$ where

$$\left(\frac{1}{a_t}, \frac{1}{b_t} \right) = t \left(\frac{1}{q}, \frac{1}{r} \right) + (1 - t) \left(\frac{1}{\infty}, \frac{1}{2} \right).$$

In the cases $t = 0, 1$ the inequality holds, because these are cases considered above. By a generalization of Riesz–Thorin, Theorem 1.6, the inequality holds also for the intermediate t's. $\square$

14 The semilinear Schrödinger equation

There is a vast literature on semilinear Schrödinger equations. For a survey, with a concise discussion of some physical motivations, we refer to [15]. Here though, we consider only the mathematical formalism and only the pure power semilinear Schrödinger equations

$$\begin{cases} \mathrm{i} u_t = -\triangle u + \lambda |u|^{p-1} u \text{ for } (t, x) \in [0, \infty) \times \mathbb{R}^d \\ u(0, x) = u_0(x) \end{cases} \tag{14.1}$$

for $\lambda \in \mathbb{R} \backslash \{0\}$ and $p > 1$. Here $p < d^*$ with $d^* = \infty$ for $d = 1, 2$ and $d^* = \frac{d+2}{d-2}$ for $d \ge 3$. We collect here a number of facts needed later.

Lemma 14.1. *We have the following facts.*

(1) For $1 < p < d^*$ we have the Gagliardo–Nirenberg inequality:

$$\|u\|_{L^{p+1}(\mathbb{R}^d)} \leq C_p \|\nabla u\|^{\alpha}_{L^2(\mathbb{R}^d)} \|u\|^{1-\alpha}_{L^2(\mathbb{R}^d)} \text{ for } \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}. \tag{14.2}$$

(2) The map $u \to |u|^{p-1}u$ is a locally Lipschitz from $H^1(\mathbb{R}^d)$ to $H^{-1}(\mathbb{R}^d)$.

(3) For $u \in W^{1,p+1}(\mathbb{R}^d, \mathbb{C})$ we have $\nabla(|u|^{p-1}u) = p|u|^{p-1}\nabla u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2 \nabla \overline{u}$ and belonging to $L^{\frac{p+1}{p}}(\mathbb{R}^d, \mathbb{C})$.

Proof. For (1) see Theorem 5.2.

We turn (2). By (14.2) we know that $u \to |u|^{p-1}u$ maps $H^1(\mathbb{R}^d) \to L^{p+1}(\mathbb{R}^d) \to L^{\frac{p+1}{p}}(\mathbb{R}^d)$. Furthermore this map is locally Lipschitz:

$$\begin{aligned} &\||u|^{p-1}u - |v|^{p-1}v\|_{L^{\frac{p+1}{p}}} \leq C\|(|u|^{p-1} + |v|^{p-1})(u-v)\|_{L^{\frac{p+1}{p}}} \\ &\leq C'(\|u\|^{p-1}_{L^{p+1}} + \|v\|^{p-1}_{L^{p+1}})\|u-v\|_{L^{p+1}} \end{aligned}$$

where we have used, for $w = v - u$,

$$|u|^{p-1}u - |v|^{p-1}v = \int_0^1 \frac{d}{dt}\left(|u+tw|^{p-1}(u+tw)\right)dt =$$
$$\int_0^1 |u+tw|^{p-1}dtw + \int_0^1 (u+tw)\frac{d}{dt}\left((u_1+tw_1)^2 + (u_2+tw_2)^2\right)^{\frac{p-1}{2}} dt = \int_0^1 |u+tw|^{p-1}dtw +$$
$$\sum_{j=1}^2 \int_0^1 (u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2 + (u_2+tw_2)^2\right)^{\frac{p-3}{2}} 2(u_j+tw_j)dtw_j$$

which from $|u+tw| \leq |u| + |v|$ for $t \in [0,1]$ and

$$\left|(u+tw)\frac{p-1}{2}\left((u_1+tw_1)^2 + (u_2+tw_2)^2\right)^{\frac{p-3}{2}} 2(u_j+tw_j)w_j\right| \leq (p-1)|u+tw|^{p-1}|w|$$

yields

$$\left||u|^{p-1}u - |v|^{p-1}v\right| \leq p(|u|+|v|)^{p-1}|u-v| \leq p2^{p-1}(|u|^{p-1} + |v|^{p-1})|u-v|,$$

where in the last step we used, for $|u| \geq |v|$,

$$(|u|+|v|)^{p-1} \leq 2^{p-1}|u|^{p-1} \leq 2^{p-1}(|u|^{p-1} + |v|^{p-1}).$$

Next, we show that we have an embedding $L^{\frac{p+1}{p}}(\mathbb{R}^d) \hookrightarrow H^{-1}(\mathbb{R}^d)$. Indeed, this is equivalent to $H^1(\mathbb{R}^d) \hookrightarrow L^{p+1}(\mathbb{R}^d)$ with in turn is a consequence of (14.2).

We turn (3). First of all we claim that if $G \in C^1(\mathbb{C},\mathbb{C})$ with $G(0)=0$ and $|\nabla G| \leq M < \infty$, then $\nabla(G(u)) = \partial_u G(u)\nabla u + \partial_{\overline{u}} G(u)\nabla \overline{u}$ in the sense of distributions. This claim can be proved like Proposition 9.5 in [2] and we skip the proof here.
Let us now consider an increasing function $g \in C^\infty(\mathbb{R}_+,\mathbb{R})$ s.t.

$$g(s) = \begin{cases} s^{\frac{p-1}{2}} \text{ for } 0 \leq s \leq 1 \\ 2^{\frac{p-1}{2}} \text{ for } s \geq 2 \end{cases}$$

and let us define $G_m(u) = m^{p-1} g\left(\frac{|u|^2}{m^2}\right) u$ for $m \in \mathbb{N}$. Then, by the claim, for all $\varphi \in C_c^\infty(\mathbb{R}^d,\mathbb{C})$ and all $u \in W^{1,p+1}(\mathbb{R}^d,\mathbb{C})$ we have

$$-\int G_m(u)\ \partial_j\varphi = \int \left(\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j \overline{u}\right)\ \varphi. \tag{14.3}$$

Let us take now the limit for $m \to \infty$. We have

$$\int G_m(u)\ \partial_j\varphi = \int |u|^{p-1}u\ \partial_j\varphi - \int_{|u|\geq m} |u|^{p-1}u\ \partial_j\varphi + \int_{|u|\geq m} G_m(u)\ \partial_j\varphi.$$

Now we have

$$\int_{|u|\geq m} |u|^{p-1}u\ \partial_j\varphi \xrightarrow{m\to\infty} 0 \text{ by Dominated Convergence}$$

since $\chi_{\{|u|\geq m\}}(x) \xrightarrow{m\to\infty} 0$ a.e. by Chebyshev's inequality. Similarly

$$\left|\int_{|u|\geq m} G_m(u)\ \partial_j\varphi\right| \leq \int_{|u|\geq m} |G_m(u)\ \partial_j\varphi| \leq 2^{p-1}\int_{|u|\geq m} m^{p-1}|u|\ \partial_j\varphi|$$
$$\leq 2^{p-1}\int_{|u|\geq m} |u|^p|\partial_j\varphi| \xrightarrow{m\to\infty} 0$$

Next, we consider the limit of the r.h.s. of (14.3). For $G(u) = |u|^{p-1}u$ we have

$$\int \left(\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j\overline{u}\right)\ \varphi = \int \left(\partial_u G(u)\partial_j u + \partial_{\overline{u}} G(u)\partial_j\overline{u}\right)$$
$$-\int_{|u|\geq m} \left(\partial_u G(u)\partial_j u + \partial_{\overline{u}} G(u)\partial_j\overline{u}\right)\ \varphi + \int_{|u|\geq m} \left(\partial_u G_m(u)\partial_j u + \partial_{\overline{u}} G_m(u)\partial_j\overline{u}\right)\ \varphi.$$

Then, like before, the terms of the 2nd line converge to 0 as $m \to \infty$ and so we conclude that all $\varphi \in C_c^\infty(\mathbb{R}^d,\mathbb{C})$ and all $u \in W^{1,p+1}(\mathbb{R}^d,\mathbb{C})$ we have

$$-\int |u|^{p-1}u\ \partial_j\varphi = \int \left(p|u|^{p-1}\partial_j u + (p-1)|u|^{p-1}\left(\frac{u}{|u|}\right)^2 \partial_j\overline{u}\right)\ \varphi.$$

The fact of belonging to $L^{\frac{p+1}{p}}(\mathbb{R}^d, \mathbb{C})$ follows immediately from Hölder inequality.

$\square$

Important are the following quantities:

$$\begin{aligned} E(u) &= \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx \\ P_j(u) &= \frac{1}{2}\operatorname{Im}\int_{\mathbb{R}^d} \partial_j u\, \overline{u} dx \\ Q(u) &= \int_{\mathbb{R}^d} |u|^2 dx. \end{aligned} \tag{14.4}$$

Here $E(u)$ is the energy, $P_j(u)$ for $j = 1, ..., d$ are the linear momenta and $Q(u)$ is the mass or charge.

Remark 14.2. Notice, passingly, that $Q, P_j \in C^\infty(H^1(\mathbb{R}^d), \mathbb{R})$ while $E \in C^1(H^1(\mathbb{R}^d), \mathbb{R})$. We will show that the above quantities are conserved for solutions in $H^1(\mathbb{R}^d, \mathbb{C})$. Here E is the hamiltonian. The system is invariant under the transformation $u \to e^{\mathrm{i}\vartheta} u$ for $\vartheta \in \mathbb{R}$ and the transformations $u(x_1, ...x_{j-1}, x_j, x_{j+1}, ..., x_d) \to u(x_1, ...x_{j-1}, x_j - \tau, x_{j+1}, ..., x_d)$ for $\tau \in \mathbb{R}$. The related Noether invariants are Q and P_j.

14.1 The local existence

We will consider the following integral formulation of (14.1):

$$u(t) = e^{\mathrm{i}t\triangle} u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle} |u(s)|^{p-1} u(s) ds. \tag{14.5}$$

Proposition 14.3 (Local well posedness in $L^2(\mathbb{R}^d)$)**.** *For any $p \in (1, 1 + 4/d)$ and any $u_0 \in L^2(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution of* (14.5) *with*

$$u \in C([-T, T], L^2(\mathbb{R}^d)) \cap L^q([-T, T], L^{p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{14.6}$$

Moreover, for any $T' \in (0, T)$ there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([-T', T'], L^2(\mathbb{R}^d)) \cap L^q([-T', T'], L^{p+1}(\mathbb{R}^d)) \tag{14.7}$$

and is Lipschitz.

Finally, we have $u \in L^a([-T, T], L^b(\mathbb{R}^d))$ for all admissible pairs (a, b).

Remark 14.4. We will prove later that for $p \in (1, 1 + 2/d)$ that we can take $T = \infty$ always.

Proof. The proof is a fixed point argument. We set

$$E(T,a)=\Big\{v\in C([-T,T],L^2(\mathbb{R}^d))\cap L^q([-T,T],L^{p+1}(\mathbb{R}^d)):\\ \|v\|_T:=\|v\|_{L^\infty([-T,T],L^2(\mathbb{R}^d))}+\|v\|_{L^q([-T,T],L^{p+1}(\mathbb{R}^d))}\le a\Big\}$$

and we denote by $\Phi(u)$ the r.h.s. of (14.5). Our first aim is to show that for $T=T(\|u_0\|_{L^2})$ sufficiently small, then $\Phi:E(T,a)\to E(T,a)$ and it is a contraction.
By Strichartz's estimates

$$\begin{aligned}\|\Phi(u)\|_T&\le c_0\|u_0\|_{L^2}+c_0|\lambda|\||u|^{p-1}u\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})}\\&=c_0\|u_0\|_{L^2}+c_0|\lambda|\|u\|^p_{L^{pq'}([-T,T],L^{p+1})}\end{aligned}$$

We will see in a moment that

$$p\in(1,1+4/d)\Longleftrightarrow pq'<q. \tag{14.8}$$

Assuming this for a moment, by Hölder we conclude that for a $\theta>0$

$$\|\Phi(u)\|_T\le c_0\|u_0\|_{L^2}+c_0(2T)^\theta|\lambda|\|u\|^p_{L^q([-T,T],L^{p+1})}\le c_0\|u_0\|_{L^2}+c_0(2T)^\theta|\lambda|a^p.$$

So for $c_0(2T)^\theta|\lambda|a^{p-1}<1/2$, which can be obtained by picking T small enough, we have

$$\|\Phi(u)\|_T\le c_0\|u_0\|_{L^2}+\frac{a}{2}\le a$$

if $a\ge 2c_0\|u_0\|_{L^2}$. Hence $\Phi(E(T,a))\subseteq E(T,a)$. Let us fix here $a=2c_0\|u_0\|_{L^2}$.
Now let us show that Φ is a contraction for T small enough. We have

$$\begin{aligned}\|\Phi(u)-\Phi(v)\|_T&\le c_0|\lambda|\||u|^{p-1}u-|v|^{p-1}v\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})}\\&\le c_0C|\lambda|\|(\|u\|^{p-1}_{L^{p+1}}+\|v\|^{p-1}_{L^{p+1}})\|u-v\|_{L^{p+1}}\|_{L^{q'}(-T,T)}\\&\le c_0C|\lambda|(\|u\|^{p-1}_{L^q([-T,T],L^{p+1}}+\|v\|^{p-1}_{L^q([-T,T],L^{p+1})})\|u-v\|_{L^\rho([-T,T],L^{p+1})}\end{aligned}$$

where $\frac{p-1}{q}+\frac{1}{\rho}=\frac{1}{q'}$. Since we are still assuming (14.8), we must have $\rho<q$, for $\rho\ge q$ would imply $pq'\ge q$, contrary to (14.8). Then by Hölder and for an appropriate $\theta>0$

$$\|\Phi(u)-\Phi(v)\|_T\le c_0C|\lambda|2a^{p-1}T^\theta\|u-v\|_{L^q([-T,T],L^{p+1})}\le c_0C|\lambda|2a^{p-1}T^\theta\|u-v\|_T.$$

So, for $c_0C|\lambda|2a^{p-1}T^\theta<1$, where $a=2c_0\|u_0\|_{L^2}$, we obtain that Φ is a contraction and we obtain the existence and uniqueness of the solution.

Next, let us prove (14.8). Obviously $pq'<q$ is equivalent to $p/q<1-1/q$, in turn to $(p+1)/q<1$, that is to $1/q<1/(p+1)$. But $1/q=d/4-d/(2p+2)$, so the last inequality is equivalent to

$$d/4<\left(\frac{d}{2}+1\right)/(p+1)\Leftrightarrow p+1<\frac{2d+4}{d}=2+\frac{4}{d}$$

and this yields the desired result.

We have proved the existence of a $T = T(\|u_0\|_{L^2})$ with the desired properties. Fix $T' \in (0,T)$. Then there exists a neighborhood V of u_0 in $L^2(\mathbb{R}^d)$ such that for any $v_0 \in V$ the corresponding solution $v(t)$ is in $C([-T',T'],L^2(\mathbb{R}^d)) \cap L^q([-T',T'],L^{p+1}(\mathbb{R}^d))$ with $\|v\|_{T'} \le 2c_0\|v_0\|_{L^2}$. This is clear because with v_0 sufficiently close to u_0, by $T' < T$ we can assume

$$
\begin{aligned}
&c_0(2T')^\theta|\lambda|(2c_0\|v_0\|_{L^2})^{p-1} < 1/2c_0(2T)^\theta|\lambda|(2c_0\|u_0\|_{L^2})^{p-1} < 1/2 \text{ and}\\
&c_0C|\lambda|2(2c_0\|v_0\|_{L^2})^{p-1}(T')^\theta < c_0C|\lambda|2(2c_0\|u_0\|_{L^2})^{p-1}T^\theta.
\end{aligned}
$$

Using the equation and proceeding like above,

$$
\begin{aligned}
\|u-v\|_{T'} &\le c_0\|u_0-v_0\|_{L^2} + c_0C|\lambda|(2T')^\theta\left(\|u\|_{T'}^{p-1} + \|v\|_{T'}^{p-1}\right)\|u-v\|_{T'}\\
&\le c_0\|u_0-v_0\|_{L^2} + c_0C|\lambda|(2T')^\theta 2\left((2c_0\|v_0\|_{L^2})^{p-1} + (2c_0\|u_0\|_{L^2})^{p-1}\right)\|u-v\|_{T'}.
\end{aligned}
$$

Adjusting T, we can assume that, in addition to the previous inequalities, T satisfies also

$$4c_0C|\lambda|(2T)^\theta(2c_0\|u_0\|_{L^2})^{p-1} < 1/2.$$

Adjusting V, we can assume that,

$$(2T')^\theta(2c_0\|v_0\|_{L^2})^{p-1} < (2T)^\theta(2c_0\|u_0\|_{L^2})^{p-1}.$$

Then from the above we get

$$\|u-v\|_{T'} \le 2c_0\|u_0-v_0\|_{L^2}$$

and this give the desired Lipschitz continuity.

Finally, the last statement follows from (14.5) and the Strichartz Estimates.

□

Proposition 14.5 (Local well posedness in $H^1(\mathbb{R}^d)$)**.** *For any $p \in (1,d^*)$ and any $u_0 \in H^1(\mathbb{R}^d)$ there exists $T > 0$ and a unique solution of (14.5) with*

$$u \in C([-T,T],H^1(\mathbb{R}^d)) \cap L^q([-T,T],W^{1,p+1}(\mathbb{R}^d)) \text{ with } \frac{2}{q} + \frac{d}{p+1} = \frac{d}{2}. \tag{14.9}$$

Moreover, for any $T' \in (0,T)$ there exists a neighborhood V of u_0 in $H^1(\mathbb{R}^d)$ s.t. the map $v_0 \to v(t)$, associating to each initial value its corresponding solution, sends

$$V \to C([-T',T'],L^2(\mathbb{R}^d)) \cap L^q([-T',T'],W^{1,p+1}(\mathbb{R}^d))$$

and is Lipschitz.

Finally, we have $u \in L^a([-T,T],W^{b,1}(\mathbb{R}^d))$ for all admissible pairs (a,b).

Proof. The proof is similar to that of Proposition 14.3. The proof is a fixed point argument.This time we set

$$E^1(T,a) = \Big\{ v \in C([-T,T], H^1(\mathbb{R}^d)) \cap L^q([-T,T], W^{1,p+1}(\mathbb{R}^d)) : \\ \|v\|_T^{(1)} := \|v\|_{L^\infty([-T,T],H^1(\mathbb{R}^d))} + \|v\|_{L^q([-T,T],W^{1,p+1}(\mathbb{R}^d))} \leq a \Big\}$$

and, as before, use $\Phi(u)$ for the r.h.s. of (14.5). We need to show that by taking T sufficiently small then $\Phi : E^1(T,a) \to E^1(T,a)$ and is a contraction. The argument is similar to the one in Proposition 14.3 and is based on the Strichartz estimates. We will only consider some of the estimates. By Lemma 14.1 and Strichartz's estimates, we have

$$\begin{aligned} \|\nabla\Phi(u)\|_T &\leq c_0\|u_0\|_{H^1} + c_0|\lambda| \||u|^{p-1}\nabla u\|_{L^{q'}([-T,T],L^{\frac{p+1}{p}})} \\ &= c_0\|u_0\|_{L^2} + c_0|\lambda|\|u\|^{p-1}_{L^\beta([-T,T],L^{p+1})}\|\nabla u\|_{L^q([-T,T],L^{p+1})}. \end{aligned}$$

where $\frac{p-1}{\beta} + \frac{1}{q} = \frac{1}{q'}$. Notice that if $\beta < q$, we can proceed exactly like in Proposition 14.3. However this works only for $p \in (1, 1+4/d)$, which is not necessarily true here. Instead, using the Sobolev Embedding we bound

$$\|u\|^{p-1}_{L^\beta([-T,T],L^{p+1})} \lesssim \|u\|^{p-1}_{L^\beta([-T,T],H^1)} \leq (2T)^{\frac{p-1}{\beta}}\|u\|^{p-1}_{L^\infty([-T,T],H^1)} \leq (2T)^{\frac{p-1}{\beta}}(\|u\|_T^{(1)})^{p-1}.$$

So, inserting this in the previous inequality we get

$$\|\nabla\Phi(u)\|_T \leq c_0\|u_0\|_{H^1} + c_0|\lambda|(2T)^{\frac{p-1}{\beta}}(\|u\|_T^{(1)})^p. \tag{14.10}$$

Here it is important to remark that the admissible pair $(q, p+1)$ is s.t. $q > 2$. Indeed, for $d = 1, 2$ it is always true that, if $p+1 < \infty$, then the q in (14.6) is $q > 2$. On the other hand, for $d \geq 3$ recall that

$$p + 1 < d^* + 1 = \frac{d+2}{d-2} + 1 = \frac{2d}{d-2}.$$

And so again, since $(q, p+1)$ differs from the endpoint admissible pair $(2, \dfrac{2d}{d-2})$, we necessarily have $q > 2$ also if $d \geq 3$.
In turn, the fact that $q > 2$ implies that the β in the above formulas is $\beta < \infty$. This implies that we can pick T small enough s.t. $(2T)^{p-1}a^{p-1} < 1/2$, which from (14.10) yields $\|\Phi(u)\|_T^{(1)} \leq c_1\|u_0\|_{H^1} + a/2 \leq a$ for $a \geq 2c_1\|u_0\|_{H^1}$. From these arguments, it is easy to conclude that there exists a $T(\|u_0\|_{H^1})$ s.t. for $T \in (0, T(\|u_0\|_{H^1}))$ we have $\Phi\left(E^1(T,a)\right) \subseteq E^1(T,a)$. Proceeding similarly and like in Proposition 14.3, it can be shown that there exists a $T_1(\|u_0\|_{H^1})$ s.t. for $T \in (0, T_1(\|u_0\|_{H^1}))$ and $a \geq 2c_1\|u_0\|_{H^1}$ the map Φ is a contraction inside $E^1(T,a)$. The Lipschitz continuity in terms of the initial data can be shown like in Proposition 14.3 and the last statement follows from the Strichartz estimates. □

Proposition 14.6 (Conservation laws)**.** *Let $u(t)$ be a solution* (14.5) *as in Proposition 14.5. Then all the three quantities in* (14.4) *are constant in t.*

Proof. For $u \in C((-T_2, T_1), H^1(\mathbb{R}^d))$ a maximal solution of (14.5) we will show that there exists $[-T, T] \subset (-T_2, T_1)$ where $E(u(t)) = E(u(0))$, $Q(u(t)) = Q(u(0))$ and $P_j(u(t)) = P_j(u(0))$. In fact this shows that $E(u(t))$, $Q(u(t))$ and $P_j(u(t))$ are locally constant in t. Since these functions are continuous in t, the set of $t \in (-T_2, T_1)$ where $E(u(t)) = E(u(0))$ is closed in $(-T_2, T_1)$; on the other hand, it is also open in $(-T_2, T_1)$ since $E(u(t))$ is locally constant, and hence we have $E(u(t)) = E(u(0))$ for all $t \in (-T_2, T_1)$. Similarly $Q(u(t)) = Q(u(0))$ and $P_j(u(t)) = P_j(u(0))$ for all $t \in (-T_2, T_1)$.

Step 1: truncations of the NLS. For $\varphi \in C_c^\infty(\mathbb{R}, [0,1])$ a function with $\varphi = 1$ near 0 and with support contained in the ball $B_{\mathbb{R}^d}(0, r_0)$, consider [2] the operators $\mathbf{Q}_n = \varphi(\sqrt{-\triangle}/n)$. The truncations $\mathbf{Q}_n(|u|^{p-1}u)$ are locally Lipschitz functions from $H^1(\mathbb{R}^d)$ into itself as they are compositions $H^1(\mathbb{R}^d) \overset{|u|^{p-1}u}{\to} H^{-1}(\mathbb{R}^d) \overset{\mathbf{Q}_n}{\to} H^1(\mathbb{R}^d))$ of a locally Lipschitz function, Lemma 14.1, and of bounded linear maps.
We consider the following truncations of the NLS

$$\begin{cases} \mathrm{i}u_{nt} = -\mathbf{P}_{nr_0}\triangle u_n + \lambda \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n) \text{ for } (t,x) \in \mathbb{R} \times \mathbb{R}^d \\ u_n(0) = \mathbf{Q}_n u_0. \end{cases} \tag{14.11}$$

By the theory of ODE's, there exists a maximal solution $u_n(t) \in C^1(-T_1(n), T_2(n)), H^1(\mathbb{R}^d))$ of (14.11) . Furthermore, if $T_2(n) < \infty$ then we must have blow up

$$\lim_{t \nearrow T_2(n)} \|u_n(t)\|_{H^1} = +\infty \text{ if } T_2(n) < \infty \tag{14.12}$$

with a similar blow up phenomenon if $T_1(n) < \infty$.
To get bounds on this sequence of functions we consider invariants of motion. The following will be proved later.

Claim 14.7. The following functions are invariants of motion of (14.11):

$$\begin{aligned} &E_n(v) := \frac{1}{2}\|P_{nr_0}\nabla v\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n v|^{p+1} dx \\ &P_j(v) \text{ with } j = 1, ..., d, \\ &Q(v). \end{aligned} \tag{14.13}$$

We assume Claim 14.7 and proceed. It is easy to check that $u_n = \mathbf{P}_{nr_0} u_n$. We claim that $T_1(n) = T_2(n) = \infty$. Indeed by $Q(u_n(t)) = Q(\mathbf{Q}_n u_0) \le Q(u_0)$ we have

$$\|u_n(t)\|_{H^1} = \|\mathbf{P}_{nr_0} u_n(t)\|_{H^1} \le nr_0 \|u_n(t)\|_{L^2} = nr_0 \|\mathbf{Q}_n u_0\|_{L^2} \le nr_0 \|u_0\|_{L^2}. \tag{14.14}$$

[2] Notice that using everywhere the projections $\mathbf{P}_n = \chi_{[0,n]}(\sqrt{-\triangle})$ would be a bad choice for this proof. Difficulties would arise from the fact proved by C.Feffermann [6] that $\mathbf{P}_n$ for $d \ge 2$ is bounded from $L^p(\mathbb{R}^d)$ into itself only if $p = 2$. On the other hand it is elementary that the $\mathbf{Q}_n$ are of the form $\rho_{\frac{1}{n}} *$ for a $\rho \in \mathcal{S}(\mathbb{R}^d)$ and so are uniformly bounded from $L^p(\mathbb{R}^d)$ into itself for all p and form a sequence converging strongly to the identity operator.

Let us now fix M such that $\|u_0\|_{H^1} < M$ and let us set

$$\theta_n := \sup\{\tau > 0 : \|u_n(t)\|_{H^1} < 2M \text{ for } |t| < \tau.\} \tag{14.15}$$

Our main focus is now to prove that there exists a fixed $T(M) > 0$ s.t. $\theta_n \geq T(M)$ for all n.

First of all we prove that $u_n \in C^{0,\frac{1}{2}}((-\theta_n, \theta_n), L^2)$ with a fixed Hölder constant $C(M)$. By an interpolation similar to Lemma 5.1

$$\begin{aligned} &\|u_n(t) - u_n(s)\|_{L^2} \lesssim \|u_n(t) - u_n(s)\|_{H^1}^{\frac{1}{2}} \|u_n(t) - u_n(s)\|_{H^{-1}}^{\frac{1}{2}} \\ &\leq \sqrt{2}\|u_n\|_{L^\infty((-\theta_n,\theta_n),H^1)}^{\frac{1}{2}} \|u_{nt}\|_{L^\infty((-\theta_n,\theta_n),H^{-1})}^{\frac{1}{2}} \sqrt{|t-s|} \\ &\leq C(M)\sqrt{|t-s|} \text{ for } t, s \in (-\theta_n, \theta_n) \end{aligned} \tag{14.16}$$

Now we want to prove

$$\|u_n(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + C(M)t^b \text{ for some fixed } b > 0 \text{ and for } t \in (-\theta_n, \theta_n). \tag{14.17}$$

From $E_n(u_n(t)) = E_n(\mathbf{Q}_n u_0)$ and $Q(u_n(t)) = Q(\mathbf{Q}_n u_0)$ we get

$$\|u_n(t)\|_{H^1}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1} dx = \|\mathbf{Q}_n u_0\|_{H^1}^2 + \frac{2\lambda}{p+1}\int_{\mathbb{R}^n} |\mathbf{Q}_n^2 u_0|^{p+1} dx.$$

Hence using Hölder and Gagliardo–Nirenberg

$$\begin{aligned} &\|u_n(t)\|_{H^1}^2 \leq \|u_0\|_{H^1}^2 + \frac{2|\lambda|}{p+1}\int_{\mathbb{R}^d} \left| \; |\mathbf{Q}_n u_n(t)|^{p+1} - |\mathbf{Q}_n^2 u_0|^{p+1} \right| dx \\ &\leq \|u_0\|_{H^1}^2 + C\int_{\mathbb{R}^d} (|\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p)|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0| dx \\ &\leq \|u_0\|_{H^1}^2 + C\| |\mathbf{Q}_n u_n(t)|^p + |\mathbf{Q}_n^2 u_0|^p\|_{L^{\frac{p+1}{p}}} \|\mathbf{Q}_n u_n(t) - \mathbf{Q}_n^2 u_0\|_{L^{p+1}} \\ &\leq \|u_0\|_{H^1}^2 + C_1\left(\|\mathbf{Q}_n u_n(t)\|_{L^{p+1}}^p + \|\mathbf{Q}_n^2 u_0\|_{L^{p+1}}^p\right) \|u_n(t) - \mathbf{Q}_n u_0\|_{\dot{H}^1}^\alpha \|u_n(t) - \mathbf{Q}_n u_0\|_{L^2}^{1-\alpha} \end{aligned}$$

Then by (14.16) with $s = 0$, the Sobolev Embedding Theorem and (14.15) we get (14.17). Now for $T(M)$ defined s.t. $C(M)T(M)^b = 2M^2$ (for the $C(M)$ in (14.17)) from (14.17) we get

$$\|u_n(t)\|_{L^\infty([-T(M),T(M)],H^1)} \leq \sqrt{3}M. \tag{14.18}$$

Since $\sqrt{3}M < 2M$ this obviously means that $T(M) < \theta_n$ since, if we had $\theta_n \leq T(M)$ then, by the fact that $u_n \in C^1(\mathbb{R}, H^1)$, the definition of θ_n in (14.15) would be contradicted. Hence we have

$$\|u_n\|_{L^\infty([-T(M),T(M)],H^1)} < 2M \tag{14.19}$$

This completes step 1, up to Claim 14.7.

The proof of Claim 14.7 is rather elementary and involves applying to (14.11) $\langle \; , u_{nt}\rangle$, $\langle \; , \mathrm{i}u_n\rangle$ and $\langle \; , \partial_{x_j} u_n\rangle$ and integration by parts. We will do this now, but then we will discuss also the fact that Claim 14.7 is just a consequence of the fact that (14.11) is a hamiltonian system with hamiltonian E_n and that the invariance of Q resp. P_j just due to Nöther principle and the invariance with respect to multiplication by $e^{\mathrm{i}\vartheta}$ resp. translation.

Indeed, applying $\langle \cdot, u_{nt}\rangle$ to (14.11)

$$\begin{aligned}0 &= -\langle \mathbf{P}_{nr_0}\triangle u_n, u_{nt}\rangle + \lambda\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), u_{nt}\rangle \\ &= -\langle \triangle u_n, u_{nt}\rangle + \lambda\langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathbf{Q}_n u_{nt}\rangle = \frac{d}{dt}E_n(u_n).\end{aligned}$$

Notice furthermore that, by $u_n = \mathbf{P}_{nr_0}u_n$, we have

$$E_n(u_n) = \frac{1}{2}\|\nabla u_n\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1}dx.$$

Similarly when we apply $\langle \cdot, \mathrm{i}u_n\rangle$ to (14.11) we get

$$\frac{1}{2}\frac{d}{dt}\|u_n(t)\|_{L^2} = -\langle \mathbf{P}_{nr_0}\triangle u_n, \mathrm{i}u_n\rangle + \lambda\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle. \tag{14.20}$$

We have to show that r.h.s. are equal to 0. We observe that the the 1st term is 0 because the bounded operator $\mathrm{i}\mathbf{P}_{nr_0}\triangle$ of $L^2(\mathbb{R}^d)$ into itself is antisymmetric: $(\mathrm{i}\mathbf{P}_{nr_0}\triangle)^* = -\mathrm{i}\mathbf{P}_{nr_0}\triangle$. For the 2nd term we use

$$\langle \mathbf{Q}_n(|\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n), \mathrm{i}u_n\rangle = \langle |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n, \mathrm{i}\mathbf{Q}_n u_n\rangle = \lambda \operatorname{Re} \mathrm{i}\int_{\mathbb{R}^d} |\mathbf{Q}_n u_n|^{p+1}dx = 0.$$

This yields $\frac{d}{dt}Q(u_n(t)) = 0$. In a similar fashion we can prove $\frac{d}{dt}P_j(u_n(t)) = 0$.

These computations obscure somewhat the following simple facts. First of all, (14.11) and, in a somewhat formal sense also (14.1), is a hamiltonian system. First of all, the symplectic form is

$$\Omega(X,Y) := \langle \mathrm{i}X, Y\rangle \tag{14.21}$$

where

$$\langle f, g\rangle = \operatorname{Re}\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx. \tag{14.22}$$

Notice that Ω satisfies the following definition for $X = L^2(\mathbb{R}^d,\mathbb{C})$ or $X = H^1(\mathbb{R}^d,\mathbb{C})$.

Definition 14.8. Let X be a Banach space on $\mathbb{R}$ and let X' be its dual. A strong symplectic form is a 2-form ω on X s.t. $d\omega = 0$ (i.e. ω is closed) and s.t. the map $X \ni x \to \omega(x,\cdot) \in X'$ is an isomorphism.

Definition 14.9 (Gradient)**.** Let $F \in C^1(L^2(\mathbb{R}^d,\mathbb{C}),\mathbb{R})$. Then the gradient $\nabla F \in C^0(L^2(\mathbb{R}^d,\mathbb{C}), L^2(\mathbb{R}^d,\mathbb{C}))$ is defined by

$$\langle \nabla F(u), Y\rangle = dF(u)Y \text{ for all } u, Y \in L^2(\mathbb{R}^d,\mathbb{C}).$$

Notice that

$$\langle \nabla E_n(u), Y\rangle = \frac{d}{dt}\left(\frac{1}{2}\|P_{nr_0}\nabla(u+tY)\|_{L^2}^2 + \frac{\lambda}{p+1}\int_{\mathbb{R}^d}|\mathbf{Q}_n(u+tY)|^{p+1}dx\right)\Bigg|_{t=0} \quad (14.23)$$
$$= \left\langle -\mathbf{P}_{nr_0}\triangle u + \lambda \mathbf{Q}_n(|\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u), Y\right\rangle .$$

We are interested in hamiltonian vector fields.

Definition 14.10 (Hamiltonian vector field)**.** Let ω be a strong symplectic form on the Banach space X and $F \in C^1(X,\mathbb{R})$. We define the Hamiltonian vector field X_F with respect to ω by

$$\omega(X_F(u), Y) := dF(u)Y \text{ for all } u, Y \in X.$$

From $\Omega(X_F, Y) = \langle \mathrm{i}X_F, Y\rangle = \langle \nabla F, Y\rangle$ we conclude $X_F = -\mathrm{i}\nabla F$. Then from (14.23) it is straightforward to conclude that (14.11) is a hamiltonian system with hamiltonian E_n.

Definition 14.11 (Poisson bracket)**.** Let ω be a strong symplectic form in a Banach space X and let $F, G \in C^1(X,\mathbb{R})$. Then the Poisson bracket $\{F, G\}$ is given by

$$\{F, G\}(u) := \omega(u)(X_F(u), X_G(u)) = dF(u)X_G(u).$$

So, for Ω we have $\{F, G\} = \langle \nabla F, -\mathrm{i}\nabla G\rangle = \langle \mathrm{i}\nabla F, \nabla G\rangle$. Now notice that if $F \in C^1(X,\mathbb{R})$ then

$$\frac{d}{dt}\left(F(u_n(t))\right) = \langle \nabla F(u_n(t)), \dot{u}_n(t)\rangle = \langle \nabla F(u_n(t)), -\mathrm{i}\nabla E_n(u_n(t))\rangle = \{F, E_n\}|_{u_n(t)} \quad (14.24)$$

Notice now that the map $u \in e^{\mathrm{i}\vartheta}u$ leaves E_n invariant. In particular the last assertion implies that

$$0 = \frac{d}{d\vartheta}E_n(u)\Big|_{\vartheta=0} = \frac{d}{d\vartheta}E_n(e^{\mathrm{i}\vartheta}u)\Big|_{\vartheta=0}$$
$$= \langle \nabla E_n(u), \mathrm{i}u\rangle = \langle \nabla E_n(u), \mathrm{i}\nabla Q(u)\rangle = \langle \mathrm{i}\nabla Q(u), \nabla E_n(u)\rangle = \{Q, E_n\}|_u$$

But then, since $\{Q, E_n\} = 0$, by (14.24) we obviously have $\frac{d}{dt}\left(Q(u_n(t))\right) = 0$.

Let us consider now, for $\{\overrightarrow{e}_j\}_{j=1}^d$ the standard basis of $\mathbb{R}^d$, the transformation $(\tau_{\lambda\overrightarrow{e}_j}F)(x) := F(x - \lambda\overrightarrow{e}_j)$. Obviously E_n is invariant by this transformation and

$$0 = \frac{d}{d\lambda}E_n(u)\Big|_{\lambda=0} = \frac{d}{d\lambda}E_n(\tau_{\lambda\overrightarrow{e}_j}u)\Big|_{\lambda=0}$$
$$= -\langle \nabla E_n(u), \partial_j u\rangle = \langle \nabla E_n(u), \mathrm{i}\nabla P_j(u)\rangle = \langle \mathrm{i}\nabla P_j(u), \nabla E_n(u)\rangle = \{P_j, E_n\}|_u$$

But then, since $\{P_j, E_n\} = 0$, by (14.24) we obviously have $\frac{d}{dt}\left(P_j(u_n(t))\right) = 0$.
The above argument gives a link between group actions and invariants.

Step 2: Convergence $u_n \to u$. Let us consider $I := [-T, T] \subseteq [-T(M), T(M)] \cap (-T_2, T_1)$. Obviously we have

$$u_n(t) = e^{\mathrm{i}t\triangle}\mathbf{Q}_n u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}\mathbf{Q}_n(|\mathbf{Q}_n u_n(s)|^{p-1}\mathbf{Q}_n u_n(s))ds.$$

Taking the difference with (14.5) we obtain

$$\begin{aligned}
u(t) - u_n(t) &= e^{\mathrm{i}t\triangle}(1-\mathbf{Q}_n)u_0 - \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}(1-\mathbf{Q}_n)|u(s)|^{p-1}u(s)ds \\
&- \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}\mathbf{Q}_n\left(|u(s)|^{p-1}u(s) - |\mathbf{Q}_n u(s)|^{p-1}\mathbf{Q}_n u(s)\right)ds \\
&- \mathrm{i}\lambda \int_0^t e^{\mathrm{i}(t-s)\triangle}\mathbf{Q}_n\left(|\mathbf{Q}_n u(s)|^{p-1}\mathbf{Q}_n u(s) - |\mathbf{Q}_n u_n(s)|^{p-1}\mathbf{Q}_n u_n(s)\right)ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
&\|u-u_n\|_{L^q(I,W^{1,p+1})} + \|u-u_n\|_{L^\infty(I,H^1)} \le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1} + c_0|\lambda|\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})} \\
&+ c_0|\lambda|\||u|^{p-1}u - |\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})} \\
&+ c_0|\lambda|\||\mathbf{Q}_n u|^{p-1}\mathbf{Q}_n u - |\mathbf{Q}_n u_n|^{p-1}\mathbf{Q}_n u_n\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})}.
\end{aligned}$$

and so, for a fixed $\vartheta > 0$

$$\begin{aligned}
&\|u-u_n\|_{L^q(I,W^{1,p+1})} + \|u-u_n\|_{L^\infty(I,H^1)} \le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1} + c_0|\lambda|\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})} \\
&+ c_0C|\lambda||I|^\vartheta\left(\|u\|^{p-1}_{L^\infty(I,H^1)} + \|\mathbf{Q}_n u\|^{p-1}_{L^\infty(I,H^1)}\right)\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})} \\
&+ c_0C|\lambda||I|^\vartheta\left(\|\mathbf{Q}_n u\|^{p-1}_{L^\infty(I,H^1)} + \|\mathbf{Q}_n u_n\|^{p-1}_{L^\infty(I,H^1)}\right)\|\mathbf{Q}_n(u-u_n)\|_{L^q(I,W^{1,p+1})} \\
&\le c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1} + c_0|\lambda|\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})} \\
&+ c_0C|\lambda||I|^\vartheta 2\|u\|^{p-1}_{L^\infty(I,H^1)}\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})} \\
&+ c_0C|\lambda||2T|^\vartheta\left(\|u\|^{p-1}_{L^\infty(I,H^1)} + (C(M))^{p-1}\right)\|u-u_n\|_{L^q(I,W^{1,p+1})}.
\end{aligned}$$

Then, taking T small so that $c_0C|\lambda||2T|^\vartheta\left(\|u\|^{p-1}_{L^\infty(I,H^1)} + (C(M))^{p-1}\right) < 1/2$ we conclude

$$\begin{aligned}
&\|u-u_n\|_{L^q(I,W^{1,p+1})} + \|u-u_n\|_{L^\infty(I,H^1)} \le 2c_0\|(1-\mathbf{Q}_n)u_0\|_{H^1} + \\
&2c_0|\lambda|\|(1-\mathbf{Q}_n)|u|^{p-1}u\|_{L^{q'}(I,W^{1,\frac{p+1}{p}})} + 2c_0C|\lambda||I|^\vartheta 2\|u\|^{p-1}_{L^\infty(I,H^1)}\|(1-\mathbf{Q}_n)u\|_{L^q(I,W^{1,p+1})}.
\end{aligned}$$

But now we have r.h.s.$\overset{n\to\infty}{\to} 0$. Hence we have proved that there exist $T > 0$ s.t.

$$\lim_{n\to+\infty}\|u-u_n\|_{L^\infty([-T,T],H^1)} = 0. \tag{14.25}$$

Now, taking the limit for $n \to +\infty$ in $Q(u_n(t)) = Q(\mathbf{Q}_n u_0)$ and $P_j(u_n(t)) = P_j(\mathbf{Q}_n u_0)$ we obtain $Q(u(t)) = Q(u_0)$ and $P_j(u(t)) = P_j(u_0)$ for all $t \in [-T, T]$. Similarly, taking the limit for $n \to +\infty$ in $E_n(u_n) = E_n(\mathbf{Q}_n u_0)$ and with a little bit of work, we obtain $E(u(t)) = E(u_0)$ for all $t \in [-T, T]$.

□

Corollary 14.12. *Let $u(t)$ be a solution (14.5) as in Proposition 14.3. Then $Q(u(t)) = Q(u_0)$. In particular, the solutions in in Proposition 14.3 are globally defined.*

Proof. As above it is enough to show that $Q(u(t)) = Q(u_0)$ for $t \in [-T, T]$ for some $T > 0$. So let us take the T in the statement of Proposition 14.3 and let us take $T' \in (0, T)$. There exists a sequence $u_0^{(n)} \in H^1(\mathbb{R}^d, \mathbb{C})$ with $u_0^{(n)} \overset{n\to\infty}{\to} u_0$ in $L^2(\mathbb{R}^d, \mathbb{C})$. So for $n \gg 1$ we have $u_0^{(n)} \in V$, the V in (14.7). In particular, for the corresponding solutions u_n we have $u^{(n)} \overset{n\to\infty}{\to} u$ in $C([-T', T'], L^2(\mathbb{R}^d))$. Then, since $Q(u^{(n)}(t)) = Q(u_0^{(n)})$ for $t \in ([-T', T']$, taking the limit we obtain $Q(u(t)) = Q(u_0)$ for $t \in ([-T', T']$. Since $T' \in (0, T)$ is arbitrary and $t \to Q(u(t))$ is continuous, we have $Q(u(t)) = Q(u_0)$ for $t \in ([-T, T]$. This implies that $t \to Q(u(t))$ is locally constant, and hence it is constant.

□

14.2 The global existence

We start with the following observation.

Lemma 14.13. *Let $u \in C^0((-S, T), H^1(\mathbb{R}^d))$ be a maximal solution as of Proposition 14.5. Then if $T < \infty$ we have*

$$\lim_{t \nearrow T} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty. \qquad (14.26)$$

Analogously, $\lim_{t \searrow -S} \|\nabla u(t)\|_{L^2(\mathbb{R}^d)} = +\infty$ *if $S < \infty$.*

Remark 14.14. Notice that it is very important for this lemma that $p < d^*$. Indeed, in the *energy* critical case $p = d^*$, the above statement is false.

Proof. Suppose by contradiction that there exists a solution with $T < \infty$ for which there is a sequence $t_j \nearrow T$ s.t. $\|u(t_j)\|_{H^1(\mathbb{R}^d)} \le M < \infty$. Then by Proposition 14.5 one can extend $u(t)$ beyond $t_j + T(M) > T$ and get a contradiction.

□

Corollary 14.15. *If $\lambda > 0$ the solutions of Proposition 14.5 are globally defined.*

Proof. Indeed if a solution has maximal interval of existence $(-S, T)$ with $T < \infty$, we must have (14.26). But for $\lambda > 0$ we have $\|\nabla u(t)\|_{L^2} \le 2E(u(t)) = 2E(u_0)$.

□

Corollary 14.16. *If $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$ the solutions of Proposition 14.5 are globally defined.*

Proof. We have

$$2E(u(t)) \geq \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} - \frac{2|\lambda|}{p+1} C_p^{p+1} \|\nabla u(t)\|^{\alpha(p+1)}_{L^2(\mathbb{R}^d)} \|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)} \text{ for } \frac{1}{p+1} = \frac{1}{2} - \frac{\alpha}{d}.$$

Notice that

$$\alpha(p+1) = \frac{d}{2}(p+1) - d < 2 \Longleftrightarrow (p+1) - 2 < \frac{4}{d} \Longleftrightarrow p < 1 + \frac{4}{d}.$$

But then, if (14.26) happens, we have

$$2E(u_0) = \lim_{t \nearrow T} 2E(u(t)) \geq \lim_{t \nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \left(1 - \frac{2|\lambda|}{p+1} C_p^{p+1} \|\nabla u(t)\|^{\alpha(p+1)-2}_{L^2(\mathbb{R}^d)} \|u_0\|^{(1-\alpha)(p+1)}_{L^2(\mathbb{R}^d)}\right)$$
$$= \lim_{t \nearrow T} \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} = +\infty,$$

which is absurd. □

Corollary 14.17. *If $\lambda < 0$ and $1 < p < 1 + \frac{4}{d}$ the solutions of Proposition 14.5 are globally defined.*

15 Fujita's Blow Up Theorem for Semilinear Heat Equations

We will consider now a particular formulation of Fujita's classical blow up result. We consider the heat equation

$$\begin{cases} u_t = \triangle u + |u|^{p-1} u \text{ with } (t,x) \in (0,T) \times \mathbb{R}^d \\ u(0,x) = u_0(x) \text{ where } u_0 \in C_0(\mathbb{R}^d, \mathbb{R}). \end{cases}$$

Here we recall that, like in (1.5),

$$C_0(\mathbb{R}^d, \mathbb{R}) := \{g \in C^0(\mathbb{R}^d, \mathbb{R}) : \lim_{x \to \infty} g(x) = 0\}.$$

We formulate this problem in the following integral form:

$$u(t) = e^{t\triangle} f + \int_0^t e^{(t-s)\triangle} |u(s)|^{p-1} u(s) ds. \tag{15.1}$$

It turns out that there exists a unique maximal solution of (15.13) with maximal lifespan T_f in $C^0([0, T_f, C_0(\mathbb{R}^d))$.
We will prove the following result.

Theorem 15.1. *Let $u_0 \in C_0(\mathbb{R}^d)$ with $u_0 \geq 0$ and $u_0 \neq 0$ and suppose $1 < p \leq 1 + \frac{2}{d}$. Consider the solution of*

$$u(t) = e^{t\triangle} u_0 + \int_0^t e^{(t-s)\triangle} u^p(s) ds \tag{15.2}$$

in $C^0([0, T_{u_0}), C_0(\mathbb{R}^d))$. Then $T_{u_0} < \infty$.

Remark 15.2. The original paper by Fujita [7] deals with the case $1 < p < 1 + \frac{2}{d}$. The proof we give is due to Weissler [16].

Somewhat related to Fujita's Theorem are theorems for dispersive equations originating from work by Fritz John, like the following, which is only a prototype of much more general results, and which we state only (for the proof see [12, p. 92]).

Theorem 15.3. *Let $u_1 \in C^2_c(\mathbb{R}^3, \mathbb{R})$ with $u_1 \geq 0$ and $u_1 \not\equiv 0$ and consider*

$$\begin{cases} (\partial_t^2 - \triangle)u - |u|^p = 0 \\ (u(0), \partial_t u(0)) = (0, u_1). \end{cases}$$

Then, if $1 < p < 1 + \sqrt{2}$ the solution blows up in finite time, in the sense that there exists a unique maximal solution $u \in C^2([0, T_{u_1}) \times \mathbb{R}^3, \mathbb{R})$ with $T_{u_1} < \infty$ where $u \notin L^\infty([0, T_{u_1}) \times \mathbb{R}^3)$.

□

15.1 Preliminaries on abstract dissipative semilinear equations

Definition 15.4 (Contraction semigroup)**.** Let X be a Banach space. A family $(S(t))_{t\geq 0} \in \mathcal{L}(X)$ is a contraction semigroup if the following happens.

(1) $\|S(t)\| \leq 1$ for all $t \geq 0$.

(2) $S(0) = I$.

(3) $S(t)S(s) = S(t+s)$ for all $t, s \geq 0$.

(4) For any $x \in X$ we have $S(t)x \in C^0([0, \infty), X)$.

Example 15.5. $S(t) := e^{t\triangle}$ is a contraction semigroup in $C_0(\mathbb{R}^d, \mathbb{R})$ (thought as a subspace of $L^\infty(\mathbb{R}^d, \mathbb{R})$). Indeed recall that for $K_t(x) := (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$ we have $e^{t\triangle} f = K_t * f$ for all $f \in C_0(\mathbb{R}^d, \mathbb{R})$. Then $\|S(t)\| \leq \|S(t)1\|_\infty = 1$. We have $S(0) = I$. We have also $S(t+s)f = S(t)S(s)f$ for any $f \in C_c(\mathbb{R}^d, \mathbb{R})$, from

$$\begin{aligned} \mathcal{F}(K_{t+s} * f) &= e^{-t|\xi|^2} e^{-s|\xi|^2} \widehat{f} = (2\pi)^{-\frac{d}{2}} \mathcal{F}\left(\underbrace{\mathcal{F}^*(e^{-t|\xi|^2})}_{(2t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}} *(K_s * f) \right) \\ &= \mathcal{F}\left(K_t * (K_s * f)\right) \Longrightarrow K_{t+s} * f = K_t * (K_s * f), \end{aligned}$$

and this extends to $f \in C_0(\mathbb{R}^d, \mathbb{R})$ by density. Finally, by Theorem 1.10 we have the continuity in $t = 0$ of $S(t)f$, and hence by (3) the continuity for all t.

Lemma 15.6. *Let $S(t)$ be a contraction semigroup, $F : X \to X$ a locally Lipschitz map, let $x \in X$ and let $u, v \in C^0([0, t_0], X)$ for $t_0 \in \mathbb{R}_+$ solve*

$$w(t) = S(t)x + \int_0^t S(t-s)F(w(s))ds. \tag{15.3}$$

Then $u = v$.

Let $M = \max_{0 \le t \le t_0}\{\|u(t)\|, \|v(t)\|\}$. Then

$$\|u(t) - v(t)\| \le \int_0^t \|F(u(s)) - F(v(s))\|ds \le L(M) \int_0^t \|u(s) - v(s)\|ds$$

and apply Gronwall's inequality. □

Proposition 15.7. *Let $x \in X$ with $\|x\| \le M$. Then there is a unique solution $u \in C^0([0, T_M], X)$ of (15.3) with*

$$T_M := \frac{1}{2L(2M + \|F(0)\|) + 2}. \tag{15.4}$$

Proof. Set $K = 2M + \|F(0)\|$ and

$$E = \{u \in C^0([0, T_M], X) : \|u(t)\| \le K \text{ for all } t \in [0, T_M]\}$$

with the distance of $L^\infty([0, T_M], X)$. E is a complete metric space. Next consider the map $u \in E \to \Phi_u$

$$\Phi_u(t) = S(t)x + \int_0^t S(t-s)F(u(s))ds \text{ for all } t \in [0, T_M].$$

By $T_M = \frac{1}{2(L(K)+1)}$ for all $t \in [0, T_M]$ we have

$$\begin{aligned} &\|F(u(t))\| \le \|F(0)\| + \|F(u(t)) - F(0)\| \le \|F(0)\| + KL(K) \\ &= \|F(0)\| + (2M + \|F(0)\|)L(K) \le 2(M + \|F(0)\|)(L(K) + 1) = \frac{M + \|F(0)\|}{T_M} \end{aligned} \tag{15.5}$$

and

$$\|S(t)x\| \le \|x\| \le M. \tag{15.6}$$

So from (15.5)–(15.6) for $t \in [0, T_M]$ we have

$$\|\Phi_u(t)\| \le M + t\frac{M + \|F(0)\|}{T_M} \le 2M + \|F(0)\| = K$$

and so $\Phi_u \in E$.
For $u, v \in E$ we have

$$\|\Phi_u(t) - \Phi_v(t)\| \le \int_0^t \|F(u(s)) - F(v(s))\|ds \le T_M L(K)\|u - v\|_{L^\infty([0,T_M],X)}.$$

So by $T_M L(K) < 2^{-1}$

$$\|\Phi_u - \Phi_v\|_{L^\infty([0,T_M],X)} \le 2^{-1}\|u - v\|_{L^\infty([0,T_M],X)}$$

Hence $u \to \Phi_u$ is a contraction in E and so it has exactly one fixed point.

□

Notice that if $F(0) = 0$ if and $\lim_{M\to 0^+} L(M) = 0$, something which happens in many important cases, we can improve the above result and get a T_M s.t. $\lim_{M\to 0^+} T_M = \infty$, as we will see now.

Proposition 15.8. *Let $x \in X$ with $\|x\| \le M$. Assume $F(0) = 0$ Then there is a unique solution $u \in C^0([0,T_M],X)$ of (15.3) with*

$$T_M := \frac{1}{2L(2M)}. \tag{15.7}$$

Proof. The argument is the same. Here we set $K = 2M$ and define E as above by

$$E = \{u \in C^0([0,T_M],X) : \|u(t)\| \le 2M \text{ for all } t \in [0,T_M]\}$$

Consider the map $u \in E \to \Phi_u$ defined as above by

$$\Phi_u(t) = S(t)x + \int_0^t S(t-s)F(u(s))ds \text{ for all } t \in [0,T_M].$$

By $T_M = \frac{1}{2L(2M)}$ for all $t \in [0,T_M]$ we have

$$\|F(u(t))\| \le 2ML(2M) = \frac{M}{T_M} \tag{15.8}$$

and

$$\|T(t)x\| \le \|x\| \le M. \tag{15.9}$$

So from (15.5)–(15.6) for $t \in [0,T_M]$ we have

$$\|\Phi_u(t)\| \le M + t\frac{M}{T_M} \le 2M$$

and so $\Phi_u \in E$.
For $u, v \in E$ we have

$$\|\Phi_u(t) - \Phi_v(t)\| \le \int_0^t \|F(u(s)) - F(v(s))\|ds \le T_M L(2M)\|u - v\|_{L^\infty([0,T_M],X)}.$$

So by $T_M L(2M) = 2^{-1}$

$$\|\Phi_u - \Phi_v\|_{L^\infty([0,T_M],X)} \le 2^{-1}\|u - v\|_{L^\infty([0,T_M],X)}$$

Hence $u \to \Phi_u$ is a contraction in E and so it has exactly one fixed point. □

We now turn to an abstract form of the *maximum principle*.

Recall that in an ordered Banach space the ordering is characterized by a convex closed cone $\mathcal{C}$ s.t.

1. $\mathcal{C} + \mathcal{C} \subseteq \mathcal{C}$,
2. $\lambda\mathcal{C} \subseteq \mathcal{C}$ for all $\lambda \geq 0$ and
3. $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

Then given $x, y \in X$ we write $y \geq x$ if $(y - x) \in \mathcal{C}$.

Lemma 15.9. *Suppose that in X there is a relation of order and that $F(u) \geq 0$ if $u \geq 0$. Suppose furthermore that $S(t)$ is positivity preserving, that is $x \geq 0 \Rightarrow S(t)x \geq 0$ for all t. Then if $x \geq 0$ the solution $u \in C^0([0, T_M], X)$ of Prop. 15.7 is $u(t) \geq 0$ for all t.*

Proof. We just rephrase the fixed point argument of Prop. 15.7 in a different set up. Indeed, if we redefine the set E writing

$$E = \{u \in C^0([0, T_M], X) : \|u(t)\| \leq K \text{ and } u(t) \geq 0 \text{ for all } t \in [0, T_M]\},$$

then E is a complete metric space. Furthermore the map $u \to \Phi_u$ with

$$\Phi_u(t) = S(t)f + \int_0^t S(t-s)F(u(s))ds \text{ for all } t \in [0, T_M].$$

is such that $u(t) \geq 0$ for all $t \in [0, T_M]$ implies $\Phi_u(t) \geq 0$ for all $t \in [0, T_M]$. Then the proof of Proposition 15.7 works out in the same way as before under this slightly more restrictive definition of E.

□

Lemma 15.10. *Assume the hypotheses of Lemma 15.9 and furthermore that $F(v) \geq F(u) \geq 0$ if $v \geq u \geq 0$. Let $x < y$. Let $u(t), v(t) \in C^0([0, T_*), X)$ be solutions with $u(0) = x$ and $v(0) = y$. Then $u(t) \leq v(t)$ in $[0, T_*)$.*

Proof. If $M = \max\{\|x\|, \|y\|\}$, then using the setup of Prop. 15.7 we consider the set

$$E = \{f \in C^0([0, T_M], X) : f(t) \geq 0 \text{ and } \|f(t)\| \leq K \text{ for all } t \in [0, T_M]\}$$

and the maps $f \in E \to \Phi_x(f)$ and $f \in E \to \Phi_y(f)$

$$\Phi_w(f)(t) = S(t)w + \int_0^t S(t-s)F(f(s))ds \text{ for all } t \in [0, T_M].$$

Let $v(t)$ be the solution with initial datum y. Then we have $\Phi_x(v) < \Phi_y(v) = v$. This can be iterated and if $0 < \Phi_x^j(v) < \Phi_x^{j-1}(v)$, then $0 < \Phi_x^{j+1}(v) < \Phi_x^j(v)$. But we know that $\Phi_x^j(v) \overset{j\to\infty}{\to} u$, with u the solution with initial datum x. Hence $u \leq v$.

So we have proved $u(t) \leq v(t)$ in $[0, T_M]$. Let now

$$T_1 := \sup\{T \in [0, T_*) \text{ such that } u(t) \leq v(t) \text{ in } [0, T]\}.$$

If $T_1 = T_*$ the theorem is finished. If $T_1 < T_*$ we have by continuity $u(T_1) \leq v(T_1)$. But then there exists a $0 < T < T_* - T_1$ with s.t. $\widetilde{u}(t) := u(t + T_1)$ and resp. $\widetilde{v}(t) := v(t + T_1)$ solve in $[0, T]$ the equation with initial data $\widetilde{x} \leq \widetilde{y}$ with $\widetilde{x} := u(T_1)$ and resp. $\widetilde{y} := v(T_1)$. But for T small enough we have $\widetilde{u}(t) \leq \widetilde{v}(t)$ in $[0, T]$ by the 1st part of the proof. But this implies than $u(t) \leq v(t)$ in $[0, T_1 + T]$. This is absurd by the definition of T_1, and so $T_1 = T_*$.

□

We will consider now the function $T : X \to (0, \infty]$ where for any $x \in X$ the interval $[0, T(x))$ is the maximal (positive) interval of existence of the unique solution of (15.3).

Theorem 15.11. *We have, for $u(t)$ the corresponding solution in $C([0, T(x)), X)$,*

$$2L(\|F(0)\| + 2\|u(t)\|) \geq \frac{1}{T(x) - t} - 2 \tag{15.10}$$

for all $t \in [0, T(x))$. We have the alternatives

(1) $T(x) = +\infty$;

(2) if $T(x) < +\infty$ then $\lim_{t \nearrow T(x)} \|u(t)\| = +\infty$.

Proof. First of all it is obvious that if $T(x) < +\infty$ then by (15.10)

$$\lim_{t \nearrow T(x)} L(\|F(0)\| + 2\|u(t)\|) = +\infty \Rightarrow \lim_{t \nearrow T(x)} \|u(t)\| = +\infty$$

where the implication follows from the fact that $M \to L(M)$ is an increasing function.

Let $x \in X$. Set $T(x) = \sup\{T > 0 : \exists u \in C^0([0, T), X) \text{ solution of (15.3) }\}$. We are left with the proof of (15.10), which is clearly true if $T(x) = \infty$. Now suppose that $T(x) < \infty$ and that (15.10) is false. This means that there exists a $t \in [0, T(x))$ with

$$\frac{1}{T_M} - 2 = 2L(\|F(0)\| + 2\|u(t)\|) < \frac{1}{T(x) - t} - 2 \Rightarrow T(x) - t < T_M$$

for $M = \|u(t)\|$, where we recall $T_M := \frac{1}{2L(2M + \|F(0)\|) + 2}$ in (15.4). Consider now $v \in C^0([0, T_M], X)$ the solution of

$$v(s) = S(s)u(t) + \int_0^s S(s - s')F(v(s'))ds' \text{ for all } s \in [0, T_M].$$

which exists by the previous Proposition 15.7. Then define

$$w(s) := \begin{cases} u(s) \text{ for } s \in [0, t] \\ v(s - t) \text{ for } s \in [t, T_M]. \end{cases}$$

We claim that $w \in C^0([0,T_M],X)$ is a solution of (15.3). In $[0,t]$ this is obvious since in $w = u$ in $[0,t]$ and $u \in C^0([0,t],X)$ is a solution of (15.3). Let now $s \in (t, T_M]$. We have

$$\begin{aligned} w(s) = v(s-t) &= S(s-t)u(t) + \int_0^{s-t} S(s-t-s')F(v(s'))ds' \\ &= S(s-t)\left[S(t)x + \int_0^t S(t-s')F(u(s'))ds'\right] + \int_0^{s-t} S(s-t-s')F(v(s'))ds' \\ &= S(s)x + \int_0^t S(s-s')F(\underbrace{u(s')}_{w(s')})ds' + \int_t^s S(s-s')F(\underbrace{v(s'-t)}_{w(s')})ds' \\ &= S(s)x + \int_0^s S(s-s')F(w(s'))ds. \end{aligned}$$

□

Remark 15.12. Notice that if $F(0) = 0$, then we can prove the improved estimate

$$2L(\|F(0)\| + 2\|u(t)\|) \geq \frac{1}{T(x)-t}. \tag{15.11}$$

The proof is exactly the same of Theorem 15.11 using the altered definitions of T_M, $T_M = (2L(2M))^{-1}$.

Proposition 15.13. *(1) $T : X \to (0,\infty]$ is lower semicontinuous;*

(2) if $x_n \to x$ in X and if $\bar{t} < T(x)$ we have $u_n \to u$ in $C^0([0,\bar{t}],X)$ with u_n the solution of (15.3) *with initial datum x_n.*

Proof. Let $u \in C^0([0,T(x)),X)$ be the solution of (15.3) and consider $\bar{t} < T(x)$. Set $M = 2\|u\|_{L^\infty([0,\bar{t}],X)}$ and let

$$\tau_n = \sup\{t \in [0,T(x_n)) : \|u_n\|_{L^\infty([0,t],X)} \leq K\} \text{ where } K = 2M + \|F(0)\|.$$

For $n \gg 1$ we have $\|x_n\| < M$. Then $u_n \in C^0([0,T_M],X)$ with $\|u_n\|_{L^\infty([0,T_M],X)} \leq K$ by Prop. 15.7. This implies $\tau_n \geq T_M$. For $0 \leq t \leq \min\{\bar{t},\tau_n\}$ we have

$$u(t) - u_n(t) = S(t)(x - x_n) + \int_0^t S(s-t)(F(u(s)) - F(u_n(s)))ds$$

and so

$$\begin{aligned} &\|u(t) - u_n(t)\| \leq \|x - x_n\| + L(K)\int_0^t \|u(s) - u_n(s)\|ds \Rightarrow \\ &\|u(t) - u_n(t)\| \leq e^{L(K)t}\|x - x_n\| \Rightarrow \|u(t) - u_n(t)\| \leq e^{L(K)\bar{t}}\|x - x_n\|. \end{aligned} \tag{15.12}$$

So $\|u_n(t)\| \leq \|u(t)\| + e^{L(K)\bar{t}}\|x - x_n\| \leq M/2 + e^{L(K)\bar{t}}\|x - x_n\| \leq M$ for $n \gg 1$ and $0 \leq t \leq \min\{\bar{t},\tau_n\}$. This and continuity imply $\tau_n > \min\{\bar{t},\tau_n\}$ and so $\tau_n > \bar{t}$. Then we have $T(x_n) > \bar{t}$. This implies the lower semi–continuity in claim (1). Furthermore by (15.12) we have also $u_n \to u$ in $C^0([0,\bar{t}],X)$. □

15.2 Proof of Fujita's Theorem

We know that $S(t) := e^{t\triangle}$ is a contraction semigroup in $C_0(\mathbb{R}^d, \mathbb{R})$. Notice that in $C_0(\mathbb{R}^d, \mathbb{R})$ there is a natural partial order, and that this is preserved by $e^{t\triangle}$. In fact, if $f \in C_0(\mathbb{R}^d, \mathbb{R})$ is $f(x) \geq 0$ for all $x \in \mathbb{R}^d$, and is not identically 0, then $e^{t\triangle} f > 0$ everywhere ($e^{t\triangle}$ is *positivity enhancing*).

By the abstract theory presented above, we can prove the following maximum principle property.

Lemma 15.14. *Let $u \in C([0,T), C_0(\mathbb{R}^d, \mathbb{R}))$ be the unique maximal solution of*

$$u(t) = e^{t\triangle} f + \int_0^t e^{(t-s)\triangle} |u(s)|^{p-1} u(s) ds \tag{15.13}$$

and let $f \geq 0$. Then $u(t,x) \geq 0$ for all $(t,x) \in [0,T) \times \mathbb{R}^n$.

□

We prove now the following version of Fujita's Theorem (compared to Theorem 15.1, we add the hypothesis $u_0 \in L^1(\mathbb{R}^d)$).

Theorem 15.15. *Let $u_0 \in L^1(\mathbb{R}^d) \cap C_0(\mathbb{R}^d)$ with $u_0 \geq 0$ and suppose $1 < p \leq 1 + \frac{2}{d}$. Suppose that $u(t) \in C^0([0, T_{u_0}), C_0(\mathbb{R}^d))$ is a positive solution of*

$$u(t) = e^{t\triangle} u_0 + \int_0^t e^{(t-s)\triangle} u^p(s) ds. \tag{15.14}$$

Then $T_{u_0} < \infty$.

Proof. We claim, and for the moment assume, the following inequality due to Weissler:

$$t^{\frac{1}{p-1}} e^{t\triangle} u_0(x) \leq C \text{ for a fixed } C = C(p) > 0, \text{ for any } x \in \mathbb{R}^d,\ t \in [0, T_{u_0}) \text{ and any } u_0 \geq 0. \tag{15.15}$$

Here, crucially, C depends only on p.
Suppose we have $T_{u_0} = \infty$ and assume (15.15).
By dominated convergence we have for any $x \in \mathbb{R}^d$

$$\lim_{t \nearrow \infty} (4\pi)^{\frac{d}{2}} t^{\frac{d}{2}} e^{t\triangle} u_0(x) = \lim_{t \nearrow \infty} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy = \int_{\mathbb{R}^n} u_0(y) dy = \|u_0\|_{L^1(\mathbb{R}^n)}. \tag{15.16}$$

In the particular case $p < 1 + \frac{2}{d}$, equivalent to $\frac{1}{p-1} - \frac{d}{2} > 0$, we see immediately that (15.16) is incompatible with (15.15) since

$$\lim_{t \nearrow \infty} t^{\frac{1}{p-1}} e^{t\triangle} u_0(x) = \lim_{t \nearrow \infty} t^{\frac{1}{p-1} - \frac{d}{2}} t^{\frac{d}{2}} e^{t\triangle} u_0(x) = \lim_{t \nearrow \infty} t^{\frac{1}{p-1} - \frac{d}{2}} (4\pi)^{-\frac{d}{2}} \|u_0\|_{L^1(\mathbb{R}^n)} = +\infty.$$

In the case $p = 1 + \frac{2}{d}$ this argument does not provide a contradiction for all u_0 (although this argument shows that if $\|u_0\|_{L^1(\mathbb{R}^d)} > (4\pi)^{\frac{d}{2}} C$ for $C = C(1 + \frac{2}{d})$ then there is blow up). We complete the argument below, but first we prove claim (15.15).

Proof of (15.15) We turn now to the proof of (15.15). We have $u(t) \geq e^{t\triangle}u_0(x)$ and

$$\begin{aligned} u(t) &\geq \int_0^t e^{(t-s)\triangle}u^p(s)ds \geq \int_0^t e^{(t-s)\triangle}(e^{s\triangle}u_0)^p ds \\ &\geq \int_0^t (e^{(t-s)\triangle}e^{s\triangle}u_0)^p ds = \int_0^t (e^{t\triangle}u_0)^p ds = t(e^{t\triangle}u_0)^p, \end{aligned} \tag{15.17}$$

where we used, for $d\mu(y) := (4\pi\tau)^{-\frac{d}{2}}e^{-\frac{|x-y|^2}{4\tau}}dy$ which gives a probability measure in $\mathbb{R}^d$,

$$\begin{aligned} &e^{\tau\triangle}(f)^p(x) = (4\pi\tau)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\tau}}f^p(y)dy = \int_{\mathbb{R}^d} f^p(y)d\mu(y) \\ &\geq \left(\int_{\mathbb{R}^d} f(y)d\mu(y)\right)^p = \left((4\pi\tau)^{-\frac{d}{2}}\int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\tau}}f(y)dy\right)^p = \left(e^{\tau\triangle}(f)(x)\right)^p, \end{aligned}$$

which follows from Jensen's inequality $\varphi(\int f d\mu) \leq \int \varphi \circ f d\mu$ for a convex function φ and a probability measure μ.

By a substitution inside (15.17) and by repeating the same argument we get

$$u(t) \geq \int_0^t e^{(t-s)\triangle}s^p(e^{s\triangle}u_0)^{p^2}ds \geq \int_0^t s^p(e^{t\triangle}u_0)^{p^2}ds = \frac{t^{p+1}}{p+1}(e^{t\triangle}u_0)^{p^2}.$$

This is the case $k = 2$ of the following inequality which for any $k \in \mathbb{N}$ with $k \geq 2$ we will obtain by induction:

$$u(t) \geq \frac{t^{1+p+...+p^{k-1}}(e^{t\triangle}u_0)^{p^k}}{(1+p)^{p^{k-2}}(1+p+p^2)^{p^{k-3}}...(1+p+...+p^{k-1})} = \frac{t^{\frac{p^k-1}{p-1}}(e^{t\triangle}u_0)^{p^k}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k-\ell}}}. \tag{15.18}$$

Indeed, assuming (15.18) for k and repeating (15.17) we have

$$\begin{aligned} u(t) &\geq \int_0^t e^{(t-s)\triangle}u^p(s)ds \geq \int_0^t \frac{s^{\frac{p^k-1}{p-1}p}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}}e^{(t-s)\triangle}(e^{s\triangle}u_0)^{p^{k+1}}ds \\ &\geq \int_0^t \frac{s^{\frac{p^k-1}{p-1}p}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}}ds(e^{t\triangle}u_0)^{p^{k+1}} = \frac{t^{\frac{p^k-1}{p-1}p+1}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}\left(\frac{p^k-1}{p-1}p+1\right)}(e^{t\triangle}u_0)^{p^{k+1}} \\ &= \frac{t^{\frac{p^{k+1}-1}{p-1}}}{\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}\frac{p^{k+1}-1}{p-1}}(e^{t\triangle}u_0)^{p^{k+1}} = \frac{t^{\frac{p^{k+1}-1}{p-1}}}{\prod_{\ell=2}^{k+1}\left(\frac{p^\ell-1}{p-1}\right)^{p^{k+1-\ell}}}(e^{t\triangle}u_0)^{p^{k+1}}. \end{aligned}$$

So (15.18) holds also for $k+1$ and hence for any $k \in \mathbb{N}$ with $k \geq 2$. Then

$$\begin{aligned} &t^{\frac{p^k-1}{(p-1)p^k}}e^{t\triangle}u_0 \leq (u(t))^{\frac{1}{p^k}}\prod_{\ell=2}^k\left(\frac{p^\ell-1}{p-1}\right)^{\frac{1}{p^\ell}} \Rightarrow t^{\frac{1}{p-1}}e^{t\triangle}u_0 \leq \prod_{\ell=2}^\infty\left(\frac{p^\ell-1}{p-1}\right)^{\frac{1}{p^\ell}} \\ &= e^{\sum_{\ell=2}^\infty p^{-\ell}\log\left(\frac{p^\ell-1}{p-1}\right)} = e^{\sum_{\ell=2}^\infty p^{-\ell}\log\left(\sum_{j=1}^{\ell-1}p^j\right)} \leq e^{\sum_{\ell=2}^\infty p^{-\ell}\log\left(\ell p^\ell\right)} < +\infty. \end{aligned}$$

This proves (15.15).

Proof of the case $p = 1 + \frac{2}{d}$ We return to the proof of Theorem 15.15 when $p = 1 + \frac{2}{d}$. If instead of looking at solutions in $C_0(\mathbb{R}^d)$ we look at solutions in $X := C_0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ then our $u \in C^0([0, T_{u_0}), C_0(\mathbb{R}^d))$ is also $u \in C^0([0, T_{u_0}), X)$. Indeed, if the lifespan in X was shorter, then for some $t_0 < T_{u_0}$ we would have

$$\lim_{t \nearrow t_0} \|u(t)\|_{L^1(\mathbb{R}^d)} = \infty \text{ while } \sup_{0 \le t \le t_0} \|u(t)\|_{L^\infty(\mathbb{R}^d)} < \infty.$$

But this is impossible because from (15.14) for $t < t_0$ we get

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} + \int_0^t \|u(s)\|^{p-1}_{L^\infty(\mathbb{R}^d)} \|u(s)\|_{L^1(\mathbb{R}^d)} ds$$

implies by the Gronwall inequality

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} e^{t_0 (\sup_{0 \le t \le t_0} \|u(t)\|_{L^\infty(\mathbb{R}^d)})^{p-1}} < \infty$$

and so

$$+\infty = \lim_{t \nearrow t_0} \|u(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} e^{t_0 (\sup_{0 \le t \le t_0} \|u(t)\|_{L^\infty(\mathbb{R}^d)})^{p-1}} < +\infty,$$

which is absurd.
Hence we conclude that $t_0 = T_{u_0}$ and we have $u \in C^0([0, T_{u_0}), L^1(\mathbb{R}^d))$, and so $u(t) \in L^1(\mathbb{R}^d)$ for all $t \in [0, T_{u_0})$. Since any such t can be taken as an initial value at time t for our solution, it follows that

$$\tau^{\frac{d}{2}} (e^{\tau \triangle} u(t))(x) \le C \text{ for a fixed } C > 0 \text{ , any } x \in \mathbb{R}^d \text{ and } 0 < \tau < T_{u_0} - t$$

and for all $t \in [0, T_{u_0})$. In particular if $T_{u_0} = \infty$, by the argument in (15.16), we have

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \le (4\pi)^{\frac{d}{2}} C \text{ for all } t \ge 0. \tag{15.19}$$

Initially we assume that $u_0 \ge k K_\alpha$, for $K_\alpha(x) := (4\pi\alpha)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4\alpha}}$. Notice that $K_\alpha = e^{\alpha \triangle} \delta_0$. Then we have (a bit formally, but can be checked)

$$u(t) \ge e^{t\triangle} u_0 \ge k e^{t\triangle} K_\alpha = k e^{t\triangle} e^{\alpha\triangle} \delta_0 = k e^{(\alpha+t)\triangle} \delta_0 = k K_{\alpha+t}.$$

Now we have

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \geq \| \int_0^t e^{(t-s)\triangle} u^p(s) ds \|_{L^1(\mathbb{R}^d)} = \int_{\mathbb{R}^d} dx \int_0^t e^{(t-s)\triangle} u^p(s)(x) ds$$

$$= \int_0^t ds \int_{\mathbb{R}^d} dx e^{(t-s)\triangle} u^p(s)(x) = \int_0^t \|e^{(t-s)\triangle} u^p(s)\|_{L^1(\mathbb{R}^d)} ds \text{ (by commuting the order of integration)}$$

$$\geq \int_0^t \|e^{(t-s)\triangle}(e^{s\triangle}u_0)^p\|_{L^1(\mathbb{R}^d)} ds$$

$$= \int_0^t ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^n} dy K_{t-s}(x-y)(e^{s\triangle}u_0)^p(y) = \int_0^t ds \int_{\mathbb{R}^d} dy (e^{s\triangle}u_0)^p(y) \underbrace{\int_{\mathbb{R}^d} dx K_{t-s}(x-y)}_{1}$$

$$= \int_0^t \|(e^{s\triangle}u_0)^p\|_{L^1(\mathbb{R}^d)} ds \geq k^p \int_0^t \|(e^{s\triangle}K_\alpha)^p ds\|_{L^1(\mathbb{R}^d)} = k^p \int_0^t \|K^p_{\alpha+s}\|_{L^1(\mathbb{R}^d)} ds.$$

Now notice that

$$K^p_\beta(x) = (4\pi\beta)^{-\frac{d}{2}p} e^{-\frac{p|x|^2}{4\beta}} = (4\pi\beta)^{-\frac{d}{2}(p-1)} p^{-\frac{d}{2}} (4\pi\beta/p)^{-\frac{d}{2}} e^{-\frac{p|x|^2}{4\beta}} \quad = (4\pi\beta)^{-\frac{d}{2}(p-1)} p^{-\frac{d}{2}} K^p_{\frac{\beta}{p}}(x)$$

$$= (4\pi\beta)^{-1} p^{-\frac{d}{2}} K^p_{\frac{\beta}{p}}(x) \qquad \text{by } p = 1 + 2/d.$$

This implies that, if by contradiction we suppose $T_{u_0} = +\infty$, then we have

$$\|u(t)\|_{L^1(\mathbb{R}^d)} \geq p^{-\frac{d}{2}} k^p \int_0^t (4\pi(\alpha+s))^{-1} \|K_{\frac{\alpha+s}{p}}\|_{L^1(\mathbb{R}^d)} ds$$

$$= p^{-\frac{d}{2}} k^p (4\pi)^{-1} \int_0^t (\alpha+s)^{-1} ds \to +\infty \text{ as } t \nearrow \infty.$$

This contradicts (15.19).
Suppose now we don't have $u_0 \geq kK_\alpha$. Let us set $v(t) = u(t+\varepsilon)$ for some $\varepsilon > 0$. Then $v(t)$ is a solution of (15.14) with initial value $u(\varepsilon)$. We have $u(\varepsilon) \geq e^{\varepsilon\triangle}u_0$

$$v(0) = u(\varepsilon) \geq e^{\varepsilon\triangle}u_0 = (4\pi\varepsilon)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4\varepsilon}} f(y) dy = (4\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \int_{\mathbb{R}^d} e^{\frac{|x+y|^2}{4\varepsilon}} e^{-\frac{|y|^2}{2\varepsilon}} f(y) dy$$

$$\geq (4\pi\varepsilon)^{-\frac{d}{2}} e^{-\frac{|x|^2}{2\varepsilon}} \int_{\mathbb{R}^d} e^{-\frac{|y|^2}{2\varepsilon}} f(y) dy = kK_{\frac{\varepsilon}{2}}$$

where we used the parallelogram formula

$$|x+y|^2 + |x-y|^2 = 2|x|^2 + 2|y|^2.$$

But then $v(t)$ blows up in finite time, and so $u(t)$ does too. This completes the proof of Theorem 15.15 also in the case $p = 1 + \frac{2}{d}$.

□

So far we have proved the blow up when $1 < p \leq 1 + \frac{2}{d}$ for positive initial data with $u_0 \in C^0_0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. But in fact the result holds for $u_0 \in C^0_0(\mathbb{R}^d)$ because of the maximum principle.

Lemma 15.16. *Suppose that* $0 \le v_0 \le u_0$ *are in* $C_0^0(\mathbb{R}^d)$ *and let* $u(t), v(t) \in C^0([0,T], C_0^0(\mathbb{R}^d))$ *be corresponding solutions of* (15.14). *Then* $u(t) \ge v(t)$.

This follows by Lemma 15.10 and means that if $u_0 \in C_0^0(\mathbb{R}^d)$ but $u_0 \notin L^1(\mathbb{R}^d)$, the solution u blows up, because we can find a $0 \le v_0 \le u_0$ with $v_0 \in C_0^0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and v_0 non zero whose corresponding $v(t)$ blows up. Then by the maximum principle also $u(t)$ blows up. ☐

This completes the proof of Theorem 15.1. ☐

Remark 15.17. The coefficient $p = 1 + \frac{2}{d}$ is critical. In fact, for any $p > 1 + \frac{2}{p}$ there exists $\epsilon_p > 0$ s.t. if $u_0 \in X := C_0^0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ satisfies $\|u_0\|_X < \epsilon_p$, then equation (15.14) admits a global solution in $C_b^0([0,\infty), C_0^0(\mathbb{R}^d) \cap L^1(\mathbb{R}^d))$.

A Appendix. On the Bochner integral

For this part see [3]. Let X be a Banach space.

Definition A.1 (Strong measurability)**.** Let I be an interval. A function $f : I \to X$ is strongly measurable if there exists a set E of measure 0 and a sequence $(f_n(t))$ in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ for any $t \in I \backslash E$.

Remark A.2. Notice that when $\dim X < \infty$ a function is measurable (in the sense that $f^{-1}(B)$ is measurable for any Borel set B) if an only if it is strongly measurable in the above sense. Indeed if f is strongly measurable in the above sense then as a point wise limit of measurable functions f is measurable, see Theorem 1.14 p. 14 Rudin [11]. Viceversa if f is measurable, then f is strongly measurable in the above sense, see the Corollary to Lusin's Theorem in Rudin [11] p. 54.

Example A.3. Consider $\{x_j\}_{j=1}^n$ in X and $\{A_j\}_{j=1}^n$ measurable sets in I with $|A_j| < \infty$ and with $A_j \cap A_k = \emptyset$ for $j \neq k$. Then we claim that the *simple* function

$$f(t) := \sum_{j=1}^n x_j \chi_{A_j}(t) : I \to X \tag{A.1}$$

is measurable. Indeed, see Rudin [11] p. 54, there are sequences $\{\varphi_{j,k}\}_{k\in\mathbb{N}}$ in $C_c^0(I,\mathbb{R})$ with $\varphi_{j,k}(t) \stackrel{k\to\infty}{\to} \chi_{A_j}(t)$ a.e. and hence

$$C_c^0(I,\mathbb{R}) \ni f_k(t) := \sum_{j=1}^n x_j \varphi_{j,k}(t) \stackrel{k\to\infty}{\to} f(t) \text{ a.e. in } I.$$

Proposition A.4. *If* (f_n) *is a sequence of strongly measurable functions from* I *to* X *convergent a.e. to a* $f : I \to X$*, then* f *is strongly measurable.*

Proof. There is an E with $|E| = 0$ s.t. $f_n(t) \overset{n\to\infty}{\to} f(t)$ for any $t \in I\backslash E$. Consider for any n a sequence $C_c(I, X) \ni f_{n,k} \overset{k\to\infty}{\to} f_n$ a.e. We will suppose first that $|I| < \infty$. By applying Egorov Theorem to $\{\|f_{n,k} - f_n\|\}_{k\in\mathbb{N}}$ there is $E_n \subset I$ with $|E_n| \le 2^{-n}$ s.t. $\|f_{n,k} - f_n\| \overset{k\to\infty}{\to} 0$ uniformly in $I\backslash E_n$ Let $k(n)$ be s.t. $\|f_{n,k(n)} - f_n\| < 1/n$ in $I\backslash E_n$ and set $g_n = f_{n,k(n)}$. Set $F := E \bigcup (\bigcap_m \bigcup_{n>m} E_n)$. Then $|F| = 0$. Indeed for any m

$$|F| \le |E| + \sum_{n=m}^{\infty} |E_n| \le |E| + \sum_{n=m}^{\infty} 2^{-n} \overset{m\to\infty}{\to} 0.$$

Let $t \in I\backslash F$. Since $t \notin E$ we have $f_n(t) \overset{n\to\infty}{\to} f(t)$. Furthermore, for n large enough we have $t \in I\backslash E_n$. Indeed

$$t \notin \bigcap_m \bigcup_{n>m} E_n \Rightarrow \exists\ m \text{ s.t. } t \notin \bigcup_{n>m} E_n \Rightarrow\ t \notin E_n\ \forall\ n > m.$$

Then $\|g_n(t) - f_n(t)\| < 1/n$ and $g_n(t) \overset{n\to\infty}{\to} f(t)$. So $f(t)$ is measurable in the case $|I| < \infty$. Now we consider the case $|I| = \infty$. We express $I = \cup_n I_n$ for an increasing sequence of intervals with $|I_n| < \infty$. Consider for any n a sequence $C_c(I_n, X) \ni f_{n,k} \overset{k\to\infty}{\to} f$ a.e. in I_n. Then by Egorov Theorem to $\|f_{n,k} - f_n\|$ there is $E_n \subset I_n$ with $|E_n| \le 2^{-n}$ s.t. $f_{n,k} \overset{k\to\infty}{\to} f_n$ uniformly in $I_n\backslash E_n$ Let $k(n)$ be s.t. $\|f_{n,k(n)} - f_n\| < 1/n$ in $I_n\backslash E_n$ and set $g_n = f_{n,k(n)}$. Then defining F like above, the remainder of the proof works exactly like for the case $|I| < \infty$.

□

Example A.5. Consider a sequence $\{x_j\}_{j\in\mathbb{N}}$ in X and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of measurable sets in I with $|A_j| < \infty$ and with $A_j \cap A_k = \emptyset$ for $j \ne k$. Then we claim

$$f(t) := \sum_{j=1}^{\infty} x_j \chi_{A_j}(t) \tag{A.2}$$

is measurable. Indeed if we set $f_n(t) := \sum_{j=1}^{n} x_j \chi_{A_j}(t)$, then we have $\lim_{n\to\infty} f_n(t) = f(t)$ for any t, since if $t \notin \cup_{j=1}^{\infty} A_j$ both sides are 0, and if $t \in A_{n_0}$ then for $n \ge n_0$ we have $f_n(t) = x_{n_0} = f(t)$. Hence by Proposition A.4 the function f is measurable.
When the sum in (A.2) is finite then the function f is called *simple*.

Example A.6. Consider a sequence $\{x_j\}_{j\in\mathbb{N}}$ in X and a sequence $\{A_j\}_{j\in\mathbb{N}}$ of measurable sets in I where again $A_j \cap A_k = \emptyset$ for $j \ne k$ but we allow $|A_j| = \infty$. Then

$$f(t) := \sum_{j=1}^{\infty} x_j \chi_{A_j}(t) \tag{A.3}$$

is measurable. To see this consider $f_n(t) = \chi_{[-n,n]}(t) f(t)$. Then

$$f_n(t) = \sum_{j=1}^{\infty} x_j \chi_{A_j \cap [-n,n]}(t)$$

and by Example A.5 we know that each $f_n(t)$ is strongly measurable. Since $f_n(t) \to f(t)$ for any $t \in I$ we conclude by Proposition A.4 that f is strongly measurable.

Another natural definition of measurability is the following one.

Definition A.7 (Weak measurability)**.** Let I be an interval. A function $f : I \to X$ is weakly measurable if for any $x' \in X'$ the function $t \to \langle x', f(t) \rangle_{X'X}$ is a measurable function $I \to \mathbb{R}$.

Obviously, strongly measurable implies weakly measurable. Let us explore the viceversa.

Definition A.8. Let I be an interval. A function $f : I \to X$ is *almost separably valuable* if there exists a 0 measure set $N \subset I$ s.t. $f(I \backslash N)$ is separable.

The following lemma shows that strongly measurable functions are almost separably valuable.

Lemma A.9. *If $f : I \to X$ is strongly measurable with $(f_n(t))$ a sequence in $C_c(I, X)$ s.t. $f_n(t) \to f(t)$ for any $t \in I \backslash E$ for a 0 measure set $E \subset I$ then $f(I \backslash E)$ is separable and there exists a separable Banach subspace $Y \subseteq X$ with $f(I \backslash E) \subseteq Y$.*

Proof. First of all $f_n(I \cap \mathbb{Q})$ is a countable dense set in $f_n(I)$. So $f_n(I)$ is separable. In a separable metric space any subspace is separable. So $f_n(I \backslash E)$ is separable. The closed vector space Y generated by $\cup_n f_n(I \backslash E)$ is separable. Indeed let $C \subseteq \cup_n f_n(I \backslash E)$ be a countable set dense in $\cup_n f_n(I \backslash E)$. Let $\mathrm{Span}_{\mathbb{Q}}(C)$ be the vector space on $\mathbb{Q}$ generated by C. Then $\mathrm{Span}_{\mathbb{Q}}(C)$ is dense in Y. For $C = \{x_1, x_2, ...\}$ we have $\mathrm{Span}_{\mathbb{Q}}(C) = \cup_{n=1}^{\infty} \mathrm{Span}_{\mathbb{Q}}(\{x_1, ..., x_n\})$. This proves that $\mathrm{Span}_{\mathbb{Q}}(C)$ is countable and that Y is separable. □

Example A.10. Let X be a Hilbert space with an orthonormal basis $\{e_t\}_{t \in \mathbb{R}}$. Then the map $f : \mathbb{R} \to X$ given by $f(t) = e_t$ is not strongly measurable. This follows from the fact that it is not almost separably valuable.
On the other hand if $x \in X$ then $t \to \langle f(t), x \rangle$ is different from 0 only on a countable subset of $\mathbb{R}$, and as such it is measurable. Hence f is weakly measurable.
Notice however that if $C \subset [0, 1]$ is the standard Cantor set (which has 0 measure and has same cardinality of $\mathbb{R}$) and if $\{\widetilde{e}_t\}_{t \in C}$ is another basis of X, then the map

$$g(t) = \begin{cases} \widetilde{e}_t \text{ for } t \in C \text{ and} \\ \quad 0 \text{ otherwise} \end{cases}$$

is weakly measurable (like f and for the same reasons) and is almost separably valuable. Pettis Theorem, which we prove below, implies that $g : \mathbb{R} \to X$ is strongly measurable.

The following lemma will be used for Pettis Theorem.

Lemma A.11. *Let X be a separable Banach space and let S' be the unit ball of the dual X'. Then X' is separable for the weak topology $\sigma(X', X)$, see Brezis [2] p.62, that is there exists a sequence $\{x'_n\}$ in S' s.t. for any $x' \in S'$ there exists a subsequence $\{x'_{n_k}\}$ s.t. for any $x \in X$ we have $\lim_{k \to \infty} \langle x'_{n_k}, x \rangle_{X'X} = \langle x', x \rangle_{X'X}$.*

Proof. Let $\{x_n\}$ be dense in X. For any n consider

$$F_n : S' \to \mathbb{R}^n \text{ defined by } F_n(x') := (\langle x', x_1\rangle_{X'X}, ..., \langle x', x_n\rangle_{X'X}).$$

Since $\mathbb{R}^n$ is separable, and so is $F_n(S')$, there exists a sequence $\{x'_{n,k}\}_k$ s.t. $\{F_n(x'_{n,k})\}_k$ is dense in $F_n(S')$. Obviously $\{x'_{n,k}\}_{n,k}$ can be put into a sequence. For any $x' \in S'$ for any n there is a k_n s.t. $|\langle x' - x'_{n,k_n}, x_i\rangle_{X'X}| < 1/n$ for all $i \le n$. This implies that for any fixed i we have $\lim_{n\to\infty} \langle x'_{n,k_n}, x_i\rangle_{X'X} = \langle x', x_i\rangle_{X'X}$. By density this holds for any $x \in X$. □

Proposition A.12 (Pettis's Theorem)**.** *Consider $f : I \to X$. Then f is strongly measurable if and only if it is weakly measurable and almost separable valuable.*

Proof. The necessity has been already proved, so we focus on the sufficiency only. By modifying f we can assume that $f(I)$ is separable. By replacing X by a smaller space, we can assume that X is separable.
Fix now $x \in X$. Then we claim that $t \to \|f(t) - x\|$ is measurable. Indeed for any $a > 0$

$$\{t \in I : \|f(t) - x\| \le a\} = \cap_{x'\in S'}\{t \in I : |\langle x', f(t) - x\rangle_{X'X}| \le a\}.\}$$

Using the fact that S' is separable in the weak topology $\sigma(X', X)$ and the notation in Lemma A.11, we have

$$\{t \in I : \|f(t) - x\| \le a\} = \cap_{n\in\mathbb{N}}\{t \in I : |\langle x'_n, f(t) - x\rangle_{X'X}| \le a\}.$$

Since the set in the r.h.s. is measurable, we conclude that $t \to \|f(t) - x\|$ is measurable and so our claim is correct.
Consider now $n \ge 1$. Since $f(I)$ is separable there is a sequence of balls $\{B(x_j, \frac{1}{n})\}_{j\ge 0}$ whose union contains $f(I)$. Set now

$$\begin{cases} \omega_0^{(n)} := \{t : f(t) \in B(x_0, \frac{1}{n})\}\,, \\ \omega_j^{(n)} := \{t : f(t) \in B(x_j, \frac{1}{n})\}\setminus \cup_{k<j}\, \omega_k^{(n)} \end{cases}$$

and

$$f_n(t) := \sum_{j=0}^{\infty} x_j \chi_{\omega_j^{(n)}}(t).$$

Notice that $\cup_{j\ge 0}\omega_j^{(n)} = I$ and they are pairwise disjoint and measurable. By Example A.6 we know that $f_n : I \to X$ is strongly measurable. Furthermore, for any $t \in I$ there is a j s.t. $t \in \omega_j^{(n)}$ and this implies

$$\frac{1}{n} > \|f(t) - x_j\| = \|f(t) - f_n(t)\|.$$

In other words, $\|f(t) - f_n(t)\| \le 1/n$ for any $t \in I$. Then $f_n(t) \to f(t)$ for any t, and so by Proposition A.4 the function $f : I \to X$ is strongly measurable. □

Example A.13. Consider the map $f : (0,1) \to L^\infty(0,1)$ defined by $t \stackrel{f}{\to} \chi_{(0,t)}$. This map is not almost separable valued. Indeed $t \neq s$ implies $\|f(t) - f(s)\|_\infty = 1$. If f was almost separable valued then there would exist a 0 measure subset E in $(0,1)$ and a countable set $\mathcal{N} = \{t_n\}_n$ in $(0,1)\backslash E$ such that for any $t \in (0,1)\backslash(E \cup \mathcal{N})$ there would exist a subsequence n_k with $f(t_{n_k}) \stackrel{k\to\infty}{\to} f(t)$ in $L^\infty(0,1)$. But this is impossible since $\|f(t) - f(t_{n_k})\|_\infty = 1$.
On the other hand $f : (0,1) \to L^2(0,1)$ defined in the same way, is strongly measurable. First of, since $L^2(0,1)$ is separable, it is almost separable valued. Next for any given any $w \in L^2(0,1)$ we have

$$\langle f(t), w\rangle_{L^2(0,1)} = \int_0^t w(x)dx$$

which is a continuous, and hence measurable, function. So f is also weakly measurable and hence it is strongly measurable by Pettis Theorem.

Recall that in Remark A.2 we mentioned another possible notion of measurability, that is that $f : I \to X$ could be defined as measurable if $f^{-1}(A)$ is a measurable set for any open subset $A \subseteq X$. We have the following fact.

Proposition A.14. *Consider $f : I \to X$. Then f is strongly measurable $\Leftrightarrow$ it almost separably valuable and $f^{-1}(A)$ is a measurable set for any open subset $A \subseteq X$.*

Proof. The "$\Leftarrow$" follows from the fact that for any $\mathfrak{a}$ open subset of $\mathbb{R}$ and for any $x' \in X$ the set $A = \{x \in X : \langle x, x'\rangle_{X,X'} \in \mathfrak{a}\}$ is open and for $g(t) := \langle f(t), x'\rangle_{X,X'}$ we have $f^{-1}(A) = g^{-1}(\mathfrak{a})$. So the latter being measurable it follows that g is measurable and hence f is weakly measurable. Hence by Pettis Theorem we conclude that f is strongly measurable.
We now assume that f is strongly measurable. We know from Lemma A.9 that f is almost separably valuable. Let U be an open subset of X. Let $(f_n)_n$ be a sequence in $C_c^0(I, X)$ with $f_n(t) \stackrel{n\to\infty}{\to} f(t)$ a.e. outside a 0 measure set $E \subset I$. Let $U_r = \{x \in X : \text{dist}(x, U^c) > r\}$. Then

$$f^{-1}(U)\backslash E = (\cup_{m\geq 1} \cup_{n\geq 1} \cap_{k\geq n} f_k^{-1}(U_{\frac{1}{m}}))\backslash E. \tag{A.4}$$

To check this, notice that if t belongs to the left hand side , then $f(t) \in U_{\frac{1}{m_0}}$ for some m_0 and, since $f_n(t) \stackrel{n\to\infty}{\to} f(t)$, for n large we have $f_k(t) \in U_{\frac{1}{m_1}}$ if $k \geq n$ for $m_1 > m_0$ preassigned. Viceversa if t belongs to the right hand side, then there exist n and m s.t. $f_k(t) \in U_{\frac{1}{m}}$ for all $k \geq n$. Then by $f_k(t) \stackrel{k\to\infty}{\to} f(t)$ it follows that $f(t) \in \overline{U_{\frac{1}{m}}}$ with the latter a subset of U. This proves (A.4). Since the r.h.s. is a measurable set, this completes the proof. ∎

Definition A.15 (Bochner integrability)**.** A strongly measurable function $f : I \to X$ is Bochner–integrable if there exists a sequence $(f_n(t))$ in $C_c(I, X)$ s.t.

$$\lim_{n\to\infty} \int_I \|f_n(t) - f(t)\|_X dt = 0. \tag{A.5}$$

Notice that $\|f_n(t) - f(t)\|_X$ is measurable.

Example A.16. Consider the situation of Example A.13 of a Hilbert space X with an orthonormal basis $\{e_t\}_{t\in\mathbb{R}}$ and the map $f : \mathbb{R} \to X$, which we saw is not strongly measurable and hence is not Bochner–integrable. Notice that f is Riemann integrable in any compact interval $[a, b]$ with $\int_a^b f(t)dt = 0$.
To see this recall that the Riemann integral is, if it exists, the limit

$$\int_a^b f(t)dt = \lim_{|\triangle|\to 0} \sum_{I_j\in\triangle} f(t_j)|I_j| \text{ with } t_j \in I_j \text{ arbitrary}$$

where $\triangle$ varies among all possible decompositions of $[a, b]$ and $|\triangle| = \max_{I\in\triangle} |I|$. We have

$$\| \sum_{I_j\in\triangle} e_{t_j} |I_j| \|^2 = \sum_{j,k} \langle e_{t_j}, e_{t_k}\rangle |I_j||I_k| \le 2 \sum_j |I_j||\triangle| = 2|\triangle|(b - a) \stackrel{|\triangle|\to 0}{\to} 0.$$

Proposition A.17. *Let $f : I \to X$ be Bochner–integrable. Then there exists an $x \in X$ s.t. if $(f_n(t))$ is a sequence in $C_c(I, X)$ satisfying* (A.5) *then we have*

$$\lim_{n\to\infty} x_n = x \text{ where } x_n := \int_I f_n(t)dt. \tag{A.6}$$

Proof. First of all we check that x_n is Cauchy. This follows immediately from (A.5) and from

$$\begin{aligned}\|x_n - x_m\|_X &= \| \int_I (f_n(t) - f_m(t))dt\|_X \le \int_I \|f_n(t) - f_m(t))\|_X dt \\ &\le \int_I \|f_n(t) - f(t))\|_X dt + \int_I \|f(t) - f_m(t))\|_X dt.\end{aligned}$$

Let us set $x = \lim x_n$. Let $(g_n(t))$ be another sequence in $C_c(I, X)$ satisfying (A.5). Then $\lim \int_I g_n = x$ by

$$\begin{aligned}\| \int_I g_n(t)dt - x\|_X &= \| \int_I (g_n(t) - f_n(t))dt + \int_I f_n(t)dt - x\|_X \\ &\le \int_I \|g_n(t) - f_n(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt \\ &\le \int_I \|g_n(t) - f(t)\|_X dt + \int_I \|f_n(t) - f(t)\|_X dt + \| \int_I f_n(t)dt - x\|_X dt.\end{aligned}$$

□

Definition A.18. Let $f : I \to X$ be Bochner–integrable and let $x \in X$ be the corresponding element obtained from Proposition A.17. The we set $\int_I f(t)dt = x$.

Theorem A.19 (Bochner's Theorem)**.** *Let $f : I \to X$ be strongly measurable. Then f is Bochner–integrable if and only if $\|f\|$ is Lebesgue integrable. Furthermore, we have*

$$\| \int_I f(t)dt\| \le \int_I \|f(t)\|dt. \tag{A.7}$$

Proof. Let f be Bochner–integrable. Then there is a sequence $(f_n(t))$ in $C_c(I,X)$ satisfying (A.5). We have $\|f\| \leq \|f_n\| + \|f - f_n\|$. Since both functions in the r.h.s. are Lebesgue integrable and $\|f\|$ is measurable it follows that$\|f\|$ is Lebesgue integrable.

Conversely let $\|f\|$ be Lebesgue integrable. Then there exist a sequence $(g_n(t))$ in $C_c(I,\mathbb{R})$ and $g \in L^1(I)$ s.t. $\int_I |g_n(t) - \|f(t)\||dt \to 0$ and $|g_n(t)| \leq g(t)$. In fact it is possible to choose such a sequence so that $\|g_n - g_m\|_{L^1(I)} < 2^{-n}$ for any n and any $m \geq n$ (just by extracting an appropriate subsequence from a starting g_n [3]). Then if we set

$$S_N(t) := \sum_{n=1}^{N} |g_n(t) - g_{n+1}(t)| \tag{A.8}$$

we have $\|S_N\|_{L^1(I)} \leq 1$. Since $\{S_N(t)\}_{N\in\mathbb{N}}$ is increasing, the limit $S(t) := \lim_{n\to+\infty} S_n(t)$ remains defined, is finite a.e. and $\|S\|_{L^1(I)} \leq 1$. Then $|g_n(t)| \leq |g_1(t)| + S(t) =: g(t)$ everywhere, where $g \in L^1(I)$. Notice that $\lim\limits_{n\to\infty} g_n(t)$ is convergent almost everywhere (it convergent in all points where $\lim_{n\to+\infty} S_n(t)$ is convergent). By dominated convergence it follows that this limit holds also in $L^1(I)$ and hence it is equal to $\|f\|$.
Let $(f_n(t))$ in $C_c(I,X)$ s.t. $f_n(t) \to f(t)$ a.e. (this sequence exists by the strong measurability of $f(t)$). Set

$$u_n(t) := \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t).$$

Notice that $(u_n(t))$ is in $C_c(I,X)$. We have

$$\|u_n(t)\| \leq \frac{|g_n(t)|\ \|f_n(t)\|}{\|f_n(t)\| + \frac{1}{n}} \leq |g_n(t)| \leq g(t).$$

We have (where the 2nd equality holds because because $\lim\limits_{n\to\infty} g_n(t) = \|f(t)\|$ and $\lim\limits_{n\to\infty} \|f_n(t)\| = \|f(t)\|$ a.e.)

$$\lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \frac{|g_n(t)|}{\|f_n(t)\| + \frac{1}{n}} f_n(t) = \lim_{n\to\infty} f_n(t) = f(t) \text{ a.e..}$$

Then we have

$$\lim_{n\to\infty} \|u_n(t) - f(t)\| = 0 \text{ a.e. with } \|u_n(t) - f(t)\| \leq g(t) + \|f(t)\| \in L^1(I).$$

By dominated convergence we conclude

$$\lim_{n\to\infty} \int_I \|u_n(t) - f(t)\|dt = 0.$$

[3] Suppose we start with a given $\{g_n\}$. Then for any 2^{-n} there exists N_n s.t. $n_1, n_2 > N_n$ implies $\|g_{n_1} - g_{n_2}\|_{L^1(I)} < 2^{-n}$. Let now $\{\varphi(n)\}$ be a strictly increasing sequence in $\mathbb{N}$ s.t. $\varphi(n) > N_n$ for any n. Then $\|g_{\varphi(n)} - g_{\varphi(m)}\|_{L^1(I)} < 2^{-n}$ for any pair $m > n$. Rename $g_{\varphi(n)}$ as g_n.

This implies that f is Bochner–integrable. Finally, we have

$$\| \int_I f(t)dt\| = \lim_{n\to\infty} \| \int_I u_n(t)dt\| \leq \lim_{n\to\infty} \int_I \|u_n(t)\|dt = \int_I \|f(t)\|dt.$$

□

Corollary A.20 (Dominated Convergence)**.** *Consider a sequence $(f_n(t))$ of Bochner–integrable functions $I \to X$, $g : I \to \mathbb{R}$ Lebesgue integrable and let $f : I \to X$. Suppose that*

$$\|f_n(t)\| \leq g(t) \text{ for all } n$$
$$\lim_{n\to\infty} f_n(t) = f(t) \text{ for almost all } t.$$

Then f is Bochner–integrable with $\int_I f(t) = \lim_n \int_I f_n(t)$.

Proof. By Dominated Convergence in $L^1(I, \mathbb{R})$ we have $\int_I \|f(t)\| = \lim_n \int_I \|f_n(t)\|$. By Proposition A.4, as a pointwise limit a.e. of a sequence of strongly measurable functions, f is strongly measurable. By Bochner's Theorem f is Bochner–integrable. By the triangular inequality

$$\limsup_n \| \int_I (f(t) - f_n(t))\| \leq \lim_n \int_I \|f(t) - f_n(t)\| = 0$$

where the last inequality follows from $\|f(t) - f_n(t)\| \leq \|f(t)\| + g(t)$ and the standard Dominated Convergence. □

Definition A.21. Let $p \in [1, \infty]$. We denote by $L^p(I, X)$ the set of equivalence classes of strongly measurable functions $f : I \to X$ s.t. $\|f(t)\| \in L^p(I, \mathbb{R})$. We set $\|f\|_{L^p(I,X)} := \|\|f\|\|_{L^p(I,\mathbb{R})}$.

Proposition A.22. *$(L^p(I, X), \| \|_{L^p})$ is a Banach space.*

Proof. The proof is similar to the case $X = \mathbb{R}$, see [2].
(**Case** $p = \infty$). Let (f_n) be Cauchy sequence in $L^\infty(I, X)$. For any $k \geq 1$ there is a N_k s.t.

$$\|f_n - f_m\|_{L^\infty(I,X)} \leq \frac{1}{k} \text{ for all } n, m \geq N_k.$$

So there exists an $E_k \subset I$ with $|E_k| = 0$ s.t.

$$\|f_n(t) - f_m(t)\|_X \leq \frac{1}{k} \text{ for all } n, m \geq N_k \text{ and for all for } t \in I\backslash E_k.$$

Set $E := \cup_k E_k$. Then for any $t \in I\backslash E$ the sequence $(f_n(t))$ is convergent. So a function $f(t)$ remains defined with

$$\|f_n(t) - f(t)\|_X \leq \frac{1}{k} \text{ for all } n \geq N_k \text{ and for all for } t \in I\backslash E. \tag{A.9}$$

By Proposition A.4 the function f is strongly measurable. By (A.9) we have $f \in L^\infty(I, X)$ and

$$\|f_n - f\|_{L^\infty(I,X)} \leq \frac{1}{k} \text{ for all } n \geq N_k$$

and so $f_n \to f$ in $L^\infty(I, X)$.

(**Case** $p < \infty$). Let (f_n) be Cauchy sequence in $L^p(I, X)$ and let (f_{n_k}) be a subsequence with

$$\|f_{n_k} - f_{n_{k+1}}\|_{L^p(I,X)} \leq 2^{-k}.$$

Set now

$$g_l(t) = \sum_{k=1}^{l} \|f_{n_k}(t) - f_{n_{k+1}}(t)\|_X$$

Then

$$\|g_l\|_{L^p(I,\mathbb{R})} \leq 1.$$

By monotone convergence we have that $(g_l(t))_l$ converges a.e. to a $g \in L^p(I, \mathbb{R})$. Furthermore, for $2 \leq k < l$

$$\|f_{n_k}(t) - f_{n_l}(t)\|_X = \sum_{j=k}^{l-1} \|f_{n_j}(t) - f_{n_{j+1}}(t)\|_X \leq g(t) - g_{k-1}(t).$$

Then a.e. the sequence $(f_{n_k}(t))$ is Cauchy in X for a.e. t and so it converges for a.e. t to some $f(t)$. By Proposition A.4 the function f is strongly measurable. Furthermore,

$$\|f(t) - f_{n_k}(t)\|_X \leq g(t).$$

It follows that $f - f_{n_k} \in L^p(I, X)$, and so also $f \in L^p(I, X)$. Finally we claim $\|f - f_{n_k}\|_{L^p(I,X)} \to 0$. First of all we have $\|f(t) - f_{n_k}(t)\|_X \to 0$ for a.e. t and

$$\|f(t) - f_{n_k}(t)\|_X^p \leq g^p(x)$$

by dominated convergence we obtain that $\|f - f_{n_k}\|_X \to 0$ in $L^p(I, \mathbb{R})$. Hence $f_{n_k} \to f$ in $L^p(I, X)$. □

Proposition A.23. *$C_c^\infty(I, X)$ is a dense subspace of $L^p(I, X)$ for $p < \infty$.*

Proof. We split the proof in two parts. We first show that $C_c^0(I, X)$ is a dense subspace of $L^p(I, X)$ for $p < \infty$. For $p = 1$ this follows from the definition of integrable functions in Definition A.15. For $1 < p < \infty$ going through the proof of Bochner's Theorem A.19, the functions u_n considered in that proof can be taken to belong to $C_c^0(I, X)$ and converge to f in $L^p(I, X)$.

The second part of the proof consists in showing that $C_c^\infty(I, X)$ is a dense subspace of $C_c^0(I, X)$ inside $L^p(I, X)$ for $p < \infty$. Let $f \in C_c^0(I, X)$. We consider $\rho \in C_c^\infty(\mathbb{R}, [0, 1])$ s.t. $\int \rho(x)dx = 1$. Set $\rho_\epsilon(x) := \epsilon^{-1}\rho(x/\epsilon)$. Then for $\epsilon > 0$ small enough $\rho_\epsilon * f \in C_c^\infty(I, X)$. We

extend both f and $\rho_\epsilon * f$ on $\mathbb{R}$ setting them 0 in $\mathbb{R}\backslash I$. In this way $\rho_\epsilon * f \in C_c^\infty(\mathbb{R}, X)$ and $f \in C_c^0(\mathbb{R}, X)$ and it is enough to show that $\rho_\epsilon * f \overset{\epsilon\to 0^+}{\to} f$ in $L^p(\mathbb{R}, X)$..
We have

$$\rho_\epsilon * f(t) - f(t) = \int_{\mathbb{R}} (f(t - \epsilon s) - f(s))\rho(s)dy$$

so that, by Minkowski inequality and for $\Delta(s) := \|f(\cdot - s) - f(\cdot)\|_{L^p}$, we have

$$\|\rho_\epsilon * f(t) - f(t)\|_{L^p} \leq \int |\rho(s)|\Delta(\epsilon\ s)ds.$$

Now we have $\lim_{s\to 0} \Delta(s) = 0$ and $\Delta(s) \leq 2\|f\|_{L^p}$. So, by dominated convergence we get

$$\lim_{\epsilon\searrow 0} \|\rho_\epsilon * f - f\|_{L^p} = \lim_{\epsilon\searrow 0} \int |\rho(s)|\Delta(\epsilon\ s)ds = 0.$$

So

$$\lim_{\epsilon\searrow 0} \rho_\epsilon * f = f \text{ in } L^p(\mathbb{R}, X). \tag{A.10}$$

□

Definition A.24. We denote by $\mathcal{D}'(I, X)$ the space $\mathcal{L}(\mathcal{D}(I, \mathbb{R}), X)$.

Proposition A.25. *Let $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}, X)$. Set*

$$T_h f(t) = h^{-1} \int_t^{t+h} f(s)ds \text{ for } t \in \mathbb{R} \text{ and } h \neq 0.$$

Then $T_h f \in L^p(\mathbb{R}, X) \cap L^\infty(\mathbb{R}, X) \cap C^0(\mathbb{R}, X)$ and $T_h f \overset{h\to 0}{\to} f$ in $L^p(\mathbb{R}, X)$ and for almost every t.

□

Corollary A.26. *Let $f \in L^1_{loc}(I, X)$ be such that $f = 0$ in $\mathcal{D}'(I, X)$. Then $f = 0$ a.e.*

Proof. First of all we have $\int_J f dt = 0$ for any $J \subset I$ compact. Indeed, let $(\varphi_n) \in \mathcal{D}(I)$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n \to \chi_J$ a.e. Then

$$\int_J f dt = \lim_{n\to+\infty} \int_J \varphi_n f dt = 0$$

where we applied Dominated Convergence for the last equality.
Set now $\overline{f}(t) = f(t)$ in J and $\overline{f}(t) = 0$ outside J. Then $T_h\overline{f} = 0$ for all $h > 0$. Then $\overline{f}(t) = 0$ for a.e. t. So $f(t) = 0$ for a.e. $t \in J$. This implies $f(t) = 0$ for a.e. $t \in \mathbb{R}$. □

Corollary A.27. *Let $g \in L^1_{loc}(I, X)$, $t_0 \in I$, and $f \in C(I, X)$ given by $f(t) = \int_{t_0}^t g(s)ds$. Then:*

(1) $f' = g$ in $\mathcal{D}'(I, X)$;

(2) f is differentiable a.e. with $f' = g$ a.e.

Proof. It is not restrictive to consider the case $I = \mathbb{R}$ and $g \in L^1(\mathbb{R}, X)$. We have

$$T_h g(t) = h^{-1} \int_t^{t+h} g(s)ds = \frac{f(t+h) - f(t)}{h}.$$

By Proposition A.25 $T_h g \overset{h\to 0}{\to} g$ for almost every t. This yields (2).

For $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\langle f', \varphi \rangle = -\int_{\mathbb{R}} f(t)\varphi'(t)dt.$$

Furthermore

$$\lim_{h\to 0} \frac{\varphi(t+h) - \varphi(t)}{h} = \varphi'(t) \text{ in } L^\infty(\mathbb{R}).$$

So

$$\langle f', \varphi \rangle = -\lim_{h\to 0} \int_{\mathbb{R}} f(t) \frac{\varphi(t+h) - \varphi(t)}{h} dt = -\lim_{h\to 0} \int_{\mathbb{R}} \varphi(t) \frac{f(t-h) - f(t)}{h} dt$$
$$= -\lim_{h\to 0} \int_{\mathbb{R}} \varphi(t) T_{-h} g(t) dt = \langle g, \varphi \rangle.$$

□

Definition A.28. Let $p \in [1, \infty]$. We denote by $W^{1,p}(I, X)$ the space formed by the $f \in L^p(I, X)$ s.t. $f' \in \mathcal{D}(I, X)$ is also $f' \in L^p(I, X)$ and we set $\|f\|_{W^{1,p}} = \|f\|_{L^p} + \|f'\|_{L^p}$.

References

[1] Bahouri, Chemin, Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 343. Springer, Heidelberg, 2011.

[2] H.Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, 2011.

[3] T.Cazenave, Hareaux *Semilinear Equations*, Oxford Univ.Press.

[4] T. Cazenave, *Semilinear Schrödinger equations,* Courant Lecture Notes in Mathematics **10**, New York University, Courant Institute of Mathematical Sciences, American Mathematical Society, Providence, RI, 2003.

[5] Chemin, Desjardins, Gallagher, Grenier, *Mathematical geophysics. An introduction to rotating fluids and the Navier-Stokes equations.* Oxford Lecture Series in Mathematics and its Applications, 32, The Clarendon Press, Oxford University Press, Oxford, 2006

[6] C. Fefferman, *The multiplier problem for the ball*, Ann. of Math. 94 (1971), 330–336.

[7] H. Fujita, *On the blow up for the Cauchy problem for $u_t = \triangle u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect. IA Math., **13** (1966), 109–124.

[8] M. Keel and T, Tao, *Endpoint Strichartz estimates*, Amer. J. Math. **120** (1998), 955–980.

[9] L.Hörmander *Estimates for translation invariant operators in L^p spaces*, Acta Math. **104** (1960), 93–140.

[10] F.Linares, G. Ponce, *Introduction to nonlinear dispersive equations* Universitext, Springer, New York, 2009.

[11] W.Rudin, *Real and Complex Analysis*, McGraw–Hill (1970).

[12] C.D.Sogge, *Lectures on Nonlinear Wave Equations*, International Press (1995)

[13] E.M.Stein, *Singular Integrals and differentiability properties of functions*, Princeton Un. Press (1970).

[14] E.M.Stein, *Harmonic Analysis*, Princeton Un. Press (1993).

[15] C.Sulem, P.L.Sulem, *The nonlinear Schrödinger equation. Self-focusing and wave collapse*, Applied Mathematical Sciences, 139. Springer-Verlag (1999).

[16] F.B.Weissler, *Existence and non–existence of global solutions for semilinear heat equations,* Israel. Jour. Math., **38** (1981), 29–40.