Inference for a mean from a Gaussian dist.

$$y_1, \dots, y_n \mid \theta \sim \text{IID } \mathcal{N}(\theta, \sigma^2)$$

known

$$p(y \mid \theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{1}{2\sigma^2} (y_i - \theta)^2 \right\}$$

$$= (2\pi)^{-n/2} (\sigma^2)^{-n/2} \exp\left\{ -\frac{1}{2\sigma^2} \sum (y_i - \theta)^2 \right\}$$

$$\propto \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i - \theta)^2 \right\}$$

$$\propto \exp\left\{ -\frac{1}{2\sigma^2} \left(\sum y_i^2 - 2\theta \underbrace{\sum y_i}_{n\bar{y}} + \theta^2 n \right) \right\}$$

$$p(y|\theta) \propto \exp\left\{-\frac{1}{2\sigma^2}\left(-2\theta \underbrace{n\bar{y}}_{t(y)} + n\theta^2\right)\right\}$$

In exponential form $t(y)$ suff stat

$$\propto \underbrace{\exp\left\{-\frac{\overset{\nu}{n}\theta^2}{2\sigma^2}\right\}}\exp\left\{+\frac{\theta}{\sigma^2} n\bar{y}\right\}$$

$$g(\sigma)^n \exp\left\{\phi(\theta)\, t(y)\right\}$$

$$\pi(\theta) \propto \exp\left\{-\frac{\eta\theta^2}{2\sigma^2}\right\}\exp\left\{\frac{\theta}{\sigma^2}\nu\right\}$$

Prior for θ

$$\pi(\theta) \sim N(\mu_0, \sigma_0^2)$$

$$p(y|\theta) \propto \exp\left\{-\frac{n}{2\sigma^2}(\bar{y}-\theta)^2\right\}$$

distr of sufficient statistic.

$$\bar{y}|\theta \sim N\left(\theta, \frac{\sigma^2}{n}\right)$$

(usual result for the sample mean)

Posterior

$$\pi(\theta|y) \propto \pi(\theta)\, p(y|\theta)$$

$$\propto \exp\left\{-\frac{1}{2\sigma_0^2}(\theta-\mu_0)^2\right\} \exp\left\{-\frac{n}{2\sigma^2}(\bar{y}-\theta)^2\right\}$$

$$\propto \exp\left\{-\frac{1}{2\sigma_0^2}(\theta^2-2\mu_0\theta)-\frac{n}{2\sigma^2}(\theta^2-2\theta\bar{y})\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\theta^2\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)-2\theta\left(\frac{\mu_0}{\sigma_0^2}+\frac{\bar{y}n}{\sigma^2}\right)\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)\left(\theta^2-2\theta\frac{\frac{\mu_0}{\sigma_0^2}+\frac{\bar{y}n}{\sigma^2}}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}\right)\right\}$$

$$\pi(\theta|y) \propto \exp\left\{-\frac{1}{2}\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)\left(\theta-\frac{\frac{\mu_0}{\sigma_0^2}+\bar{y}\frac{n}{\sigma^2}}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}\right)^2\right\}$$

$$\theta|y \sim N\left(\underbrace{\frac{\frac{\mu_0}{\sigma_0^2}+\bar{y}\frac{n}{\sigma^2}}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}}_{\mu_n}, \underbrace{\frac{1}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}}_{\sigma_n^2}\right)$$

$$\mu_n \xrightarrow{n\to\infty} \bar{y} \qquad \sigma_n^2 \xrightarrow{n\to\infty} 0$$

$$\mu_n \xrightarrow{\sigma_0^2\to 0} \mu_0 \qquad \sigma_n^2 \xrightarrow{\sigma_0^2\to 0} 0$$

$$\to \mu_n \xrightarrow{\sigma_0\to\infty} \bar{y}. \qquad \to \sigma_n^2 \xrightarrow{\sigma_0^2\to\infty} \frac{\sigma^2}{n}$$

Predictive distribution $\tilde{y} \mid \theta \sim N(\theta, \sigma^2)$

exchang. $\rightarrow$ indep. of $y_1, \dots, y_n \mid \theta$

$$p(\tilde{y} \mid y) \propto \int p(\tilde{y} \mid \theta, \cancel{y})\, \pi(\theta \mid y)\, d\theta$$

$$\propto \int \exp\left\{-\frac{1}{2\sigma^2}(\tilde{y}-\theta)^2\right\} \exp\left\{-\frac{1}{2\sigma_n^2}(\theta-\mu_n)^2\right\} d\theta$$

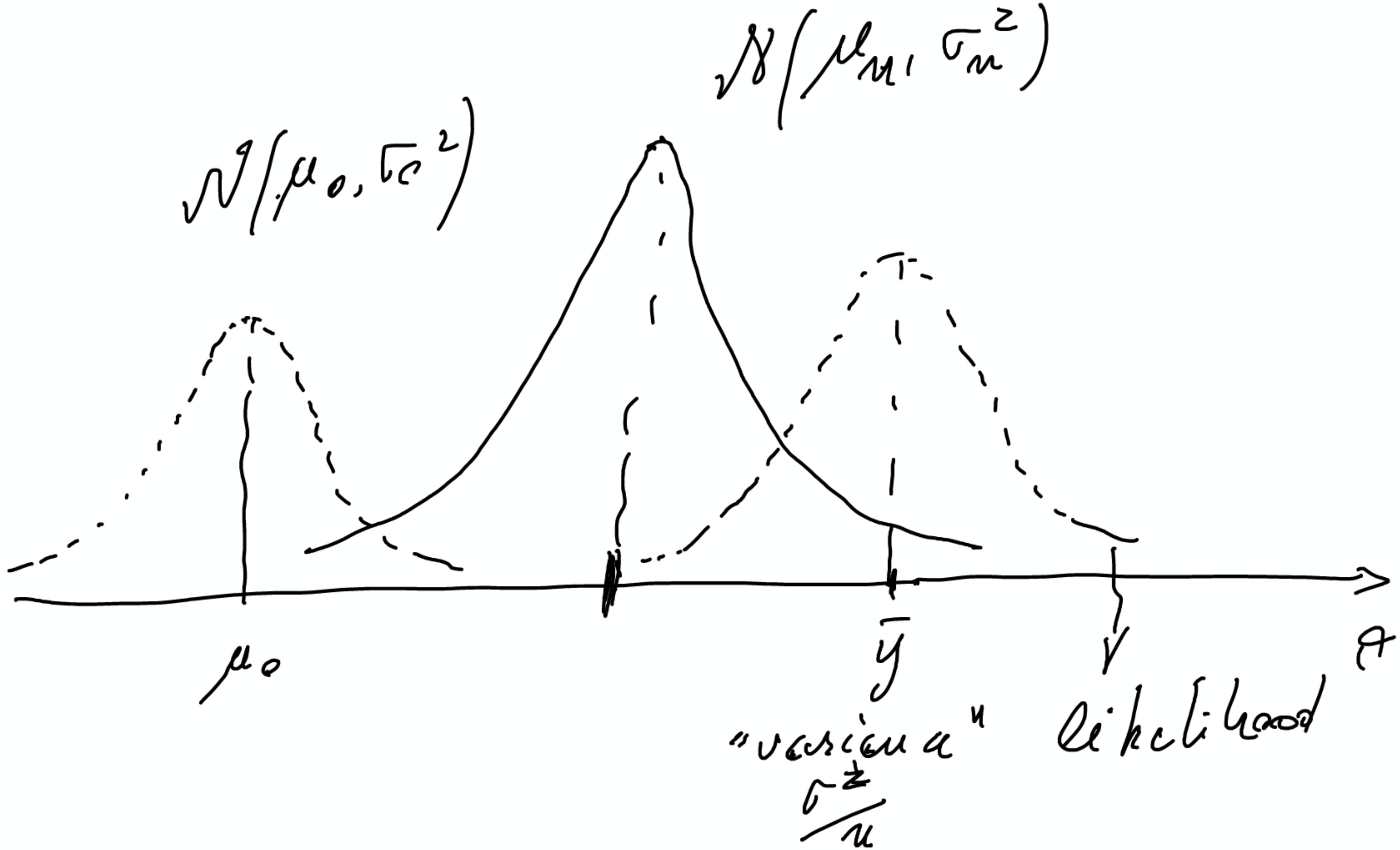
$$\underbrace{\int \exp\left\{ \dots \text{ quadratic in } \tilde{y}\ \theta \dots \right\} d\theta}$$

joint dist. of $(\tilde{y}, \theta)$ is Gaussian $\Rightarrow$ marginal for $\tilde{y}$ is Gaussian

$$E(\tilde{y} \mid y) = E\Big(\underbrace{E(\tilde{y} \mid \cancel{y}, \theta)}_{\theta} \Big| \, y \Big) = \mu_\mu$$

$$V(\tilde{y} \mid y) = E\Big(\underbrace{V(\tilde{y} \mid \cancel{y}, \theta)}_{= \sigma^2} \Big| \, y \Big) + V\Big(\underbrace{E(\tilde{y} \mid \theta)}_{\theta} \Big| \, y \Big)$$

$$= \sigma^2 + \sigma_\mu^2$$



$\mathcal{N}(\mu_0, \sigma_0^2)$
$\mathcal{N}(\mu_n, \sigma_n^2)$
μ_0
$\bar{y}$
θ
"variance"
σ^2/n
likelihood

Interval estimate for the mean θ

posterior intervals (quantiles)

→ high posterior density regions

$z_\alpha = \Phi^{-1}(\alpha)$

α quantile of standard normal dist

$$\theta \mid y \sim N(\mu_n, \sigma_n^2)$$

Posterior interval prob. $1-\alpha$

$$\mu_n \pm z_{1-\frac{\alpha}{2}} \, \sigma_n$$

if $\sigma_0 \to \infty$ then approxim.

$$\bar{y} \pm z_{1-\alpha/2} \frac{\sigma}{\sqrt{n}}$$