INTRODUCTION TO ALGEBRAIC GEOMETRY

Notes of the course

Advanced Geometry 3

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Introduction

Algebraic Geometry is the field of mathematics that studies the sets of solutions of systems of algebraic equations, i.e. of equations given by polynomials. The origins of Algebraic Geometry go back to the Ancient Babylonians and Greeks and, since them, this fascinating subject has attracted mathematicians of every times and countries. During the 19^{th} and the beginning of last century, important progress has been made, mainly by the so-called Italian School of Algebraic Geometry. Then, starting from 1950, the subject was completely refounded, taking into account the advent of Modern Algebra. This work was initiated by Oscar Zariski (1899-1986), a mathematician of Russian origin, who studied in Italy and then moved to the USA, and pushed on mainly by the French mathematician Alexander Grothendieck (1928-2014). In the last fifty years, important results and answers to classical problems have been given.

An asterisk * near an exercise denotes that it is quoted in the text.

1. Affine and projective space.

Let K be a field. By definition, the *affine space* of dimension n over K is simply the set K^n : on it, the additive group of K^n acts naturally by translation. The affine space will be denoted $\mathbb{A}^n_K$ or simply $\mathbb{A}^n$. So the points of $\mathbb{A}^n_K$ are $n-$tuples $(a_1, \ldots, a_n)$, where $a_i \in K$ for $i = 1, \ldots, n$.

The natural *action* of K^n on $\mathbb{A}^n_K$, is the map t defined by

$$t : K^n \times \mathbb{A}^n_K \longrightarrow \mathbb{A}^n_K$$

$$((x_1, \ldots, x_n), (a_1, \ldots, a_n)) \longrightarrow (x_1 + a_1, \ldots, x_n + a_n).$$

Note that: $t(0, P) = P$, where 0 is the zero vector of K^n and $P \in \mathbb{A}^n_K$, and $t(w, t(v, P)) = t(v + w, P)$, for $v, w \in K^n$ and $P \in \mathbb{A}^n_K$.

The action of a vector v on a point P is "by translation". The point $t(v, P)$ will be denoted $P + v$. The action t is *faithful* and *transitive*: this means that, for any choice of $P, Q \in \mathbb{A}^n_K$, there exists one and only one $v \in V$ such that $Q = t(v, P)$: for this vector, the notation $Q - P$ will be sometimes used.

Let $Q \in \mathbb{A}^n_K$ be a point, and $W \subset K^n$ be a vector subspace. We define the *affine subspace* of $\mathbb{A}^n_K$ passing through Q with orienting space W (or of direction W) as follows:

$$S = \{P \in \mathbb{A}^n_K \mid P = Q + w, w \in W\}.$$

S can be seen as "W translated in Q". Note that affine subspaces of $\mathbb{A}^n_K$ do not necessarily pass through the origin. Two affine subspaces of $\mathbb{A}^n$ with a common orienting space are called parallel. If $\dim W = m$, we also define $\dim S = m$. The subspaces of dimension 1 are called *lines*, those of dimension 2 *planes*, those of dimension $n - 1$ (or of codimension 1) *hyperplanes*.

The points of an affine subspace of $\mathbb{A}^n$ can be characterised as solutions of a system of equations. These are of two types:

a) Parametric equations of a subspace.

Let S be the subspace passing through $Q(y_1, \ldots, y_n)$ with orienting space W, and let $w_1, \ldots, w_s$ be a basis of W, with $w_i = (w_{i1}, \ldots, w_{in})$. Then $P(x_1, \ldots, x_n) \in S$ if and only if there exist $t_1, \ldots, t_s \in K$ such that

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) + t_1 w_1 + \ldots + t_s w_s,$$

or equivalently

$$\begin{cases} x_1 = y_1 + t_1 w_{11} + \ldots + t_s w_{s1} \\ x_2 = y_2 + t_1 w_{12} + \ldots + t_s w_{s2} \\ \ldots \\ \ldots \end{cases}$$

As $(t_1, \ldots, t_s)$ varies in K^s we get in this way all points of S.

For example, if S is the line through Q of direction $W = \langle w \rangle$, with $w = (b_1, \ldots, b_n)$, then

$$\begin{cases} x_1 = y_1 + tb_1 \\ x_2 = y_2 + tb_2 \\ \ldots \\ x_n = y_n + tb_n \end{cases}$$

are parametric equations of S.

b) Cartesian equations of a subspace.

Let $s = \dim W$, $W \subset K^n$, a vector subspace. Then W is the set of vectors whose coordinates are solutions of a homogeneous linear system of rank $n - s$ in n indeterminates $z_1, \ldots, z_n$:

$$\begin{cases} a_{11}z_1 + \ldots + a_{1n}z_n = 0 \\ \ldots \\ a_{n-s,1}z_1 + \ldots + a_{n-s,n}z_n = 0. \end{cases}$$

Hence $P(x_1, \ldots, x_n)$ belongs to S if and only if $P = Q + w$, where w is a solution of the previous system, i.e. if and only if the following equations are satisfied:

$$\begin{cases} a_{11}(x_1 - y_1) + \ldots + a_{1n}(x_n - y_n) = 0 \\ \ldots \\ a_{n-s,1}(x_1 - y_1) + \ldots + a_{n-s,n}(x_n - y_n) = 0 \end{cases}$$

i.e. $(x_1, \ldots, x_n)$ is a solution of the system:

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n + b_1 = 0 \\ \ldots \qquad\qquad\qquad\qquad\qquad = 0 \\ a_{n-s,1}x_1 + \ldots + a_{n-s,n}x_n + b_n \end{cases}$$

where we have put $b_i = -(a_{i1}y_1 + \ldots + a_{in}y_n)$, for $i = 1, \ldots, n - s$. For example a hyperplane is represented by a unique linear equation of the form:

$$a_1x_1 + \ldots + a_nx_n + b = 0.$$

Let V be a $K-$vector space, of dimension $n + 1$. Let $V^* = V \setminus \{0\}$ be the subset of non–zero vectors. The following relation in V^* is an equivalence relation (relation of proportionality):

$v \sim v'$ if and only if $\exists \lambda \neq 0, \lambda \in K$ such that $v' = \lambda v$.

The quotient set $V^*/\sim$ is called the *projective space* associated to V and denoted $\mathbb{P}(V)$. The points of $\mathbb{P}(V)$ are the lines of V (through the origin) deprived of the origin. In particular, $\mathbb{P}(K^{n+1})$ is denoted $\mathbb{P}^n_K$ (or simply $\mathbb{P}^n$) and called the *numerical projective n-space.* By definition, the dimension of $\mathbb{P}(V)$ is equal to $\dim V - 1$.

There is a canonical surjection $p : V^* \to \mathbb{P}(V)$ which takes a vector v to its equivalence class $[v]$. If $(x_0, \dots, x_n) \in (K^{n+1})^*$, then the corresponding point of $\mathbb{P}^n$ is denoted $[x_0, \dots, x_n]$. So $[x_0, \dots, x_n]$=$[x'_0, \dots, x'_n]$ if and only if $\exists \lambda \in K^*$ such that $x'_0 = \lambda x_0, \dots, x'_n = \lambda x_n$.

If a basis $e_0, \dots, e_n$ of V is fixed, then a system of *homogeneous coordinates* is introduced in V, in the following way: if $v = x_0 e_0 + \dots + x_n e_n$, then $x_0, \dots, x_n$ are called homogeneous coordinates of the corresponding point P =$[v]$= $p(v)$ in $\mathbb{P}(V)$. We also write $P[x_0, \dots, x_n]$. Note that homogeneous coordinates of a point P are not uniquely determined by P, but are defined only up to multiplication by a non–zero constant. If $\dim V = n + 1$, a system of homogeneous coordinates allows to define a *bijection*

$$\mathbb{P}(V) \longrightarrow \mathbb{P}^n$$

$$P = [v] \longrightarrow [x_0, \dots, x_n]$$

where $v = x_0 e_0 + \dots + x_n e_n$.

The points $E_0[1, 0, \dots, 0], \dots, E_n[0, 0, \dots, 1]$ are called the fundamental points and $U[1, \dots, 1]$ the unit point for the given system of coordinates.

A *projective* (or *linear*) *subspace* of $\mathbb{P}(V)$ is a subset of the form $\mathbb{P}(W)$, where $W \subset V$ is a subspace.

Assume that $\dim W = s + 1$ and that W is represented by a linear homogeneous system

$$(*) \begin{cases} a_{10}x_0 + \dots + a_{1n}x_n = 0 \\ \dots \\ a_{n-s,0}x_0 + \dots + a_{n-s,n}x_n = 0. \end{cases}$$

Note that a $(n + 1)$-tuple $(\bar{x}_0, \dots, \bar{x}_n)$ is a solution of the system if and only if $(\lambda\bar{x}_0, \dots, \lambda\bar{x}_n)$ is, with $\lambda \neq 0$. So these solutions can also be interpreted as representing the points of $\mathbb{P}(W)$ and the equations (*) as a system of Cartesian equations of $\mathbb{P}(W)$. To write down parametric equations of $\mathbb{P}(W)$ it is enough to fix a basis of W, formed by vectors $w_0, \dots, w_s$. Then a general point of $\mathbb{P}(W)$ is parametrically represented by $[\lambda_0 w_0 + \dots + \lambda_s w_s]$, as $\lambda_0, \dots, \lambda_s$ vary in $\mathbb{P}^s$.

If W, U are vector subspaces of V, the following *Grassmann relation* holds:

$$\dim U + \dim W = \dim(U \cap W) + \dim(U + W).$$

From this relation, observing that $\mathbb{P}(U \cap W) = \mathbb{P}(U) \cap \mathbb{P}(W)$, we get in $\mathbb{P}(V)$:

$$\dim \mathbb{P}(U) + \dim \mathbb{P}(W) = \dim(\mathbb{P}(U) \cap \mathbb{P}(W)) + \dim \mathbb{P}(U + W).$$

Note that $\mathbb{P}(U + W)$ is the minimal linear subspace of $\mathbb{P}(V)$ containing both $\mathbb{P}(U)$ and $\mathbb{P}(W)$: it is denoted $\mathbb{P}(U) + \mathbb{P}(W)$.

1.1. Example. Let $V = K^3$, $\mathbb{P}(V) = \mathbb{P}^2$, $U, W \subset K^3$ subspaces of dimension 2. Then $\mathbb{P}(U), \mathbb{P}(V)$ are lines in the projective plane. There are two cases:

(i) $U = W = U + W = U \cap W$;
(ii) $U \neq W$, $\dim U \cap W = 1$, $U + W = K^3$.

In case (i) the two lines in $\mathbb{P}^3$ coincide; in case (ii) $\mathbb{P}(U) \cap \mathbb{P}(W) = \mathbb{P}(U \cap W) = [v]$, if $v \neq 0$ is a vector generating $U \cap W$. Observe that *never* $\mathbb{P}(U) \cap \mathbb{P}(W) = \emptyset$.

Let $T \subset \mathbb{P}(V)$ be a non–empty set. The linear span $\langle T \rangle$ of T is the intersection of the projective subspaces of $\mathbb{P}(V)$ containing T, i.e. the minimum subspace containing T. For example, if $T = \{P_1, \dots, P_t\}$, a finite set, then $\langle P_1, \dots, P_t \rangle = \mathbb{P}(W)$, where W is the vector subspace of V generated by vectors $v_1, \dots, v_t$ such that $P_1 = [v_1], \dots, P_t = [v_t]$. So $\dim\langle P_1, \dots, P_t \rangle \leq t - 1$ and equality holds if and only if $v_1, \dots, v_t$ are linearly independent; in this case, also the points $P_1, \dots, P_t$ are called *linearly independent*. In particular, for $t = 2$, two points are linearly independent if they generate a line, for $t = 3$, three points are linearly independent if they generate a plane, etc. It is clear that, if $P_1, \dots, P_t$ are linearly independent, then $t \leq n + 1$, and any subset of $\{P_1, \dots, P_t\}$ is formed by linearly independent points.

$P_1, \dots, P_t$ are said to be *in general position* if either $t \leq n + 1$ and they are linearly independent or $t > n + 1$ and any $n + 1$ points among them are linearly independent.

1.2. Proposition. *The fundamental points $E_0, \dots, E_n$ and the unit point U of a system of homogeneous coordinates on $\mathbb{P}^n$ are $n + 2$ points in general position. Conversely, if $P_0, \dots, P_n, P_{n+1}$ are $n + 2$ points in general position, then there exists a system of homogeneous coordinates in which $P_0, \dots, P_n$ are the fundamental points and P_{n+1} is the unit point.*

Proof. If $e_0, \dots, e_n$ is a basis, then clearly the $n+1$ vectors $e_0, \dots, \hat{e_i}, \dots, e_n, e_0 + \dots + e_n$ are linearly independent: this proves the first claim. To prove the second claim, we fix vectors $v_0, \dots, v_{n+1}$ such that $P_i = [v_i]$ for all i. So $v_0, \dots, v_n$ is a basis and there exist $\lambda_0, \dots, \lambda_n$ in K such that $v_{n+1} = \lambda_0 v_0 + \dots + \lambda_n v_n$. The assumption of general position easily implies that $\lambda_0, \dots, \lambda_n$ are all different from 0, hence $\lambda_0 v_0, \dots, \lambda_n v_n$ is a new basis such $[\lambda_i v_i] = P_i$ and P_{n+1} is the corresponding unit point. $\square$

Let $H_0 = \langle E_1, \dots, E_n \rangle, H_1 = \langle E_0, E_2, \dots, E_n \rangle, \dots, H_n = \langle E_0, \dots, E_{n-1} \rangle$ be $n+1$ hyperplanes in $\mathbb{P}^n$. Note that the equation of H_i is simply $x_i = 0$. These hyperplanes are called the *fundamental hyperplanes*. Let $U_i = \mathbb{P}^n \setminus H_i = \{P[x_0, \dots, x_n] \mid x_i \neq 0\}$. Note that $\mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n$, because no point in $\mathbb{P}^n$ has all coordinates equal to zero. There is a map $\phi_0 : U_0 \longrightarrow \mathbb{A}^n (= K^n)$ defined by $\phi_0([x_0, \dots, x_n]) = (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$. ϕ_0 is bijective and the inverse map is $j_0 : \mathbb{A}^n \longrightarrow U_0$ such that $j_0(y_1, \dots, y_n) = [1, y_1, \dots, y_n]$.

So ϕ_0 and j_0 establish a bijection between the affine space $\mathbb{A}^n$ and the subset U_0 of the projective space $\mathbb{P}^n$. There are other similar maps ϕ_i and j_i for $i = 1, \dots, n$. So $\mathbb{P}^n$ is covered by $n + 1$ subsets, each one in natural bijection with $\mathbb{A}^n$.

There is a natural way of thinking of $\mathbb{P}^n$ as a completion of $\mathbb{A}^n$; this is done by identifying $\mathbb{A}^n$ with U_i via ϕ_i, and by interpreting the points of $H_i(=\mathbb{P}^n \setminus U_i)$ as *points at infinity* of $\mathbb{A}^n$, or directions in $\mathbb{A}^n$. We do this explicity for $i = 0$. First of all we identify $\mathbb{A}^n$ with U_0 via ϕ_0 and j_0. So if $P[a_0, \dots, a_n] \in \mathbb{P}^n$, either $a_0 \neq 0$ and $P \in \mathbb{A}^n$, or $a_0 = 0$ and $P[0, a_1, \dots, a_n] \notin \mathbb{A}^n$. Then we consider in $\mathbb{A}^n$ the line L, passing through $O(0, \dots, 0)$ and of direction given by the vector $(a_1, \dots, a_n)$. Parametric equations for L are the following:

$$\begin{cases} x_1 = a_1 t \\ x_2 = a_2 t \\ \dots \\ x_n = a_n t \end{cases}$$

with $t \in K$. The points of L are identified with points of U_0 (via j_0) with homogeneous coordinates $x_0, \dots, x_n$ given by:

$$\begin{cases} x_0 = 1 \\ x_1 = a_1 t \\ x_2 = a_2 t \\ \dots \end{cases}$$

or equivalently, if $t \neq 0$, by:

$$\begin{cases} x_0 = \frac{1}{t} \\ x_1 = a_1 \\ x_2 = a_2 \\ \dots \end{cases}.$$

Now, roughly speaking, if t *tends to infinity*, this point goes to $P[0, a_1, \dots, a_n]$. Clearly this is not a rigorous argument, but just a hint to the intuition.

In this way $\mathbb{P}^n$ can be interpreted as $\mathbb{A}^n$ with the points at infinity added, each point at infinity corresponding to one direction in $\mathbb{A}^n$.

Exercise to §1.

1*. Let V be a vector space of finite dimension over a field K. Let $\check{V}$ denote the dual of V. Prove that $\mathbb{P}(\check{V})$ can be put in bijection with the set of the hyperplanes of $\mathbb{P}(V)$ (hint: the kernel of a non-zero linear form on V is a subvector space of V of codimension one).

2. Algebraic sets.

Roughly speaking, algebraic subsets of the affine or of the projective space are sets of solutions of systems of algebraic equations, i.e. common roots of sets of polynomials.

Examples of algebraic sets are: linear subspaces of both the affine and the projective space, plane algebraic curves, quadrics, graphics of polynomials functions, ...

Algebraic geometry is the branch of mathematics which studies algebraic sets (and their generalizations). Our first aim is *to give a formal definition of algebraic sets.*

Let $K[x_1,\dots,x_n]$ be the polynomial ring in n variables over the field K. If $P(a_1,\dots,a_n)\in\mathbb{A}^n$, and $F=F(x_1,\dots,x_n)\in K[x_1,\dots,x_n]$, we can consider the value of F at P, i.e. $F(P)=F(a_1,\dots,a_n)\in K$. We say that P is a *zero of F* if $F(P)=0$.

For example the points $P_1(1,0)$, $P_2(-1,0)$, $P_3(0,1)$ are zeroes of $F=x_1^2+x_2^2-1$ over any field. If $G=x_1^2+x_2^2+1$ then G has no zeroes in $\mathbb{A}^2_{\mathbb{R}}$, but it does have zeroes in $\mathbb{A}^2_{\mathbb{C}}$.

2.1. Definition. A subset X of $\mathbb{A}^n_K$ is an *affine algebraic set* if X is the set of common zeroes of a family of polynomials of $K[x_1,\dots,x_n]$.

This means that there exists a subset $S\subset K[x_1,\dots,x_n]$ such that

$$X=\{P\in\mathbb{A}^n \mid F(P)=0\ \forall\ F\in S\}.$$

In this case X is called the zero set of S and is denoted $V(S)$ (or in some books $Z(S)$, e.g. this is the notation of Hartshorne's book). In particular, if $S=\{F\}$, then $V(S)$ will be simply denoted by $V(F)$.

2.2. Examples and remarks.

1. $S=K[x_1,\dots,x_n]$: then $V(S)=\emptyset$, because S contains non–zero constants.
2. $S=\{0\}$: then $V(S)=\mathbb{A}^n$.
3. $S=\{xy-1\}$: then $V(xy-1)$ is the hyperbola.
4. If $S\subset T$, then $V(S)\supset V(T)$.

Let $S\subset K[x_1,\dots,x_n]$ be a set of polynomials, let $\alpha:=\langle S\rangle$ be the ideal generated by S. Recall that $\alpha=\{$finite sums of products of the form HF where $F\in S$, $H\in K[x_1,\dots,x_n]\}$.

2.3. Proposition. $V(S)=V(\alpha)$.

Proof. If $P\in V(\alpha)$, then $F(P)=0$ for any $F\in\alpha$; in particular for any $F\in S$ because $S\subset\alpha$.

Conversely, if $P\in V(S)$, let $G=\sum_i H_iF_i$ be a polynomial of α ($F_i\in S\ \forall\ i$). Then $G(P)=(\sum H_iF_i)(P)=\sum H_i(P)F_i(P)=0$. □

The above Proposition is important in view of the following:

Hilbert' Basis Theorem. *If R is a Noetherian ring, then the polynomial ring $R[x]$ is Noetherian.*

Proof. Assume by contradiction that $R[x]$ is not Noetherian. Let $I\subset R[x]$ be a not finitely generated ideal. Let $f_1\in I$ be a non-zero polynomial of minimum degree. We define by induction as follows a sequence $\{f_k\}_{k\in\mathbb{N}}$ of polynomials: if

f_k ($k \geq 1$) has already been chosen, let f_{k+1} be a polynomial of minimum degree in $I \setminus \langle f_1, \ldots, f_k \rangle$. Let n_k be the degree of f_k and a_k be its leading coefficient. Note that, by the very choice of f_k, the chain of the degrees is increasing: $n_1 \leq n_2 \leq \ldots$.

We will prove now that $\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \ldots$ is a chain of ideals, that does not become stationary: this will give the required contradiction. Indeed, if $\langle a_1, \ldots, a_r \rangle = \langle a_1, \ldots, a_r, a_{r+1} \rangle$, then $a_{r+1} = \sum_{i=1}^{r} b_i a_i$, for suitable $b_i \in R$. In this case, we consider the element $g := f_{r+1} - \sum_{i=1}^{r} b_i x^{n_{r+1} - n_i} f_i$: g belongs to I, but $g \notin \langle f_1, \ldots, f_r \rangle$, and its degree is strictly lower than the degree of f_{r+1}: contradiction. □

2.4. Corollary. *Any affine algebraic set $X \subset \mathbb{A}^n$ is the zero set of a finite number of polynomials, i.e. there exist $F_1, \ldots, F_r \in K[x_1, \ldots, x_n]$ such that $X = V(F_1, \ldots, F_r)$.*

□

Note that $V(F_1, \ldots, F_r) = V(F_1) \cap \ldots \cap V(F_r)$, so every algebraic set is a finite intersection of algebraic sets of the form $V(F)$, i.e. zeroes of a unique polynomial F. If $F = 0$, then $V(0) = \mathbb{A}^n$; if $F = c \in K \setminus \{0\}$, then $V(c) = \emptyset$; if *deg* $F > 0$, then $V(F)$ is called a *hypersurface*.

2.5. Proposition. *The affine algebraic sets of $\mathbb{A}^n$ satisfy the axioms of the closed sets of a topology, called the Zariski topology.*

Proof. It is enough to check that finite unions and arbitrary intersections of algebraic sets are again algebraic sets.

Let $V(\alpha), V(\beta)$ be two algebraic sets, with α, β ideals of $K[x_1, \ldots, x_n]$. Then $V(\alpha) \cup V(\beta) = V(\alpha \cap \beta) = V(\alpha\beta)$, where $\alpha\beta$ is the product ideal, defined by:

$$\alpha\beta = \{\sum_{\text{fin}} a_i b_i \mid a_i \in \alpha, b_i \in \beta\}.$$

In fact: $\alpha\beta \subset \alpha \cap \beta$ so $V(\alpha \cap \beta) \subset V(\alpha\beta)$, and both $\alpha \cap \beta \subset \alpha$ and $\alpha \cap \beta \subset \beta$ so $V(\alpha) \cup V(\beta) \subset V(\alpha \cap \beta)$. Assume now that $P \in V(\alpha\beta)$ and $P \notin V(\alpha)$: hence $\exists F \in \alpha$ such that $F(P) \neq 0$; on the other hand, if $G \in \beta$ then $FG \in \alpha\beta$ so $(FG)(P) = 0 = F(P)G(P)$, which implies $G(P) = 0$.

Let $V(\alpha_i), i \in I$, be a family of algebraic sets, $\alpha_i \subset K[x_1, \ldots, x_n]$. Then $\cap_{i \in I} V(\alpha_i) = V(\sum_{i \in I} \alpha_i)$, where $\sum_{i \in I} \alpha_i$ is the sum ideal of α_i's. In fact $\alpha_i \subset \sum_{i \in I} \alpha_i$ $\forall i$, hence $V(\sum_i \alpha_i) \subset V(\alpha_i)$ $\forall i$ and $V(\sum_i \alpha_i) \subset \cap_i V(\alpha_i)$. Conversely, if $P \in V(\alpha_i)$ $\forall i$, and $F \in \sum_i \alpha_i$, then $F = \sum_i F_i$; therefore $F(P) = \sum F_i(P) = 0$.

□

2.6. Examples.

1. The Zariski topology of the affine line $\mathbb{A}^1$.

Let us recall that the polynomial ring $K[x]$ in one variable is a PID (principal ideal domain), so every ideal $I \subset K[x]$ is of the form $I = \langle F \rangle$. Hence every closed subset of $\mathbb{A}^1$ is of the form $X = V(F)$, the set of zeroes of a unique polynomial $F(x)$. If $F = 0$, then $V(F) = \mathbb{A}^1$, if $F = c \in K^*$, then $V(F) = \emptyset$, if $\deg F = d > 0$, then F can be decomposed in linear factors in polynomial ring over the algebraic closure of K; it follows that $V(F)$ has at most d points.

We conclude that the closed sets in the Zariski topology of $\mathbb{A}^1$ are: $\mathbb{A}^1$, $\emptyset$ and the finite sets.

2. If $K = \mathbb{R}$ or $\mathbb{C}$, then the Zariski topology and the Euclidean topology on $\mathbb{A}^n$ can be compared, and it results that the Zariski topology is coarser. Indeed every open set in the Zariski topology is open also in the usual topology. Let $X = V(F_1, \ldots, F_r)$ be a closed set in the Zariski topology, and $U := \mathbb{A}^n \setminus X$; if $P \in U$, then $\exists\ F_i$ such that $F_i(P) \neq 0$, so there exists an open neighbourhood of P in the usual topology in which F_i does not vanish.

Conversely, there exist closed sets in the usual topology which are not Zariski closed, for example the balls. The first case, of an interval in the real affine line, follows from part 1.

We want to define now the projective algebraic sets in $\mathbb{P}^n$. Let $K[x_0, x_1, \ldots, x_n]$ be the polynomial ring in $n+1$ variables. Fix a polynomial $G(x_0, x_1, \ldots, x_n) \in K[x_0, x_1, \ldots, x_n]$ and a point $P[a_0, a_1, \ldots, a_n] \in \mathbb{P}^n$: then, in general,

$$G(a_0, \ldots, a_n) \neq G(\lambda a_0, \ldots, \lambda a_n),$$

so the value of G at P is not defined.

2.7. Example. Let $G = x_1 + x_0 x_1 + x_2^2$, $P[0,1,2] = [0,2,4] \in \mathbb{P}^2_{\mathbb{R}}$. So $G(0,1,2) = 1 + 4 \neq G(0,2,4) = 2 + 16$. But if $Q = [1,0,0] = [\lambda,0,0]$, then $G(1,0,0) = G(\lambda,0,0) = 0$ for all λ.

2.8. Definition. Let $G \in K[x_0, x_1, \ldots, x_n]$: G is *homogeneous of degree d*, or G is *a form* of degree d, if G is a linear combination of monomials of degree d.

2.9. Lemma. *If G is homogeneous of degree d, $G \in K[x_0, x_1, \ldots, x_n]$, and t is a new variable, then $G(tx_0, \ldots, tx_n) = t^d G(x_0, \ldots, x_n)$.*

Proof. It is enough to prove the equality for monomials, i.e. for

$$G = a x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n} \text{ with } i_0 + i_1 + \ldots + i_n = d :$$

$$G(tx_0, \ldots, tx_n) = a(tx_0)^{i_0}(tx_1)^{i_1} \ldots (tx_n)^{i_n} = a t^{i_0+i_1+\ldots+i_n} x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n} = \\ = t^d G(x_0, \ldots, x_n).$$

□

2.10. Definition. Let G be a homogeneous polynomial of $K[x_0, x_1, \dots, x_n]$. A point $P[a_0, \dots, a_n] \in \mathbb{P}^n$ is a *zero of* G if $G(a_0, \dots, a_n) = 0$. In this case we write $G(P) = 0$.

Note that by Lemma 2.9 if $G(a_0, \dots, a_n) = 0$, then

$$G(\lambda a_0, \dots, \lambda a_n) = \lambda^{\deg G} G(a_0, \dots, a_n) = 0$$

for every choice of $\lambda \in K^*$.

2.11. Definition. A subset Z of $\mathbb{P}^n$ is a *projective algebraic set* if Z is the set of common zeroes of a set of homogeneous polynomials of $K[x_0, x_1, \dots, x_n]$.

If T is such a subset of $K[x_0, x_1, \dots, x_n]$, then the corresponding algebraic set will be denoted by $V_P(T)$.

Let $\alpha = \langle T \rangle$ be the ideal generated by the (homogeneous) polynomials of T. If $F \in \alpha$, then $F = \sum_i H_i F_i$, $F_i \in T$: if $P \in V_P(T)$, and $P[a_0, \dots, a_n]$, then $F(a_0, \dots, a_n) = \sum H_i(a_0, \dots, a_n) F_i(a_0, \dots, a_n) = 0$, for any choice of coordinates of P, regardless if F is homogeneous or not. We say that P is a *projective zero* of F.

If F is a polynomial, then F can be written *in a unique way* as a sum of homogeneous polynomials, called the homogeneous components of F: $F = F_0 + F_1 + \ldots + F_d$. More in general, we give the following:

2.12. Definition. Let A be a ring. A is called *a graded ring over* $\mathbb{Z}$ if there exists a family of additive subgroups $\{A_i\}_{i \in \mathbb{Z}}$ such that $A = \oplus_{i \in \mathbb{Z}} A_i$ and $A_i A_j \subset A_{i+j}$ for all pair of indices.

The elements of A_i are called *homogeneous of degree* i and A_i is the homogeneous component of degree i. The standard example of graded ring is the polynomial ring with coefficients in a ring R. In this case the homogeneous components of negative degrees are all zero.

2.13 Proposition - Definition. *Let $I \subset A$ be an ideal of a graded ring. I is called* **homogeneous** *if the following equivalent conditions are fulfilled:*
(i) I is generated by homogeneous elements;
(ii) $I = \oplus_{k \in \mathbb{Z}} (I \cap A_k)$, i.e. if $F = \Sigma_{k \in \mathbb{Z}} F_k \in I$, then all homogeneous components F_k of F belong to I.

Proof of the equivalence.

"(ii)⇒(i)": given a system of generators of I, write each of them as sum of its homogeneous components: $F_i = \Sigma_{k \in \mathbb{Z}} F_{ik}$. Then a set of homogeneous generators of I is formed by all the elements F_{ik}.

"(i)⇒(ii)": let I be generated by a family of homogeneous elements $\{G_\alpha\}$, with $\deg G_\alpha = d_\alpha$. If $F \in I$, then F is a combination of the elements G_α with suitable coefficients H_α; write each H_α as sum of its homogeneous components: $H_\alpha = \Sigma H_{\alpha k}$. Note that the product $H_{\alpha k} G_\alpha$ is homogeneous of degree $k + d_\alpha$. By the unicity of the expression of F as sum of homogeneous elements, it follows that all of them are combinations of the generators $\{G_\alpha\}$ and therefore they belong to I. $\square$

Let $I \subset K[x_0, x_1, \dots, x_n]$ be a homogeneous ideal. Note that, by the noetherianity, I admits a finite set of homogeneous generators.

Let $P[a_0, \dots, a_n] \in \mathbb{P}^n$. If $F \in I$, $F = F_0 + \dots + F_d$, then $F_0 \in I, \dots, F_d \in I$. We say that P *is a zero of* I if P is a projective zero of any polynomial of I or, equivalently, of any homogeneous polynomial of I. This also means that P is a zero of any homogeneous polynomial of a set generating I. The set of zeroes of I will be denoted $V_P(I)$: all projective algebraic subsets of $\mathbb{P}^n$ are of this form.

As in the affine case, the projective algebraic subsets of $\mathbb{P}^n$ satisfy the axioms of the closed sets of a topology called the Zariski topology of $\mathbb{P}^n$ (see also Exercise 3).

Note that also all subsets of $\mathbb{A}^n$ and $\mathbb{P}^n$ have a structure of topological space, with the induced topology, which is still called the Zariski topology.

Exercises to §2.

1. Let $F \in K[x_1, \dots, x_n]$ be a non–constant polynomial. The set $\mathbb{A}^n \setminus V(F)$ will be denoted $\mathbb{A}^n_F$. Prove that $\{\mathbb{A}^n_F | F \in K[x_1, \dots, x_n] \setminus K\}$ is a topology basis for the Zariski topology.

2. Let $B \subset \mathbb{R}^n$ be a ball. Prove that B is not Zariski closed.

3*. Let I, J be homogeneous ideals of $K[x_0, x_1, \dots, x_n]$. Prove that $I + J$, IJ and $I \cap J$ are homogeneous ideals.

4*. Prove that the map $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ defined by $t \to (t, t^2, t^3)$ is a homeomorphism between $\mathbb{A}^1$ and its image, for the Zariski topology.

5. Let $X \subset \mathbb{A}^2_{\mathbb{R}}$ be the graph of the map $\mathbb{R} \to \mathbb{R}$ such that $x \to \sin x$. Is X closed in the Zariski topology? (hint: intersect X with a line....)

3. Examples of algebraic sets.

a) In the Zariski topology both of $\mathbb{A}^n$ and of $\mathbb{P}^n$ all points are closed.

If $P(a_1, \dots, a_n) \in \mathbb{A}^n$: $P = V(x_1 - a_1, \dots, x_n - a_n)$. If $P[a_0, \dots, a_n] \in \mathbb{P}^n$: $P = V_P(\langle a_i x_j - a_j x_i \rangle_{i,j=0,\dots,n})$.

Note that in the projective case the polynomials defining P as closed set are homogeneous. They can be seen as minors of order 2 of the matrix

$$\begin{pmatrix} a_0 & a_1 & \dots & a_n \\ x_0 & x_1 & \dots & x_n \end{pmatrix}$$

with entries in $K[x_0, x_1, \dots, x_n]$.

b) Hypersurfaces.

Let us recall that the polynomial ring $K[x_1, \dots, x_n]$ is a UFD (unique factorization domain), i. e. every non-constant polynomial F can be expressed in a unique way (up to the order and up to units) as $F = F_1^{r_1} F_2^{r_2} \dots F_s^{r_s}$, where $F_1, \dots, F_s$ are irreducible polynomials, two by two distinct, and $r_i \geq 1 \forall\ i = 1, \dots, s$. Hence the hypersurface of $\mathbb{A}^n$ defined by F is

$$X := V(F) = V(F_1^{r_1} F_2^{r_2} \dots F_s^{r_s}) = V(F_1 F_2 \dots F_s) = V(F_1) \cup V(F_2) \cup \dots \cup V(F_s).$$

The equation $F_1 F_2 \dots F_s = 0$ is called the reduced equation of X. Note that $F_1 F_2 \dots F_s$ generates the radical $\sqrt{F}$. If $s = 1$, X is called an irreducible hypersurface; by definition its degree is the degree of its reduced equation. Any hypersurface is a finite union of irreducible hypersurfaces.

In a similar way one defines hypersurfaces of $\mathbb{P}^n$, i. e. projective algebraic sets of the form $Z = V_P(G)$, with $G \in K[x_0, x_1, \dots, x_n]$, G homogeneous. Since the irreducible factors of G are homogeneous (see Exercise 3.6), any projective hypersurface Z has a reduced equation (whose degree is, by definition, the degree of Z) and Z is a finite union of irreducible hypersurfaces. The degree of a projective hypersurface has the following important geometrical meaning.

3.1. Proposition. *Let K be an algebraically closed field. Let $Z \subset \mathbb{P}^n$ be a projective hypersurface of degree d. Then a line of $\mathbb{P}^n$, not contained in Z, meets Z at exactly d points, counting multiplicities.*

Proof. Let G be the reduced equation of Z and $L \subset \mathbb{P}^n$ be any line.

We fix two points on L: $A = [a_0, \dots, a_n], B = [b_0, \dots, b_n]$. So L admits parametric equations of the form

$$\begin{cases} x_0 = \lambda a_0 + \mu b_0 \\ x_1 = \lambda a_1 + \mu b_1 \\ \dots \\ x_n = \lambda a_n + \mu b_n \end{cases}$$

The points of $Z \cap L$ are obtained from the homogeneous pairs $[\lambda, \mu]$ which are solutions of the equation $G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n) = 0$. If $L \subset Z$, then this equation is identical. Otherwise, $G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n)$ is a non-zero homogeneous polynomial of degree d in two variables. Being K algebraically closed, it can be factorized in linear factors:

$$G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n) = (\mu_1 \lambda - \lambda_1 \mu)^{d_1} (\mu_2 \lambda - \lambda_2 \mu)^{d_2} \dots (\mu_r \lambda - \lambda_r \mu)^{d_r}$$

with $d_1 + d_2 + \dots + d_r = d$. Every factor corresponds to a point in $Z \cap L$, to be counted with the same multiplicity as the factor. $\square$

If K is not algebraically closed, considering the algebraic closure of K and using Proposition 3.1, we get that d is un upper bound on the number of points of $Z \cap L$.

c) Affine and projective subspaces.

The subspaces introduced in §1, both in the affine and in the projective case, are examples of algebraic sets.

d) Product of affine spaces.

Let $\mathbb{A}^n$, $\mathbb{A}^m$ be two affine spaces over the field K. The cartesian product $\mathbb{A}^n \times \mathbb{A}^m$ is the set of pairs (P, Q), $P \in \mathbb{A}^n$, $Q \in \mathbb{A}^m$: it is in natural bijection with $\mathbb{A}^{n+m}$ via the map

$$\phi : \mathbb{A}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^{n+m}$$

such that $\phi((a_1, \dots, a_n), (b_1, \dots, b_m)) = (a_1, \dots, a_n, b_1, \dots, b_m)$.

From now on we will always identify $\mathbb{A}^n \times \mathbb{A}^m$ with $\mathbb{A}^{n+m}$. We get two topologies on $\mathbb{A}^n \times \mathbb{A}^m$: the Zariski topology and the product topology.

3.1. Proposition. *The Zariski topology is strictly finer than the product topology.*

Proof. If $X = V(\alpha) \subset \mathbb{A}^n$, $\alpha \subset K[x_1, \dots, x_n]$ and $Y = V(\beta) \subset \mathbb{A}^m$, $\beta \subset K[y_1, \dots, y_m]$, then $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m$ is Zariski closed, precisely $X \times Y = V(\alpha \cup \beta)$ where the union is made in the polynomial ring in $n + m$ variables $K[x_1, \dots, x_n, y_1, \dots, y_m]$. Hence, if $U = \mathbb{A}^n \setminus X$, $V = \mathbb{A}^m \setminus Y$ are open subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ in the Zariski topology, then $U \times V = \mathbb{A}^n \times \mathbb{A}^m \setminus ((\mathbb{A}^n \times Y) \cup (X \times \mathbb{A}^m))$ is open in $\mathbb{A}^n \times \mathbb{A}^m$ in the Zariski topology.

Conversely, we prove that $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ contains some subsets which are Zariski open but are not open in the product topology. The proper open subsets in the product topology are of the form $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{$ finite unions of "vertical" and "horizontal" lines$\}$.

– Fig. 1 –

Let $X = \mathbb{A}^2 \setminus V(x-y)$: it is Zariski open but does not contain any non-empty subset of the above form, so it is not open in the product topology. There are similar examples in $\mathbb{A}^n \times \mathbb{A}^m$ for any n, m. $\square$

Note that there is no similar construction for $\mathbb{P}^n \times \mathbb{P}^m$.

e) Embedding of $\mathbb{A}^n$ in $\mathbb{P}^n$.

Let H_i be the hyperplane of $\mathbb{P}^n$ of equation $x_i = 0$, $i = 0, \ldots, n$; it is closed in the Zariski topology, and the complementar set U_i is open. So we have an open covering of $\mathbb{P}^n$: $\mathbb{P}^n = U_0 \cup U_1 \cup \ldots \cup U_n$. Let us recall that for all i there is a bijection $\phi_i : U_i \to \mathbb{A}^n$ such that $\phi_i([x_0, \ldots, x_i, \ldots, x_n]) = (\frac{x_0}{x_i}, \ldots, \hat{1}, \ldots, \frac{x_n}{x_i})$. The inverse map is $j_i : \mathbb{A}^n \to U_i$ such that $j_i(y_1, \ldots, y_n) = [y_1, \ldots, 1, \ldots, y_n]$.

3.2. Proposition. *The map ϕ_i is a homeomorphism, for $i = 0, \ldots, n$.*

Proof. Assume $i = 0$ (the other cases are similar).

We introduce two maps:

(i) *dehomogeneization* of polynomials with respect to x_0.

It is a map ${}^a : K[x_0, x_1, \ldots, x_n] \to K[y_1, \ldots, y_n]$ such that

$${}^a(F(x_0, \ldots, x_n)) = {}^aF(y_1, \ldots, y_n) := F(1, y_1, \ldots, y_n).$$

Note that ^{a} is a ring homomorphism.

(ii) *homogeneization* of polynomials with respect to x_0.

It is a map ${}^h : K[y_1, \ldots, y_n] \to K[x_0, x_1, \ldots, x_n]$ defined by

$${}^h(G(y_1, \ldots, y_n)) = {}^hG(x_0, \ldots, x_n) := x_0^{\deg G} G(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$$

.

${}^hG$ is always a homogeneous polynomial of the same degree as G. The map ^{h} is clearly not a ring homomorphism. Note that always ${}^a({}^hG) = G$ but in general ${}^h({}^aF) \neq F$; what we can say is that, if $F(x_0, \ldots, x_n)$ is homogeneous, then $\exists r \geq 0$ such that $F = x_0^r({}^h({}^aF))$.

Let $X \subset U_0$ be closed in the topology induced by the Zariski topology of the projective space, i.e. $X = U_0 \cap V_P(I)$ where I is a homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$. Define ${}^aI = \{{}^aF \mid F \in I\}$: it is an ideal of $K[y_1, \ldots, y_n]$ (because ^{a} is a ring homomorphism). We prove that $\phi_0(X) = V({}^aI)$. For: let $P[x_0, \ldots, x_n]$ be a point of U_0; then $\phi_0(P) = (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) \in \phi_0(X) \iff P[x_0, \ldots, x_n] = [1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}] \in X = V_P(I) \iff F(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}) = 0 \ \forall \ {}^aF \in {}^aI \iff \phi_0(P) \in V({}^aI)$.

Conversely: let $Y = V(\alpha)$, α ideal of $K[y_1, \ldots, y_n]$, be a Zariski closed set of $\mathbb{A}^n$. Let ${}^h\alpha$ be the homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$ generated by the set $\{{}^hG \mid G \in \alpha\}$. We prove that $\phi_0^{-1}(Y) = V_P({}^h\alpha) \cap U_0$. In fact: $[1, x_0, \ldots, x_n] \in \phi_0^{-1}(Y) \iff (x_1, \ldots, x_n) \in Y \iff G(x_1, \ldots, x_n) = {}^hG(1, x_1, \ldots, x_n) = 0 \ \forall \ G \in \alpha \iff [1, x_1, \ldots, x_n] \in V_P({}^h\alpha)$. $\square$

From now on we will often identify $\mathbb{A}^n$ with U_0 via ϕ_0 (and similarly with U_i via ϕ_i). So if $P[x_0, \dots, x_n] \in U_0$, we will refer to $x_0, \dots, x_n$ as the homogeneous coordinates of P and to $\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}$ as the non–homogeneous or affine coordinates of P.

Exercises to §3.

1*. Let $n \geq 2$. Prove that, if K is an algebraically closed field, then in $\mathbb{A}^n_K$ both any hypersurface and any complementar set of a hypersurface have infinitely many points.

2. Prove that the Zariski topology on $\mathbb{A}^n$ is T_1.

3*. Let $F \in K[x_0, x_1, \dots, x_n]$ be a homogeneous polynomial. Check that its irreducible factors are homogeneous. (hint: consider a product of two polynomials not both homogeneous...)

4. The ideal of an algebraic set and the Hilbert Nullstellensatz.

Let $X \subset \mathbb{A}^n$ be an algebraic set, $X = V(\alpha)$, $\alpha \subset K[x_1, \dots, x_n]$. The ideal α defining X is not unique: for example, let $X = \{0\} \subset \mathbb{A}^2$; then $0 = V(x_1, x_2) = V(x_1^2, x_2) = V(x_1^2, x_2^3) = V(x_1^2, x_1, x_2, x_2^2) = \dots$ Nevertheless, there is an ideal we can canonically associate to X, i.e. the biggest one. Precisely:

4.1. Definition. Let $Y \subset \mathbb{A}^n$ be any set.

The *ideal of* Y is $I(Y) = \{F \in K[x_1, \dots, x_n] \mid F(P) = 0 \text{ for any } P \in Y\} = \{F \in K[x_1, \dots, x_n] \mid Y \subset V(F)\}$: it is formed by all polynomials vanishing on Y. Note that $I(Y)$ is in fact an ideal.

For instance, if $P(a_1, \dots, a_n)$ is a point, then $I(P) = \langle x_1 - a_1, \dots, x_n - a_n \rangle$. Indeed all its polynomials vanish on P, and, on the other side, it is maximal.

The following relations follow immediately by the definition:
(i) if $Y \subset Y'$, then $I(Y) \supset I(Y')$;
(ii) $I(Y \cup Y') = I(Y) \cap I(Y')$;
(iii) $I(Y \cap Y') \supset I(Y) + I(Y')$.

Similarly, if $Z \subset \mathbb{P}^n$ is any set, the *homogeneous ideal of* Z is, by definition, the homogeneous ideal of $K[x_0, x_1, \dots, x_n]$ generated by the set $\{G \in K[x_0, x_1, \dots, x_n] \mid G \text{ is homogeneous and } V_P(G) \supset Z\}$. It is denoted $I_h(Z)$.

Relations similar to (i),(ii),(iii) are satisfied. $I_h(Z)$ is also the set of polynomials $F(x_0, \dots, x_n)$ such that every point of Z is a projective zero of F.

Let $\alpha \subset K[x_1, \dots, x_n]$ be an ideal. Let $\sqrt{\alpha}$ denote the radical of α, i.e. the ideal $\{F \in K[x_1, \dots, x_n] \mid \exists r \geq 1 \ s.t. \ F^r \in \alpha\}$. Note that always $\alpha \subset \sqrt{\alpha}$; if equality holds, then α is called a *radical ideal*.

4.2. Proposition.

1) For any $X \subset \mathbb{A}^n$, $I(X)$ is a radical ideal.

2) For any $Z \subset \mathbb{P}^n$, $I_h(Z)$ is a homogeneous radical ideal.

Proof. 1) If $F \in \sqrt{I(X)}$, let $r \geq 1$ such that $F^r \in I(X)$: hence if $P \in X$, then $(F^r)(P) = 0 = (F(P))^r$ in the base field K. Therefore $F(P) = 0$.

2) is similar, taking into account that $I_h(Z)$ is a homogeneous ideal (see Exercise 4.7.). □

We can interpret I as a map from $\mathcal{P}(\mathbb{A}^n)$, the set of subsets of the affine space, to $\mathcal{P}(K[x_1, \dots, x_n])$. On the other hand, V can be seen as a map in the opposite sense. We have:

4.3. Proposition. *Let $\alpha \subset K[x_1, \dots, x_n]$ be an ideal, $Y \subset \mathbb{A}^n$ be any subset. Then:*

(i) $\alpha \subset I(V(\alpha))$;

(ii) $Y \subset V(I(Y))$;

(iii) $V(I(Y)) = \overline{Y}$: the closure of Y in the Zariski topology of $\mathbb{A}^n$.

Proof. (i) If $F \in \alpha$ and $P \in V(\alpha)$, then $F(P) = 0$, so $F \in I(V(\alpha))$.

(ii) If $P \in Y$ and $F \in I(Y)$, then $F(P) = 0$, so $P \in V(I(Y))$.

(iii) Taking closures in (ii), we get: $\overline{Y} \subset \overline{V(I(Y))} = V(I(Y))$. Conversely, let $X = V(\beta)$ be any closed set containing Y: $X = V(\beta) \supset Y$. Then $I(Y) \supset I(V(\beta)) \supset \beta$ by (i); we apply V again: $V(\beta) = X \supset V(I(Y))$ so any closed set containing Y contains $V(I(Y))$ so $\overline{Y} \supset V(I(Y))$. □

Similar properties relate homogeneous ideals of $K[x_0, x_1, \dots, x_n]$ and subsets of $\mathbb{P}^n$; in particular, if $Z \subset \mathbb{P}^n$, then $V_P(I_h(Z)) = \overline{Z}$, the closure of Z in the Zariski topology of $\mathbb{P}^n$.

There does not exist any characterization of $I(V(\alpha))$ *in general*. We can only say that it is a radical ideal containing α, so it contains also $\sqrt{\alpha}$. To characterise $I(V(\alpha))$ we need some extra assumption on the base field.

4.4. Hilbert Nullstellensatz (Theorem of zeroes). *Let K be an algebraically closed field. Let $\alpha \subset K[x_1, \dots, x_n]$ be an ideal. Then $I(V(\alpha)) = \sqrt{\alpha}$.*

Remark. The assumption on K is necessary. Let me recall that K is algebraically closed if any non–constant polynomial of $K[x]$ has at least one root in K, or, equivalently, if any irreducible polynomial of $K[x]$ has degree 1. So if K is not algebraically closed, there exists $F \in K[x]$, irreducible of degree $d > 1$. Therefore F has no zero in K, hence $V(F) \subset \mathbb{A}^1_K$ is empty. So $I(V(F)) = I(\emptyset) = \{G \in K[x] \mid \emptyset \subset V(G)\} = K[x]$. But $\langle F \rangle$ is a maximal ideal of $K[x]$, and $\langle F \rangle \subset \sqrt{\langle F \rangle}$. If $\langle F \rangle \neq \sqrt{\langle F \rangle}$, by the maximality $\sqrt{\langle F \rangle} = \langle 1 \rangle$, so $\exists r \geq 1$ such that $1^r = 1 \in \langle F \rangle$, which is false. Hence $\sqrt{\langle F \rangle} = \langle F \rangle \neq K[x] = I(V(F))$.

We will deduce the proof of Hilbert Nullestellensatz, after several steps, from another very important theorem, known as the "Emmy Noether normalization Lemma".

We start with some definitions.

Let $K \subset E$ be fields, K a subfield of E. Let $\{z_i\}_{i\in I}$ be a family of elements of E.

4.5.Definition. The family $\{z_i\}_{i\in I}$ is *algebraically free* over K or, equivalently, the elements z_i's are *algebraically independent* over K if there does not exist any non–zero polynomial $F \in K[x_i]_{i\in I}$, the polynomial ring in a set of variables indexed on I, such that F vanishes in the elements of the family $\{z_i\}$.

For example: if the family is formed by one element z, $\{z\}$ is algebraically free over K if and only if z is transcendental over K. The family $\{\pi, \sqrt{\pi}\}$ is not algebraically free over $\mathbb{Q}$: it satisfies the non–trivial relation $x_1^2 - x_2 = 0$.

By convention, the empty family is free over any field K.

Let $\mathcal{S}$ be the set of the families of elements of E, which are algebraically free over K. $\mathcal{S}$ is a non–empty set, partially ordered by inclusion and inductive. By Zorn's lemma, there exist in $\mathcal{S}$ maximal elements, i.e. algebraically free families such that they do not remain free if any element of E is added. Any such maximal algebraically free family is called a *transcendence basis* of E over K. It can be proved that, if B, B' are two transcendence bases, then they have the same cardinality, called the *transcendence degree* of E over K. It is denoted $tr.d.E/K$.

4.6. Definition. A *K–algebra* is a ring A containing (a subfield isomorphic to) K.

Let $y_1, \dots, y_n$ be elements of E: the *K–algebra generated by* $y_1, \dots, y_n$ is, by definition, the minimum subring of E containing $K, y_1, \dots, y_n$: it is denoted $K[y_1, \dots, y_n]$ and its elements are polynomials in the elements $y_1, \dots, y_n$ with coefficients in K. Its quotient field $K(y_1, \dots, y_n)$ is the minimum subfield of E containing $K, y_1, \dots, y_n$.

A *finitely generated K–algebra* A is a K–algebra such that there exist elements of A $y_1, \dots, y_r$ which verify the condition $A = K[y_1, \dots, y_r]$.

4.7. Proposition. *There exists a transcendence basis of $K(y_1, \dots, y_n)$ over K contained in the set $\{y_1, \dots, y_n\}$.*

Proof. Let $\mathcal{S}$ be the set of the subfamilies of $\{y_1, \dots, y_n\}$ formed by algebraically independent elements: $\mathcal{S}$ is a finite set so it possesses maximal elements with respect to the inclusion. We can assume that $\{y_1, \dots, y_r\}$ is such a maximal family. Then $y_{r+1}, \dots, y_n$ are each one algebraic over $K(y_1, \dots, y_r)$ so $K(y_1, \dots, y_n)$ is an algebraic extension of $K(y_1, \dots, y_r)$. If $z \in K(y_1, \dots, y_n)$ is any element, then z is algebraic over $K(y_1, \dots, y_r)$, so the family $\{y_1, \dots, y_r, z\}$ is not algebraically free.

□

4.8. Corollary. *$tr.d.K(y_1,\dots,y_n)/K \le n$.* □

Let now $A \subset B$ be rings, A a subring of B. Let $b \in B$: b is *integral over* A if it is a root of a monic polynomial of $A[x]$, i.e. there exist $a_1,\dots,a_n \in A$ such that

$$b^n + a_1 b^{n-1} + a_2 b^{n-2} + \ldots + a_n = 0.$$

Such a relation is called an integral equation for b over A.

Note that, if A is a field, then b is integral over A if and only if b is algebraic over A.

B is called *integral over* A, or an integral extension of A, if and only if b is integral over A for every $b \in B$.

We can state now the

4.9. Normalization Lemma. *Let A be a finitely generated K–algebra and an integral domain. Let $r := tr.d.K(y_1,\dots,y_n)/K$. Then there exist elements $z_1,\dots,z_r \in A$, algebraically independent over K, such that A is integral over $K[z_1,\dots,z_r]$.*

Proof. See, for instance, Lang [6]. □

We start now the proof of the Nullstellensatz.

1^{st} **Step.**

Let K be an algebraically closed field, let $\mathcal{M} \subset K[x_1,\dots,x_n]$ be a maximal ideal. Then, there exist $a_1,\dots,a_n \in K$ such that $\mathcal{M} = \langle x_1 - a_1,\dots,x_n - a_n\rangle$.

Proof. Let K' be the quotient ring $\frac{K[x_1,\dots,x_n]}{\mathcal{M}}$: it is a field because $\mathcal{M}$ is maximal, and a finitely generated K–algebra (by the residues in K' of $x_1,\dots,x_n$). By the Normalization Lemma, there exist $z_1,\dots,z_r \in K'$, algebraically independent over K', such that K' is integral over $A := K[z_1,\dots,z_r]$. We claim that *A is a field*: let $f \in A$, $f \neq 0$; $f \in K'$ so there exists $f^{-1} \in K'$, and f^{-1} is integral over A; we fix an integral equation for f^{-1} over A:

$$(f^{-1})^s + a_{s-1}(f^{-1})^{s-1} + \ldots + a_0 = 0$$

where $a_0,\dots,a_{s-1} \in A$. We multiply this equation by f^{s-1}:

$$f^{-1} + a_{s-1} + \ldots + a_0 f^{s-1} = 0$$

hence $f^{-1} \in A$. So A is both a field and a polynomial ring over K, so $r = 0$ and $A = K$. Therefore K' is an algebraic extension of K, which is algebraically closed, so $K' \simeq K$. Let us fix an isomorphism $\psi : \frac{K[x_1,\dots,x_n]}{\mathcal{M}} \xrightarrow{\sim} K$ and let $p : K[x_1,\dots,x_n] \to \frac{K[x_1,\dots,x_n]}{\mathcal{M}}$ be the canonical epimorphism.

Let $a_i = \psi(p(x_i))$, $i = 1, \dots, n$. The kernel of $\psi \circ p$ is $\mathcal{M}$, and $x_i - a_i \in \ker(\psi \circ p)$ for any i. So $\langle x_1 - a_1, \dots, x_n - a_n \rangle \subset \ker(\psi \circ p) = \mathcal{M}$. Since $\langle x_1 - a_1, \dots, x_n - a_n \rangle$ is maximal (see Exercise 4.5.), we conclude the proof of the 1^{st} Step.

2^{nd} **Step** (Weak Nullstellensatz).

Let K be an algebraically closed field, let $\alpha \subset K[x_1, \dots, x_n]$ be a *proper* ideal. Then $V(\alpha) \neq \emptyset$ i.e. the polynomials of α have at least one common zero in $\mathbb{A}^n_K$.

Proof. Since α is proper, there exists a maximal ideal $\mathcal{M}$ containing α. Then $V(\alpha) \supset V(\mathcal{M})$. By 1^{st} Step, $\mathcal{M} = \langle x_1 - a_1, \dots, x_n - a_n \rangle$, so $V(\mathcal{M}) = \{P\}$ with $P(a_1, \dots, a_n)$, hence $P \in V(\alpha)$.

3^{rd} **Step** (Rabinowitch method).

Let K be an algebraically closed field: we will prove that $I(V(\alpha)) \subset \sqrt{\alpha}$. Since the reverse inclusion always holds, this will conclude the proof.

Let $F \in I(V(\alpha))$, $F \neq 0$ and let $\alpha = \langle G_1, \dots, G_r \rangle$. The assumption on F means: if $G_1(P) = \dots = G_r(P) = 0$, then $F(P) = 0$. Let us consider the polynomial ring in $n+1$ variables $K[x_1, \dots, x_{n+1}]$ and let β be the ideal $\beta = \langle G_1, \dots, G_r, x_{n+1}F - 1 \rangle$: β has no zeroes in $\mathbb{A}^{n+1}$, hence, by Step 1, $1 \in \beta$, i.e. there exist $H_1, \dots, H_{r+1} \in K[x_1, \dots, x_{n+1}]$ such that

$$1 = H_1 G_1 + \dots + H_r G_r + H_{r+1}(x_{n+1}F - 1).$$

We introduce the K-homomorphism $\psi : K[x_1, \dots, x_{n+1}] \to K(x_1, \dots, x_n)$ defined by $H(x_1, \dots, x_{n+1}) \to H(x_1, \dots, x_n, \frac{1}{F})$.

The polynomials $G_1, \dots, G_r$ do not contain x_{n+1} so $\psi(G_i) = G_i \; \forall \, i = 1, \dots, r$. Moreover $\psi(x_{n+1}F - 1) = 0$, $\psi(1) = 1$. Therefore

$$1 = \psi(H_1 G_1 + \dots + H_r G_r + H_{r+1}(x_{n+1}F - 1)) = \psi(H_1)G_1 + \dots + \psi(H_r)G_r$$

where $\psi(H_i)$ is a rational function with denominator a power of F. By multiplying this relation by a common denominator, we get an expression of the form:

$$F^m = H'_1 G_1 + \dots + H'_r G_r,$$

so $F \in \sqrt{\alpha}$. □

4.10. Corollaries. *Let K be an algebraically closed field.*

1. There is a bijection between algebraic subsets of $\mathbb{A}^n$ and radical ideals of $K[x_1, \dots, x_n]$. The bijection is given by $\alpha \to V(\alpha)$ and $X \to I(X)$. In fact, if X is closed in the Zariski topology, then $V(I(X)) = X$; if α is a radical ideal, then $I(V(\alpha)) = \alpha$.

2. Let $X, Y \subset \mathbb{A}^n$ be closed sets. Then

(i) $I(X \cap Y) = \sqrt{I(X) + I(Y)}$;

(ii) $I(X \cup Y) = I(X) \cap I(Y) = \sqrt{I(X)I(Y)}$.

Proof. 2. follows from next lemma, using the Nullstellensatz.

4.11. Lemma. *Let α, β be ideals of $K[x_1, \ldots, x_n]$. Then*
a) $\sqrt{\sqrt{\alpha}} = \sqrt{\alpha}$;
b) $\sqrt{\alpha + \beta} = \sqrt{\sqrt{\alpha} + \sqrt{\beta}}$;
c) $\sqrt{\alpha \cap \beta} = \sqrt{\alpha\beta} = \sqrt{\alpha} \cap \sqrt{\beta}$.

Proof.
a) if $F \in \sqrt{\sqrt{\alpha}}$, there exists $r \geq 1$ such that $F^r \in \sqrt{\alpha}$, hence there exists $s \geq 1$ such that $F^{rs} \in \alpha$.
b) $\alpha \subset \sqrt{\alpha}$, $\beta \subset \sqrt{\beta}$ imply $\alpha + \beta \subset \sqrt{\alpha} + \sqrt{\beta}$ hence $\sqrt{\alpha + \beta} \subset \sqrt{\sqrt{\alpha} + \sqrt{\beta}}$.

Conversely, $\alpha \subset \alpha + \beta$, $\beta \subset \alpha + \beta$ imply $\sqrt{\alpha} \subset \sqrt{\alpha + \beta}$, $\sqrt{\beta} \subset \sqrt{\alpha + \beta}$, hence $\sqrt{\alpha} + \sqrt{\beta} \subset \sqrt{\alpha + \beta}$ so $\sqrt{\sqrt{\alpha} + \sqrt{\beta}} \subset \sqrt{\sqrt{\alpha + \beta}} = \sqrt{\alpha + \beta}$.
c) $\alpha\beta \subset \alpha \cap \beta \subset \alpha$ (resp. $\subset \beta$) therefore $\sqrt{\alpha\beta} \subset \sqrt{\alpha \cap \beta} \subset \sqrt{\alpha} \cap \sqrt{\beta}$. If $F \in \sqrt{\alpha} \cap \sqrt{\beta}$, then $F^r \in \alpha$, $F^s \in \beta$ for suitable $r, s \geq 1$, hence $F^{r+s} \in \alpha\beta$, so $F \in \sqrt{\alpha\beta}$. □

Part 2.*(i)* of 4.10. implies that, i$I(X \cap Y) \neq I(X) + I(Y)$, if and only if $I(X) + I(Y)$ is not radical.

We move now to projective space. There exist *proper* homogeneous ideals of $K[x_0, x_1, \ldots, x_n]$ without zeroes in $\mathbb{P}^n$, also assuming K algebraically closed: for example the maximal ideal $\langle x_0, x_1, \ldots, x_n \rangle$. The following characterization holds:

4.12. Proposition. *Let K be an algebraically closed field and let I be a homogeneous ideal of $K[x_0, x_1, \ldots, x_n]$.*

The following are equivalent:
(i) $V_P(I) = \emptyset$;
(ii) either $I = K[x_0, x_1, \ldots, x_n]$ *or* $\sqrt{I} = \langle x_0, x_1, \ldots, x_n \rangle$;
(iii) $\exists d \geq 1$ *such that* $I \supset K[x_0, x_1, \ldots, x_n]_d$, *the subgroup of* $K[x_0, x_1, \ldots, x_n]$ *formed by the homogeneous polynomials of degree* d.

Proof.

(i)⇒(ii) Let $p : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$ be the canonical surjection. We have: $V_P(I) = p(V(I) - \{0\})$, where $V(I) \subset \mathbb{A}^{n+1}$. So if $V_P(I) = \emptyset$, then either $V(I) = \emptyset$ or $V(I) = \{0\}$. If $V(I) = \emptyset$ then $I(V(I)) = I(\emptyset) = K[x_0, x_1, \ldots, x_n]$; if $V(I) = \{0\}$, then $I(V(I)) = \langle x_0, x_1, \ldots, x_n \rangle = \sqrt{I}$ by the Nullstellensatz.

(ii)⇒(iii) Let $\sqrt{I} = K[x_0, x_1, \ldots, x_n]$, then $1 \in \sqrt{I}$ so $1^r = 1 \in I (r \geq 1)$. If $\sqrt{I} = \langle x_0, x_1, \ldots, x_n \rangle$, then for any variable x_k there exists an index $i_k \geq 1$ such that $x_k^{i_k} \in I$. If $d \geq i_0 + i_1 + \ldots + i_n$, then any monomial of degree d is in I, so $K[x_0, x_1, \ldots, x_n]_d \subset I$.

(iii)⇒(i) because no point in $\mathbb{P}^n$ has all coordinates equal to 0. □

4.13. Theorem. *Let K be an algebraically closed field and I be a homogeneous* *ideal of $K[x_0, x_1, \dots, x_n]$. If F is a homogeneous non–constant polynomial such that $V_P(F) \supset V_P(I)$ (i.e. F vanishes on $V_P(I)$, then $F \in \sqrt{I}$.*

Proof. We have $p(V(I) - \{0\}) = V_P(I) \subset V_P(F)$. Since F is non–constant, we have also $V(F) = p^{-1}(V_P(F)) \cup \{0\}$, so $V(F) \supset V(I)$; by the Nullstellensatz $I(V(I)) = \sqrt{I} \supset I(V(F)) = \sqrt{(F)} \ni F$. □

4.14. Corollary (homogeneous Nullstellensatz). *Let I be a homogeneous ideal of $K[x_0, x_1, \dots, x_n]$ such that $V_P(I) \neq \emptyset$, K algebraically closed. Then $\sqrt{I} = I_h(V_P(I))$.* □

4.15. Definition. A homogeneous ideal of $K[x_0, x_1, \dots, x_n]$ such that $\sqrt{I} = \langle x_0, x_1, \dots, x_n \rangle$ is called *irrelevant*.

4.16. Corollary. *Let K be an algebraically closed field. There is a bijection between the set of projective algebraic subsets of $\mathbb{P}^n$ and the set of radical homogeneous non–irrelevant ideals of $K[x_0, x_1, \dots, x_n]$.* □

Remark. Let $X \subset \mathbb{P}^n$ be an algebraic set, $X \neq \emptyset$. The affine cone of X, denoted $C(X)$, is the following subset of $\mathbb{A}^{n+1}$: $C(X) = p^{-1}(X) \cup \{0\}$. If $X = V_P(F_1, \dots, F_r)$, with $F_1, \dots, F_r$ homogeneous, then $C(X) = V(F_1, \dots, F_r)$. By the Nullstellensatz, if K is algebraically closed, $I(C(X)) = I_h(X)$.

Exercises to §4.

1. Give a non-trivial example of an ideal α of $K[x_1, \dots, x_n]$ such that $\alpha \neq \sqrt{\alpha}$.

2. Show that the following closed subsets of the affine plane $Y = V(x^2+y^2-1)$ and $Y' = V(y-1)$ are such that equality does not hold in the following relation: $I(Y \cap Y') \supset I(Y) + I(Y')$.

3. Let $\alpha \subset K[x_1, \dots, x_n]$ be an ideal. Prove that $\alpha = \sqrt{\alpha}$ if and only if the quotient ring $K[x_1, \dots, x_n]/\alpha$ does not contain any non–zero nilpotent.

4. Consider $\mathbb{Z} \subset \mathbb{Q}$. Prove that if an element $y \in \mathbb{Q}$ is integral over $\mathbb{Z}$, then $y \in \mathbb{Z}$.

5. Let $a_1, \dots a_n \in K$ (K any field). Prove that the ideal

$$I = \langle x_1 - a_1, \dots, x_n - a_n \rangle$$

is maximal. (Hint: every polynomial F can be written in the form

$$F = F(a_1, \dots, a_n) + \sum F_i(a_1, \dots, a_n)(x_i - a_i) + \dots,$$

where F_i is the i-th partial derivative of F. If $F \notin I \dots$

Remember that it makes sense to consider derivatives of polynomials over any field.)

6. Let us recall that a prime ideal of a ring R is an ideal $\mathcal{P}$ such that $a \notin \mathcal{P}$, $b \notin \mathcal{P}$ implies $ab \notin \mathcal{P}$. Prove that any prime ideal is a radical ideal.

7*. Let I be a homogeneous ideal of $K[x_1, \dots, x_n]$ satisfying the following condition: *if F is a homogeneous polynomial such that $F^r \in I$ for some positive integer r, then $F \in I$.* Prove that I is a radical ideal.

5. The projective closure of an affine algebraic set.

Let $X \subset \mathbb{A}^n$ be Zariski closed. Fix an index $i \in \{0, \dots, n\}$ and embed $\mathbb{A}^n$ into $\mathbb{P}^n$ as the open subset U_i. So $X \subset \mathbb{A}^n \overset{\phi_i}{\hookrightarrow} \mathbb{P}^n$.

5.1. Definition. The *projective closure* of X, $\overline{X}$, is the closure of X in the Zariski topology of $\mathbb{P}^n$.

Since the map ϕ_i is a homeomorphism (see Proposition 3.2.), we have: $\overline{X} \cap \mathbb{A}^n = X$ because X is closed in $\mathbb{A}^n$. The points of $\overline{X} \cap H_i$, where $H_i = V_P(x_i)$, are called the "points at infinity" of X in the fixed embedding.

Note that, if K is an infinite field, then the projective closure of $\mathbb{A}^n$ is $\mathbb{P}^n$: indeed, let F be a homogeneous polynomial vanishing along $\mathbb{A}^n = U_0$. We can write $F = F_0 x_0^d + F_1 x_0^{d-1} + \dots + F_d$. By assumption, for every $P(a_1, \dots, a_n) \in \mathbb{A}^n$, $P \in V_P(F)$, i.e. $F(1, a_1, \dots, a_n) = 0 = {}^aF(a_1, \dots, a_n)$. So ${}^aF \in I(\mathbb{A}^n)$. We claim that $I(\mathbb{A}^n) = (0)$: if $n = 1$, this follows from the principle of identity of polynomials, because K is infinite. If $n \geq 2$, assume that $F(a_1, ..., a_n) = 0$ for all $(a_1, ..., a_n) \in K^n$ and consider $F(a_1, ..., a_{n-1}, x)$: either it has positive degree in x for some choice of $(a_1, ..., a_n)$, but then it has finitely many zeroes against the assumption; or it is always constant in x, so F belongs to $K[x_1, ..., x_{n-1}]$ and we can conclude by induction. So the claim is proved. We get therefore that $F_0 = F_1 = \dots = F_d = 0$ and $F = 0$.

5.2. Proposition. *Let $X \subset \mathbb{A}^n$ be an affine algebraic set, $\overline{X}$ be the projective closure of X. Then*

$$I_h(\overline{X}) = {}^hI(X) := \langle {}^hF | F \in I(X) \rangle.$$

Proof. Assume $\mathbb{A}^n = U_0 \subset \mathbb{P}^n$.

Let $F \in I_h(\overline{X})$ be a homogeneous polynomial. If $P(a_1, \dots, a_n) \in X$, then $[1, a_1, \dots, a_n] \in \overline{X}$, so $F(1, a_1, \dots, a_n) = 0 = {}^aF(a_1, \dots, a_n)$. Hence ${}^aF \in X$. There exists $k \geq 0$ such that $F = (x_0^k)^h({}^aF)$ (see Proposition 3.2), so $F \in {}^hI(X)$. Hence $I_h(\overline{X}) \subset {}^hI(X)$.

Conversely, if $G \in I(X)$ and $P(a_1, \dots, a_n) \in X$, then $G(a_1, \dots, a_n) = 0 = {}^hG(1, a_1, \dots, a_n)$, so ${}^hG \in I_h(X)$ (here X is seen as a subset of $\mathbb{P}^n$). So ${}^hI(X) \subset I_h(X)$. Since $I_h(X) = I_h(\overline{X})$ (see Exercise 5.1), we have the claim.

□

In particular, if X is a hypersurface and $I(X) = \langle F \rangle$, then $I_h(\overline{X}) = \langle {}^hF \rangle$.

Next example will show that, *in general*, it is not true that, if $I(X) = \langle F_1, \dots, F_r \rangle$, then ${}^hI(X) = \langle {}^hF_1, \dots, {}^hF_r \rangle$. Only in the last twenty years, thanks to the development of symbolic algebra and in particular of the theory of Groebner bases, the problem of characterizing the systems of generators of $I(X)$, whose homogeneization generates ${}^hI(X)$, has been solved.

5.3. Example. *The skew cubic.*

Let K be an algebraically closed field. The affine skew cubic is the following closed subset of $\mathbb{A}^3$: $X = V(y - x^2, z - x^3)$ (we use variables x, y, z). X is the image of the map $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ such that $\phi(t) = (t, t^2, t^3)$. Note that $\phi : \mathbb{A}^1 \to X$ is a homeomorphism (see Exercise 2.4). The ideal $\alpha = \langle y - x^2, y - x^3 \rangle$ defines X and is prime: indeed the quotient ring $K[x, y, z]/\alpha$ is isomorphic to $K[x]$, hence an integral domain. Therefore α is radical so $\alpha = I(X)$.

Let $\overline{X}$ be the projective closure of X in $\mathbb{P}^3$. We are going to prove that $\overline{X}$ is the image of the map $\psi : \mathbb{P}^1 \to \mathbb{P}^3$ such that $\psi([\lambda, \mu]) = [\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3]$. We identify $\mathbb{A}^1$ with the open subset of $\mathbb{P}^1$ defined by $\lambda \neq 0$ i.e. U_0, and $\mathbb{A}^3$ with the open subset of $\mathbb{P}^3$ defined by $x_0 \neq 0$ (U_0 too). Note that $\psi|_{\mathbb{A}^1} = \phi$, because $\psi([1, t]) = [1, t, t^2, t^3] =$ via the identification of $\mathbb{A}^3$ with $U_0 = (t, t^2, t^3) = \phi(t)$. Moreover $\psi([0, 1]) = [0, 0, 0, 1]$. So $\psi(\mathbb{P}^1) = X \cup \{[0, 0, 0, 1]\}$.

If G is a homogeneous polynomial of $K[x_0, x_1, \dots, x_3]$ such that $X \subset V_P(G)$, then $G(1, t, t^2, t^3) = 0\ \forall t \in K$, so $G(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3) = 0\ \forall \mu \in K$, $\forall \lambda \in K^*$. Since K is infinite, then $G(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3)$ is the zero polynomial in λ and μ, so $G(0, 0, 0, 1) = 0$ and $V_P(G) \supset \psi(\mathbb{P}^1)$, therefore $\overline{X} \supset \psi(\mathbb{P}^1)$.

Conversely, it is easy to prove that $\psi(\mathbb{P}^1)$ is Zariski closed, in fact that $\psi(\mathbb{P}^1) = V_P(x_1^2 - x_0x_2, x_1x_2 - x_0x_3, x_2^2 - x_1x_3)$. So $\psi(\mathbb{P}^1) = \overline{X}$.

The three polynomials $F_0 := x_1x_3 - x_2^2$, $F_1 := x_1x_2 - x_0x_3$, $F_2 := x_0x_2 - x_1^2$ are the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

with entries in $K[x_0, x_1, \dots, x_3]$. Let $F = y - x^2$, $G = z - x^3$ be the two generators of $I(X)$; ${}^hF = x_0x_2 - x_1^2$, ${}^hG = x_0^2x_3 - x_1^3$, hence $V_P({}^hF, {}^hG) = V_P(x_0x_2 - x_1^2, x_0^2x_3 - x_1^3) \neq \overline{X}$, because $V_P({}^hF, {}^hG)$ contains the whole line $V_P(x_0, x_1)$.

We shall prove now the non-trivial fact:

5.4. Proposition. $I_h(\overline{X}) = \langle F_0, F_1, F_2 \rangle$.

Proof. For all integer number $d \geq 0$, let $I_h(\overline{X})_d := I_h(\overline{X}) \cap K[x_0, x_1, \dots, x_3]_d$: it is a K-vector space of dimension $\leq \binom{d+3}{3}$. We define a K-linear map ρ_d having $I_h(\overline{X})_d$ as kernel:

$$\rho_d : K[x_0, x_1, \dots, x_3]_d \to K[\lambda, \mu]_{3d}$$

such that $\rho_d(F) = F(\lambda^3, \lambda^2\mu, \lambda^2\mu^2, \mu^3)$. Since ρ_d is clearly surjective, we compute

$$\dim I_h(\overline{X})_d = \binom{d+3}{3} - (3d+1) = (d^3 + 6d^2 - 7d)/6.$$

For $d \geq 2$, we define now a second K-linear map

$$\phi_d : K[x_0, x_1, \dots, x_3]_{d-2} \oplus K[x_0, x_1, \dots, x_3]_{d-2} \oplus K[x_0, x_1, \dots, x_3]_{d-2} \to I_h(\overline{X})_d$$

such that $\phi_d(G_0, G_1, G_2) = G_0F_0 + G_1F_1 + G_2F_2$. Our aim is to prove that ϕ_d is surjective. The elements of its kernel are called the *syzygies of degree* d among the polynomials F_0, F_1, F_2. Two obvious syzygies of degree 3 are constructed by developing, according to the Laplace rule, the determinant of the matrix obtained repeating one of the rows of M, for example

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

We put $H_1 = (x_0, x_1, x_2)$ and $H_2 = (x_1, x_2, x_3)$, they both belong to $\ker \phi_3$. Note that H_1 and H_2 give raise to syzygies of all degrees ≥ 3, in fact we can construct a third linear map

$$\psi_d : K[x_0, x_1, \dots, x_3]_{d-3} \oplus K[x_0, x_1, \dots, x_3]_{d-3} \to \ker \phi_d$$

putting $\psi_d(A, B) = H_1A + H_2B = (x_0, x_1, x_2)A + (x_1, x_2, x_3)B = (x_0A + x_1B, x_1A + x_2B, x_2A + x_3B)$.

Claim. ψ_d *is an isomorphism.*
Assuming the claim, we are able to compute $\dim \ker \phi_d = 2\binom{d}{3}$, therefore

$$\dim Im\ \phi_d = 3\binom{d+1}{3} - 2\binom{d}{3}$$

which coincides with the dimension of $I_h(\overline{X})_d$ previously computed. This proves that ϕ_d is surjective for all d and concludes the proof of the Proposition.

Proof of the Claim. Let (G_0, G_1, G_2) belong to $\ker \phi_d$. This means that the following matrix N with entries in $K[x_0, x_1, \dots, x_3]$ is degenerate:

$$N := \begin{pmatrix} G_0 & G_1 & G_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

Therefore, the rows of N are linearly dependent over the quotient field of the polynomial ring $K(x_0, \dots, x_3)$. Since the last two rows are independent, there exist reduced rational functions $\frac{a_1}{a_0}, \frac{b_1}{b_0} \in K(x_0, x_1, x_2, x_3)$, such that

$$G_0 = \frac{a_1}{a_0}x_0 + \frac{b_1}{b_0}x_1 = \frac{a_1b_0x_0 + a_0b_1x_1}{a_0b_0}$$

and similarly

$$G_1 = \frac{a_1b_0x_1 + a_0b_1x_2}{a_0b_0}, G_2 = \frac{a_1b_0x_2 + a_0b_1x_3}{a_0b_0}$$

The G_i's are polynomials, therefore the denominator a_0b_0 divides the numerator in each of the three expressions on the right hand side. Moreover, if p is a prime factor of a_0, then p divides the three products b_0x_0, b_0x_1, b_0x_2, hence p divides b_0. We can repeat the reasoning for a prime divisor of b_0, so obtaining that $a_0 = b_0$ (up to invertible constants). We get:

$$G_0 = \frac{a_1x_0 + b_1x_1}{b_0}, G_1 = \frac{a_1x_1 + b_1x_2}{b_0}, G_2 = \frac{a_1x_2 + b_1x_3}{b_0},$$

therefore b_0 divides the numerators

$$c_0 := a_1x_0 + b_1x_1, c_1 := a_1x_1 + b_1x_2, c_2 := a_1x_2 + b_1x_3.$$

Hence b_0 divides also $x_1c_0 - x_0c_1 = b_1(x_1^2 - x_0x_1) = -b_1F_2$, and similarly $x_2c_0 - x_0c_2 = b_1F_1$, $x_2c_1 - x_1c_2 = -b_1F_0$. But F_0, F_1, F_2 are irreducible and coprime, so we conclude that $b_0 \mid b_1$. But b_0 and b_1 are coprime, so finally we get $b_0 = a_0 = 1$. $\square$

As a by-product of the proof of Proposition 5.4 we have the minimal free resolution of the R-module $I_h(\overline{X})$, where $R = K[x_0, x_1, \dots, x_3]$:

$$0 \to R^{\oplus 2} \xrightarrow{\psi} R^{\oplus 3} \xrightarrow{\phi} I_h(\overline{X}) \to 0$$

where ψ is represented by the transposed of the matrix M and ϕ by the triple of polynomials (F_0, F_1, F_2).

Exercises to §5.

1*. Let $X \subset \mathbb{A}^n$ be a closed subset, $\overline{X}$ be its projective closure in $\mathbb{P}^n$. Prove that $I_h(X) = I_h(\overline{X})$.

2. Find a system of generators of the ideal of the affine skew cubic X, such that, if you homogeneize them, you get a system of generators for $I_h(\overline{X})$.

6. Irreducible components.

6.1. Definition. Let $X \neq \emptyset$ be a topological space. X is *irreducible* if the following condition holds: if X_1, X_2 are closed subsets of X such that $X = X_1 \cup X_2$, then either $X = X_1$ or $X = X_2$. Equivalently, X is irreducible if for all pair of non–empty open subsets U, V we have $U \cap V \neq \emptyset$. By definition, $\emptyset$ is not irreducible.

6.2. Proposition. *X is irreducible if and only if any non–empty open subset U of X is dense.*

Proof. Let X be irreducible, let P be a point of X and I_P be an open neighbourhood of P in X. I_P and U are non–empty and open, so $I_P \cap U \neq \emptyset$, therefore $P \in \overline{U}$. This proves that $\overline{U} = X$.

Conversely, assume that open subsets are dense. Let U, $V \neq \emptyset$ be open subsets. Let $P \in U$ be a point. By assumption $P \in \overline{V} = X$, so $V \cap U \neq \emptyset$ (U is an open neighbourhood of P). $\square$

Examples.

1. If $X = \{P\}$ a unique point, then X is irreducible.

2. Let K be an infinite field. Then $\mathbb{A}^1$ is irreducible, because proper closed subsets are finite sets. The same holds for $\mathbb{P}^1$.

3. Let $f : X \to Y$ be a continuous map of topological spaces. If X is irreducible and f is surjective, then Y is irreducible.

4. Let $Y \subset X$ be a subset, give it the induced topology. Then Y is irreducible if and only if the following holds: if $Y \subset Z_1 \cup Z_2$, with Z_1 and Z_2 closed in X, then either $Y \subset Z_1$ or $Y \subset Z_2$; equivalently: if $Y \cap U \neq \emptyset$, $Y \cap V \neq \emptyset$, with U, V open subsets of X, then $Y \cap U \cap V \neq \emptyset$.

6.3. Proposition. *Let X be a topological space, Y a subset of X. Y is irreducible if and only if $\overline{Y}$ is irreducible.*

Proof. Note first that if $U \subset X$ is open and $U \cap Y = \emptyset$ then $\overline{U} \cap \overline{Y} = \emptyset$. Otherwise, if $P \in U \cap \overline{Y}$, let A be an open neighbourhood of P: then $A \cap Y \neq \emptyset$. In particular, U is an open neighbourhood of P so $U \cap Y \neq \emptyset$.

Let Y be irreducible. If U and V are open subsets of X such that $U \cap \overline{Y} \neq \emptyset$, $V \cap \overline{Y} \neq \emptyset$, then $U \cap Y \neq \emptyset$ and $V \cap Y \neq \emptyset$ so $Y \cap U \cap V \neq \emptyset$ by irreducibility of Y. Hence $\overline{Y} \cap (U \cap V) \neq \emptyset$. So $\overline{Y}$ is irreducible. If $\overline{Y}$ is irreducible, we get the irreducibility of Y in a completely analogous way. $\square$

6.4. Corollary. *Let X be an irreducible topological space and U be a non–empty open subset of X. Then U is irreducible.*

Proof. By Proposition 6.2 $\overline{U} = X$ which is irreducible. By Proposition 6.3 U is irreducible. $\square$

For algebraic sets (both affine and projective) irreducibility can be expressed in a purely algebraic way.

6.5. Proposition. *Let $X \subset \mathbb{A}^n$ (resp. $\mathbb{P}^n$) be an algebraic set. X is irreducible if and only if $I(X)$ (resp. $I_h(X)$) is prime.*

Proof. Assume first that X is irreducible, $X \subset \mathbb{A}^n$. Let F, G polynomials of $K[x_1, \dots, x_n]$ such that $FG \in I(X)$: then

$$V(F) \cup V(G) = V(FG) \supset V(I(X)) = X$$

hence either $X \subset V(F)$ or $X \subset V(G)$. In the former case, if $P \in X$ then $F(P) = 0$, so $F \in I(X)$, in the second case $G \in I(X)$; hence $I(X)$ is prime.

Assume now that $I(X)$ is prime. Let $X = X_1 \cup X_2$ be the union of two closed subsets. Then $I(X) = I(X_1) \cap I(X_2)$ (see §4). Assume that $X_1 \neq X$, then $I(X_1)$ strictly contains $I(X)$ (otherwise $V(I(X_1)) = V(I(X))$. So there exists $F \in I(X_1)$ such that $F \notin I(X)$. But for every $G \in I(X_2)$, $FG \in I(X_1) \cap I(X_2) = I(X)$ prime: since $F \notin I(X)$, then $G \in I(X)$. So $I(X_2) \subset I(X)$ hence $I(X_2) = I(X)$.

If $X \subset \mathbb{P}^n$, the proof is similar, taking into account the following:

6.6. Lemma *Let $\mathcal{P} \subset K[x_0, x_1, \dots, x_n]$ be a homogeneous ideal. Then $\mathcal{P}$ is prime if and only if, for every pair of homogeneous polynomials F, G such that $FG \in \mathcal{P}$, either $F \in \mathcal{P}$ or $G \in \mathcal{P}$.*

Proof of the Lemma. Let H, K be any polynomials such that $HK \in \mathcal{P}$. Let $H = H_0 + H_1 + \ldots + H_d$, $K = K_0 + K_1 + \ldots + K_e$ (with $H_d \neq 0 \neq K_e$) be their expressions as sums of homogeneous polynomials. Then $HK = H_0K_0 + (H_0K_1 + H_1K_0) + \ldots + H_dK_e$: the last product is the homogeneous component of degree $d+e$ of HK. $\mathcal{P}$ being homogeneous, $H_dK_e \in \mathcal{P}$; by assumption either $H_d \in \mathcal{P}$ or $K_e \in \mathcal{P}$. In the former case, $HK - H_dK = (H - H_d)K$ belongs to $\mathcal{P}$ while in the second one $H(K - K_e) \in \mathcal{P}$. So in both cases we can proceed by induction. □

We list now some consequences of the previous Proposition.

1. Let K be an infinite field. Then $\mathbb{A}^n$ and $\mathbb{P}^n$ are irreducible, because $I(\mathbb{A}^n) = I_h(\mathbb{P}^n) = (0)$.

2. Let $Y \subset \mathbb{P}^n$ be closed. Y is irreducible if and only if its affine cone $C(Y)$ is irreducible.

3. Let $Y = V(F) \subset \mathbb{A}^n$, be a hypersurface over an algebraically closed field K. If F is irreducible, then Y is irreducible.

4. Let K be algebraically closed. There is a bijection between prime ideals of $K[x_1, \dots, x_n]$ and irreducible algebraic subsets of $\mathbb{A}^n$. In particular, the maximal ideals correspond to the points. Similarly, there is a bijection between homogeneous non–irrelevant prime ideals of $K[x_0, x_1, \dots, x_n]$ and irreducible algebraic subsets of $\mathbb{P}^n$.

6.7. Definition. A topological space X is called *noetherian* if it satisfies the following equivalent conditions:

(i) the ascending chain condition for open subsets;

(ii) the descending chain condition for closed subsets;

(iii) any non–empty set of open subsets of X has maximal elements;

(iv) any non–empty set of closed subsets of X has minimal elements.

The proof of the equivalence is standard.

Example. $\mathbb{A}^n$ is noetherian: if the following is a descending chain of closed subsets

$$Y_1 \supset Y_2 \supset \ldots \supset Y_k \supset \ldots,$$

then

$$I(Y_1) \subset I(Y_2) \subset \ldots \subset I(Y_k) \subset \ldots$$

is an ascending chain of ideals of $K[x_1, \ldots, x_n]$ hence stationary from a suitable m on; therefore $V(I(Y_m)) = Y_m = V(I(Y_m)) = Y_{m+1} = \ldots$.

6.8. Proposition. *Let X be a noetherian topological space and Y be a non–empty closed subset of X. Then Y can be written as a finite union $Y = Y_1 \cup \ldots \cup Y_r$ of irreducible closed subsets. The maximal Y_i's in the union are uniquely determined by Y and called the "irreducible components" of Y. They are the maximal irreducible subsets of Y.*

Proof. By contradiction. Let S be the set of the non–empty closed subsets of X which are not a finite union of irreducible closed subsets: assume $S \neq \emptyset$. By noetherianity S has minimal elements, fix one of them Z. Z is not irreducible, so $Z = Z_1 \cup Z_2$, $Z_i \neq Z$ for $i = 1, 2$. So $Z_1, Z_2 \notin S$, hence Z_1, Z_2 are both finite unions of irreducible closed subsets, so such is Z: a contradiction.

Now assume that $Y = Y_1 \cup \ldots \cup Y_r$, with $Y_i \not\subseteq Y_j$ if $i \neq j$ and Y_i irreducible closed for all i. If there is another similar expression $Y = Y'_1 \cup \ldots \cup Y'_s$, $Y'_i \not\subseteq Y'_j$ for $i \neq j$, then $Y'_1 \subset Y_1 \cup \ldots Y_r$, so $Y'_1 = \bigcup_{i=1}^r (Y'_1 \cup Y_i)$, hence $Y'_1 \subset Y_i$ for some i, and we can assume $i = 1$. Similarly, $Y_1 \subset Y'_j$, for some j, so $Y'_1 \subset Y_1 \subset Y'_j$, so $j = 1$ and $Y_1 = Y'_1$. Now let $Z = \overline{Y - Y_1} = Y_2 \cup \ldots \cup Y_r = Y'_2 \cup \ldots \cup Y'_s$ and proceed by induction.

□

6.9. Corollary. *Any algebraic subset of $\mathbb{A}^n$ (resp. of $\mathbb{P}^n$) is in a unique way the finite union of its irreducible components.* □

Note that the irreducible components of X are its maximal algebraic subsets. They correspond to the minimal prime ideals over $I(X)$. Since $I(X)$ is radical, these minimal prime ideals coincide with the primary ideals appearing in the primary decomposition of $I(X)$.

6.10. Definition. An irreducible closed subset of $\mathbb{A}^n$ is called an *affine variety*. Similarly, an irreducible closed subset of $\mathbb{P}^n$ is a *projective variety*. A locally closed subset in $\mathbb{P}^n$ is the intersection of an open and a closed subset. An irreducible locally closed subset of $\mathbb{P}^n$ is a *quasi–projective variety*.

6.11. Proposition. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be affine varieties. Then $X \times Y$ is irreducible, i.e. a subvariety of $\mathbb{A}^{n+m}$.*

Proof. Let $X \times Y = W_1 \cup W_2$, with W_1, W_2 closed. For all $P \in X$ the map $\{P\} \times Y \to Y$ which takes (P, Q) to Q is a homeomorphism, so $\{P\} \times Y$ is irreducible. $\{P\} \times Y = (W_1 \cap (\{P\} \times Y)) \cup (W_2 \cap (\{P\} \times Y))$, so $\exists i \in \{1, 2\}$ such that $\{P\} \times Y \subset W_i$. Let $X_i = \{P \in X \mid \{P\} \times Y \subset W_i\}$, $i = 1, 2$. Note that $X = X_1 \cup X_2$.

Claim. X_i is closed in X.

Let $X^i(Q) = \{P \in X \mid (P, Q) \in W_i\}$, $Q \in Y$. We have: $(X \times \{Q\}) \cap W_i = X^i(Q) \times \{Q\} \simeq X^i(Q)$; $X \times \{Q\}$ and W_i are closed in $X \times Y$, so $X^i(Q) \times \{Q\}$ is closed in $X \times Y$ and also in $X \times \{Q\}$, so $X^i(Q)$ is closed in X. Note that $X_i = \bigcap_{Q \in Y} X^i(Q)$, hence X_i is closed, which proves the Claim.

Since X is irreducible, $X = X_1 \cup X_2$ implies that either $X = X_1$ or $X = X_2$, so either $X \times Y = W_1$ or $X \times Y = W_2$. $\square$

Exercises to §6.

1. Let $X \neq \emptyset$ be a topological space. Prove that X is irreducible if and only if all non–empty open subsets of X are connected.

2*. Prove that the *cuspidal cubic* $Y \subset \mathbb{A}^2_{\mathbb{C}}$ of equation $x^3 - y^2 = 0$ is irreducible. (Hint: express Y as image of $\mathbb{A}^1$ in a continuous map...)

3. Give an example of two irreducible subvarieties of $\mathbb{P}^3$ whose intersection is reducible.

4. Find the irreducible components of the following algebraic sets over the complex field:

a) $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subset \mathbb{A}^2$;

b) $V(y^2 - xz, z^2 - y^3) \subset \mathbb{A}^3$.

5*. Let Z be a topological space and $\{U_\alpha\}_{\alpha \in I}$ be an open covering of Z such that $U_\alpha \cap U_\beta \neq \emptyset$ for $\alpha \neq \beta$ and that all U_α's are irreducible. Prove that Z is irreducible.

7. Dimension.

Let X be a topological space.

7.1. Definition. The *topological dimension* of X is the supremum of the lengths of the chains of distinct irreducible closed subsets of X, where by definiton the following chain has length n:

$$X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n.$$

The topological dimension of X is denoted by $\dim X$. It is also called combinatorial or Krull dimension.

Example.
1. $\dim \mathbb{A}^1 = 1$: the maximal length chains have the form $\{P\} \subset \mathbb{A}^1$.
2. $\dim \mathbb{A}^n = n$: a chain of length n is

$$\{0\} = V(x_1, \ldots, x_n) \subset V(x_1, \ldots, x_{n-1}) \subset \ldots \subset V(x_1) \subset \mathbb{A}^n;$$

note that $V(x_1, \ldots, x_i)$ is irreducible for any $i \leq n$, because the ideal $\langle x_1, \ldots, x_i \rangle$ is prime. Indeed the quotient ring $K[x_1, \ldots, x_n]/\langle x_1, \ldots, x_i \rangle$ is isomorphic to $K[x_{i+1}, \ldots, x_n]$. Therefore $\dim \mathbb{A}^n \geq n$. On the other hand, from every chain of irreducible closed subsets of $\mathbb{A}^n$, passing to their ideals, we get a chain of the same length of prime ideals in $K[x_1, \ldots, x_n]$.

We define the Krull dimension of a ring A, and denote it by $\dim A$, to be the supremum of the lengths of the chains of distinct prime ideals of A. Therefore, we can reformulate the previous fact by saying that $\dim \mathbb{A}^n \leq \dim K[x_1, \ldots, x_n]$. We will see in a next chapter that $\dim K[x_1, \ldots, x_n] = n$. More in general, if A is a noetherian ring, then $\dim A[x] = \dim A + 1$.

3. Let X be irreducible. Then $\dim X = 0$ if and only if X is the closure of every point of it.

We prove now some useful relations between the dimension of X and the dimensions of its subspaces.

7.2. Proposition.
1. If $Y \subset X$, then $\dim Y \leq \dim X$. In particular, if $\dim X$ is finite, then also $\dim Y$ is (in this case, the number $\dim X - \dim Y$ is called the codimension of Y in X).

2. If $X = \bigcup_{i \in I} U_i$ is an open covering, then $\dim X = \sup\{\dim U_i\}$.

3. If X is noetherian and $X_1, \ldots, X_s$ are its irreducible components, then $\dim X = \sup_i \dim X_i$.

4. If $Y \subset X$ is closed, X is irreducible, $\dim X$ is finite and $\dim X = \dim Y$, then $Y = X$.

Proof.

1. Let $Y_0 \subset Y_1 \subset \ldots \subset Y_n$ be a chain of irreducible closed subsets of Y. Then their closures are irreducible and form the following chain: $\overline{Y_0} \subseteq \overline{Y_1} \subseteq \ldots \subseteq \overline{Y_n}$. Note that for all i $\overline{Y_i} \cap Y = Y_i$, because Y_i is closed into Y, so if $\overline{Y_i} = \overline{Y_{i+1}}$, then $Y_i = Y_{i+1}$. Therefore the two chains have the same length and we can conclude that $\dim Y \leq \dim X$.

2. Let $X_0 \subset X_1 \subset \ldots \subset X_n$ be a chain of irreducible closed subsets of X. Let $P \in X_0$ be a point: there exists an index $i \in I$ such that $P \in U_i$. So $\forall k = 0, \ldots, n$ $X_k \cap U_i \neq \emptyset$: it is an irreducible closed subset of U_i, irreducible because open in X_k which is irreducible. Consider $X_0 \cap U_i \subset X_1 \cap U_i \subset \ldots \subset X_n \cap U_i$; it is a chain of length n, because $\overline{X_k \cap U_i} = X_k$: in fact $X_k \cap U_i$ is open in X_k hence dense. Therefore, for all chain of irreducible closed subsets of X, there exists a chain of the same length of irreducible closed subsets of some U_i. So $\dim X \leq \sup \dim U_i$. By 1., equality holds.

3. Any chain of irreducible closed subsets of X is completely contained in an irreducible component of X. The conclusion follows as in 2.

4. If $Y_0 \subset Y_1 \subset \ldots \subset Y_n$ is a chain of maximal length in Y, then it is a maximal chain in X, because $\dim X = \dim Y$. Hence $X = Y_n \subset Y$. □

7.3. Corollary. $\dim \mathbb{P}^n = \dim \mathbb{A}^n$.

Proof. Because $\mathbb{P}^n = U_0 \cup \ldots \cup U_n$, and U_i is homeomorphic to $\mathbb{A}^n$ for all i. □

If X is noetherian and all its irreducible components have the same dimension r, then X is said to have *pure dimension* r.

Note that the topological dimension is invariant by homeomorphism. By definition, a *curve* is an algebraic set of pure dimension 1; a *surface* is an algebraic set of pure dimension 2.

We want to study the dimensions of affine algebraic sets. The following definition results to be very important.

7.4. Definition. Let $X \subset \mathbb{A}^n$ be an algebraic set. The *coordinate ring* of X is

$$K[X] := K[x_1, \ldots, x_n]/I(X).$$

It is a finitely generated K–algebra that has no non–zero nilpotents, because $I(X)$ is radical. This can be expressed by saying that $K[X]$ is a *reduced ring*. There is the canonical epimorphism $K[x_1, \ldots, x_n] \to K[X]$ such that $F \to [F]$. The elements of $K[X]$ can be interpreted as *polynomial functions* on X: to a polynomial F, we can associate the function $f : X \to K$ such that $P(a_1, \ldots, a_n) \to F(a_1, \ldots, a_n)$.

Two polynomials F, G define the same function on X if, and only if, $F(P) = G(P)$ for every point $P \in X$, i.e. if $F - G \in I(X)$, which means exactly that F and G have the same image in $K[X]$.

$K[X]$ is generated as K–algebra by $[x_1], \ldots, [x_n]$: these can be interpreted as the functions on X called *coordinate functions*, and generally denoted $t_1, \ldots, t_n$.

In fact $t_i : X \to K$ is the function which associates to $P(a_1, \dots, a_n)$ the constant a_i. Note that the function f can be interpreted as $F(t_1, \dots, t_n)$: the polynomial F evalued at the n– tuple of the coordinate functions.

In the projective space we can do an analogous construction. If $Y \subset \mathbb{P}^n$ is closed, then the *homogeneous coordinate ring* of Y is

$$S(Y) := K[x_0, x_1, \dots, x_n]/I_h(Y).$$

It is also a finitely generated K–algebra, but its elements have no interpretation as functions on Y. They are functions on the cone $C(Y)$.

7.5. Definition. Let R be a ring. The *Krull dimension* of R is the supremum of the lengths of the chains of prime ideals of R

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \dots \subset \mathcal{P}_r.$$

Similarly, the *heigth* of a prime ideal $\mathcal{P}$ is the sup of the lengths of the chains of prime ideals contained in $\mathcal{P}$: it is denoted $\mathrm{ht}\mathcal{P}$.

7.6. Proposition. *Let K be an algebraically closed field. Let X be an affine algebraic set contained in $\mathbb{A}^n$. Then* $\dim X = \dim K[X]$.

Proof.

By the Nullstellensatz and by 6.5 the chains of irreducible closed subsets of X correspond bijectively to the chains of prime ideals of $K[x_1, \dots, x_n]$ containing $I(X)$, hence to the chains of prime ideals of the quotient ring $K[X]$. $\square$

The dimension theory for commutative rings contains some important theorems about dimension of K–algebras. The following two results are very useful.

7.7. Theorem. *Let K be any field.*

1. Let B be a finitely generated K–algebra and an integral domain. Then $\dim B = tr.d.Q(B)/K$, *where $Q(B)$ is the quotient field of B. In particular* $\dim B$ *is finite.*

2. Let B be as above and $\mathcal{P} \subset B$ be any prime ideal. Then $\dim B = \mathrm{ht}\mathcal{P} + \dim B/\mathcal{P}$.

Proof. For 1. see Portelli's notes. For a proof of 2., see for instance [4], Ch. II, Proposition 3.4. It relies on the normalization lemma and the lying over theorem. $\square$

7.8. Corollary. *Let K be an algebraically closed field.*

1. $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.

2. If X is an affine variety, then $\dim X = tr.d.K(X)/K$, *where $K(X)$ denotes the quotient field of $K[X]$.*

2. If $X \subset \mathbb{A}^n$ *is closed and irreducible, then* $\dim X = n - \mathrm{ht} I(X)$. $\square$

The following is an important characterization of the algebraic subsets of $\mathbb{A}^n$ of codimension 1.

7.9. Proposition. *Let* $X \subset \mathbb{A}^n$ *be closed. Then* X *is a hypersurface if and only if* X *is of pure dimension* $n-1$.

Proof. We give here an elementary direct proof. It can be proved more quickly using the Krull principal ideal theorem.

Let $X \subset \mathbb{A}^n$ be a hypersurface, with $I(X) = (F) = (F_1 \dots F_s)$, where $F_1, \dots, F_s$ are the irreducible factors of F all of multiplicity one. Then $V(F_1)$,…, $V(F_s)$ are the irreducible components of X, whose ideals are (F_1), …, (F_s). So it is enough to prove that $\mathrm{ht}(F_i) = 1$, for $i = 1, \dots, s$.

If $\mathcal{P} \subset (F_i)$ is a prime ideal, then either $\mathcal{P} = (0)$ or there exists $G \in \mathcal{P}$, $G \neq 0$. In the second case, let A be an irreducible factor of G belonging to $\mathcal{P}$: $A \in (F_i)$ so $A = HF_i$. Since A is irreducible, either H or F_i is invertible; F_i is irreducible, so H is invertible, hence $(A) = (F_i) \subset \mathcal{P}$. Therefore either $\mathcal{P} = (0)$ or $\mathcal{P} = (F_i)$, and $\mathrm{ht}(F_i) = 1$.

Conversely, assume that X is irreducible of dimension $n-1$. Since $X \neq \mathbb{A}^n$, there exists $F = F_1 \dots F_s \in I(X)$, $F \neq 0$. Hence $X \subset V(F) = V(F_1) \cup \dots \cup V(F_s)$. By the irreducibility of X, some irreducible factor of F, call it F_i, also vanishes along X. Therefore $X \subset V(F_i)$, which is irreducible of dimension $n-1$, by the first part. So $X = V(F_i)$ (by Proposition 7.2, 3). $\square$

This proposition does not generalise to higher codimension. There exist codimension 2 algebraic subsets of $\mathbb{A}^n$ whose ideal is not generated by two polynomials. An example in $\mathbb{A}^3$ is the curve X parametrised by (t^3, t^4, t^5). A system of generators of $I(X)$ is $\langle x^3 - yz, y^2 - xz, z^2 - x^2 y \rangle$. One can easily show that $I(X)$ cannot be generated by two polynomials. For a discussion of this and other similar examples, see [4], Chapter V.

7.10. Proposition. *Let* $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ *be irreducible closed subsets. Then* $\dim X \times Y = \dim X + \dim Y$.

Proof. Let $r = \dim X$, $s = \dim Y$; let $t_1, \dots, t_n$ (resp. $u_1, \dots, u_m$) be coordinate functions on $\mathbb{A}^n$ (resp. $\mathbb{A}^m$). We can assume that $t_1, \dots, t_r$ be a transcendence basis of $Q(K[X])$ and $u_1, \dots, u_s$ be a transcendence basis of $Q(K[Y])$. By definition, $K[X \times Y]$ is generated as K–algebra by $t_1, \dots, t_n, u_1, \dots, u_m$: we want to show that $t_1, \dots, t_r, u_1, \dots, u_s$ is a transcendence basis of $Q(K[X \times Y])$ over K. Assume that $F(x_1, \dots, x_r, y_1, \dots, y_s)$ is a polynomial which vanishes on $t_1, \dots, t_r, u_1, \dots, u_s$, i.e. F defines the zero function on $X \times Y$. Then, $\forall\ P \in X$, $F(P; y_1, \dots, y_s)$ is zero on Y, i.e. $F(P; u_1, \dots, u_s) = 0$. Since $u_1, \dots, u_s$ are algebraically independent, every coefficient $a_i(P)$ of $F(P; y_1, \dots, y_s)$ is zero, $\forall\ P \in X$. Since $t_1, \dots, t_r$ are algebraically independent, the polynomials $a_i(x_1, \dots, x_r)$ are zero, so

$F(x_1, \dots, x_r, y_1, \dots, y_s) = 0$. So $t_1, \dots, t_r, u_1, \dots, u_s$ are algebraically independent. Since this is certainly a maximal algebraically free set, it is a transcendence basis. $\square$

Exercises to §7.

1*. Prove that a proper closed subset of an irreducible curve is a finite set. Deduce that any bijection between irreducible curves is a homeomorphism.

2*. Let $X \subset \mathbb{A}^2$ be the cuspidal cubic of equation: $x^3 - y^2 = 0$, let $K[X]$ be its coordinate ring. Prove that all elements of $K[X]$ can be written in a unique way in the form $f(x) + yg(x)$, where f, g are polynomial in the variable x. Deduce that $K[X]$ is not isomorphic to a polynomial ring.

8. Regular and rational functions.

a) Regular functions

Let $X \subset \mathbb{P}^n$ be a locally closed subset and P be a point of X. Let $\phi : X \to K$ be a function.

8.1. Definition. ϕ is *regular at* P if there exists a suitable neighbourhood of P in which ϕ can be expressed as a quotient of homogeneous polynomials of the same degree; more precisely, if there exist an open neighbourhood U of P in X and homogeneous polynomials F, $G \in K[x_0, x_1, \dots, x_n]$ with $\deg F = \deg G$, such that $U \cap V_P(G) = \emptyset$ and $\phi(Q) = F(Q)/G(Q)$, for all $Q \in U$. Note that the quotient $F(Q)/G(Q)$ is well defined.

ϕ is *regular on* X if ϕ is regular at every point P of X.

The set of regular functions on X is denoted $\mathcal{O}(X)$: it contains K (identified with the set of constant functions), and can be given the structure of a K–algebra, by the definitions:

$$(\phi + \psi)(P) = \phi(P) + \psi(P)$$

$$(\phi\psi)(P) = \phi(P)\psi(P),$$

for $P \in X$. (Check that $\phi + \psi$ and $\phi\psi$ are indeed regular on X.)

8.2. Proposition. *Let $\phi : X \to K$ be a regular function. Let K be identified with $\mathbb{A}^1$ with Zariski topology. Then ϕ is continuous.*

Proof. It is enough to prove that $\phi^{-1}(c)$ is closed in X, $\forall\ c \in K$. For all $P \in X$, choose an open neighbourhood U_P and homogeneous polynomials F_P, G_P such that $\phi|_P = F_P/G_P$. Then

$$\phi^{-1}(c) \cap U_P = \{Q \in U_P | F_P(Q) - cG_P(Q) = 0\} = U_P \cap V_P(F_P - cG_P)$$

is closed in U_P. The proposition then follows from:

8.3. Lemma. *Let T be a topological space, $T = \cup_{i\in I} U_i$ be an open covering of T, $Z \subset T$ be a subset. Then Z is closed if and only if $Z \cap U_i$ is closed in U_i for all i.*

Proof. Assume that $U_i = X \setminus C_i$ and $Z \cap U_i = Z_i \cap U_i$, with C_i and Z_i closed in X.

Claim: $Z = \bigcap_{i\in I}(Z_i \cup C_i)$, hence it is closed.
In fact: if $P \in Z$, then $P \in Z \cap U_i$ for a suitable i. Therefore $P \in Z_i \cap U_i$, so $P \in Z_i \cup C_i$. If $P \notin Z_j \cap U_j$ for some j, then $P \notin U_j$ so $P \in C_j$ and therefore $P \in Z_j \cup C_j$.

Conversely, if $P \in \bigcap_{i\in I}(Z_i \cup C_i)$, then $\forall\, i$, either $P \in Z_i$ or $P \in C_i$. Since $\exists j$ such that $P \in U_j$, hence $P \notin C_j$, so $P \in Z_j$, so $P \in Z_j \cap U_j = Z \cap U_j$.

□

8.4. Corollary.

1. Let $\phi \in \mathcal{O}(X)$: then $\phi^{-1}(0)$ is closed. It is denoted $V(\phi)$ and called the set of zeroes of ϕ.

2. Let X be a quasi–projective variety and ϕ, $\psi \in \mathcal{O}(X)$. Assume that there exists U, open non –empty subset such that $\phi|_U = \psi|_U$. Then $\phi = \psi$.

Proof. $\phi - \psi \in \mathcal{O}(X)$ so $V(\phi - \psi)$ is closed. By assumption $V(\phi - \psi) \supset U$, which is dense, because X is irreducible. So $V(\phi - \psi) = X$.

□

If $X \subset \mathbb{A}^n$ is locally closed, we can use on X both homogeneous and non–homogeneous coordinates. In the second case, a regular function is locally represented as a quotient F/G, with F and $G \in K[x_1, \dots, x_n]$. In particular all polynomial functions are regular, so, if X is closed, $K[X] \subset \mathcal{O}(X)$.

If $\alpha \subset K[X]$ is an ideal, we can consider $V(\alpha) := \bigcap_{\phi\in\alpha} V(\phi)$: it is closed into X. Note that α is of the form $\alpha = \overline{\alpha}/I(X)$, where $\overline{\alpha}$ is the inverse image of α in the canonical epimorphism, it is an ideal of $K[x_1, \dots, x_n]$ containing $I(X)$, hence $V(\alpha) = V(\overline{\alpha}) \cap X = V(\overline{\alpha})$.

If K is algebraically closed, from the Nullstellensatz it follows that, if α is proper, then $V(\alpha) \neq \emptyset$. Moreover the following relative form of the Nullstellensatz holds: if $f \in K[X]$ and f vanishes at all points $P \in X$ such that $g_1(P) = \ldots = g_m(P) = 0$ ($g_1, \dots, g_m \in K[X]$), then $f^r \in \langle g_1, \dots, g_m \rangle \subset K[X]$, for some $r \geq 1$.

8.5. Theorem. *Let K be an algebraically closed field. Let $X \subset \mathbb{A}^n_K$ be closed in the Zariski topology. Then $\mathcal{O}(X) \simeq K[X]$. It is an integral domain if and only if X is irreducible.*

Proof. Let $f \in \mathcal{O}(X)$.

(i) Assume first that X is irreducible. For all $P \in X$ fix an open neighbourhood U_P of P and polynomials F_P, G_P such that $V_P(G_P) \cap U_P = \emptyset$ and $f|_{U_P} = F_P/G_P$. Let f_P, g_P be the functions in $K[X]$ defined by F_P and G_P. Then $g_P f = f_P$ holds on U_P, so it holds on X (by Corollary 8.3, because X is irreducible). Let $\alpha \subset K[X]$ be the ideal $\alpha = \langle g_P \rangle_{P \in X}$; α has no zeroes on X, because $g_P(P) \neq 0$, so $\alpha = K[X]$. Therefore there exists $h_P \in K[X]$ such that $1 = \sum_{P \in X} h_P g_P$ (sum with finite support). Hence in $\mathcal{O}(X)$ we have the relation: $f = f \sum h_P g_P = \sum h_P (g_P f) = \sum h_P f_P \in K[X]$.

(ii) Let X be reducible: for any $P \in X$, there exists $R \in K[x_1, \ldots, x_n]$ such that $R(P) \neq 0$ and $R \in I(X \setminus U_P)$, so $r \in \mathcal{O}(X)$ is zero outside U_P. So $r g_P f = f_P r$ on X and we conclude as above by replacing g_P with $g_P r$ and f_P with $f_P r$.

$\square$

The characterization of regular functions on projective varieties is completely different: we will see in §12 that, if X is a projective variety, then $\mathcal{O}(X) \simeq K$, i.e. the unique regular functions are constant.

This gives the motivation for introducing the following weaker concept.

b) Rational functions

8.6. Definition. Let X be a quasi–projective variety. A *rational function* on X is a germ of regular functions on some open non–empty subset of X.

Precisely, let $\mathcal{K}$ be the set $\{(U, f) | U \neq \emptyset, \text{open subset of } X, f \in \mathcal{O}(U)\}$. The following relation on $\mathcal{K}$ is an equivalence relation:

$$(U, f) \sim (U', f') \text{ if and only if } f|_{U \cap U'} = f'|_{U \cap U'}.$$

Reflexive and symmetric properties are quite obvious. Transitive property: let $(U, f) \sim (U', f')$ and $(U', f') \sim (U'', f'')$. Then $f|_{U \cap U'} = f'|_{U \cap U'}$ and $f'|_{U' \cap U''} = f''|_{U' \cap U''}$, hence $f|_{U \cap U' \cap U''} = f''|_{U \cap U' \cap U''}$. $U \cap U' \cap U''$ is a non–empty open subset of $U \cap U''$ (which is irreducible and quasi–projective), so by Corollary 8.4 $f|_{U' \cap U''} = f''|_{U' \cap U''}$.

Let $K(X) := \mathcal{K}/\sim$: its elements are by definition rational functions on X. $K(X)$ can be given the structure of a field in the following natural way.

Let $\langle U, f \rangle$ denote the class of (U, f) in $K(X)$. We define:

$$\langle U, f \rangle + \langle U', f' \rangle = \langle U \cap U', f + f' \rangle,$$

$$\langle U, f \rangle \langle U', f' \rangle = \langle U \cap U', f f' \rangle$$

(check that the definitions are well posed!).

There is a natural inclusion: $K \to K(X)$ such that $c \to \langle X, c \rangle$. Moreover, if $\langle U, f \rangle \neq 0$, then there exists $\langle U, f \rangle^{-1} = \langle U \setminus V(f), f^{-1} \rangle$: the axioms of a field are all satisfied.

There is also an injective map: $\mathcal{O}(X) \to K(X)$ such that $\phi \to \langle X, \phi \rangle$.

8.7. Proposition. *If $X \subset \mathbb{A}^n$ is affine, then $K(X) \simeq Q(\mathcal{O}(X)) = K(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ are the coordinate functions on X.*

Proof. The isomorphism is as follows:
(i) $\psi : K(X) \to Q(\mathcal{O}(X))$
If $\langle U, \phi \rangle \in K(X)$, then there exists $V \subset U$, open and non–empty, such that $\phi\ |_V = F/G$, where $F, G \in K[x_1, \dots, x_n]$ and $V(G) \cap V = \emptyset$. We set $\psi(\langle U, \phi \rangle) = f/g$.
(ii) $\psi' : Q(\mathcal{O}(X)) \to K(X)$
If $f/g \in Q(\mathcal{O}(X))$, we set $\psi'(f/g) = \langle X \setminus V(g), f/g \rangle$.
It is easy to check that ψ and ψ' are well defined and inverse each other. $\square$

8.8. Corollary. *If X is an affine variety, then $\dim X$ is equal to the transcendence degree over K of its field of rational functions..*

8.9. Proposition. *If X is quasi–projective and $U \neq \emptyset$ is an open subset, then $K(X) \simeq K(U)$.*

Proof. We have the maps: $K(U) \to K(X)$ such that $\langle V, \phi \rangle \to \langle V, \phi \rangle$, and $K(X) \to K(U)$ such that $\langle A, \psi \rangle \to \langle A \cap U, \psi\ |_{A \cap U} \rangle$: they are K–homomorphisms inverse each other. $\square$

8.10. Corollary. *If X is a projective variety contained in $\mathbb{P}^n$, if i is an index such that $X \cap U_i \neq \emptyset$ (where U_i is the open subset where $x_i \neq 0$), then $\dim X = \dim X \cap U_i = tr.d.K(X)/K$.*

Proof. By Proposition 7.2 $\dim X = \sup \dim(X \cap U_i)$. By 8.8 and 8.9, if $X \cap U_i$ is non–empty, $\dim(X \cap U_i) = tr.d.K(X \cap U_i)/K = tr.d.K(X)/K$ is independent of i. $\square$

If $\langle U, \phi \rangle \in K(X)$, we can consider all possible representatives of it, i.e. all pairs $\langle U_i, \phi_i \rangle$ such that $\langle U, \phi \rangle = \langle U_i, \phi_i \rangle$. Then $\overline{U} = \bigcup_i U_i$ is the maximum open subset of X on which ϕ can be seen as a function: it is called the *domain of definition* (or of regularity) of $\langle U, \phi \rangle$, or simply of ϕ. It is sometimes denoted $\mathrm{dom}\phi$. If $P \in \overline{U}$, we say that ϕ *is regular at* P.

We can consider the set of rational functions on X which are regular at P: it is denoted by $\mathcal{O}_{P,X}$. It is a subring of $K(X)$ containing $\mathcal{O}(X)$, called the *local ring of X at P*. In fact, $\mathcal{O}_{P,X}$ is a local ring, whose maximal ideal, denoted $\mathcal{M}_{P,X}$, is the set of rational functions ϕ such that $\phi(P)$ is defined and $\phi(P) = 0$. To see this, observe that an element of $\mathcal{O}_{P,X}$ can be represented as $\langle U, F/G \rangle$: its inverse in $K(X)$ is $\langle U \setminus V_P(G), G/F \rangle$, which belongs to $\mathcal{O}_{P,X}$ if and only if $F(P) \neq 0$. We'll see in 8.12 that $\mathcal{O}_{P,X}$ is the localization $K[X]_{I_X(P)}$.

As in Proposition 8.9 for the fields of rational functions, also for the local rings of points it can easily be proved that, if $U \neq \emptyset$ is an open subset of X containing P, then $\mathcal{O}_{P,X} \simeq \mathcal{O}_{P,U}$. So the ring $\mathcal{O}_{P,X}$ only depends on the local behaviour of X in the neighbourhood of P.

The *residue field* of $\mathcal{O}_{P,X}$ is the quotient $\mathcal{O}_{P,X}/\mathcal{M}_{P,X}$: it is a field which results to be naturally isomorphic to the base field K. In fact consider the evaluation map $\mathcal{O}_{P,X} \to K$ such that ϕ goes to $\phi(P)$: it is surjective with kernel $\mathcal{M}_{P,X}$, so $\mathcal{O}_{P,X}/\mathcal{M}_{P,X} \simeq K$.

8.11. Examples.

1. Let $Y \subset \mathbb{A}^2$ be the curve $V(x_1^3 - x_2^2)$. Then $F = x_2$, $G = x_1$ define the function $\phi = x_2/x_1$ which is regular at the points $P(a_1, a_2)$ such that $a_1 \neq 0$. Another representation of the same function is: $\phi = x_1^2/x_2$, which shows that ϕ is regular at P if $a_2 \neq 0$. If ϕ admits another representation F'/G', then $G'x_2 - F'x_1$ vanishes on an open subset of X, which is irreducible (see Exercise 6.2), hence $G'x_2 - F'x_1$ vanishes on X, and therefore $G'x_2 - F'x_1 \in \langle x_1^3 - x_2^2 \rangle$. This shows that there are essentially only the above two representations of ϕ. So $\phi \in K(X)$ and its domain of regularity is $Y \setminus \{0, 0\}$.

2. The stereographic projection.
Let $X \subset \mathbb{P}^2$ be the curve $V_P(x_1^2 + x_2^2 - x_0^2)$. Let $f := x_1/(x_0 - x_2)$ denote the germ of the regular function defined by $x_1/(x_0 - x_2)$ on $X \setminus V_P(x_0 - x_2) = X \setminus \{[1, 0, 1]\} = X \setminus \{P\}$. On X we have $x_1^2 = (x_0 - x_2)(x_0 + x_2)$ so f is represented also as $(x_0 + x_2)/x_1$ on $X \setminus V_P(x_1) = X \setminus \{P, Q\}$, where $Q = [1, 0, -1]$. If we identify K with the affine line $V_P(x_2) \setminus V_P(x_0)$ (the points of the x_1–axis lying in the affine plane U_0), then f can be interpreted as the stereographic projection of X centered at P, which takes $A[a_0, a_1, a_2]$ to the intersection of the line AP with the line $V_P(x_2)$. To see this, observe that AP has equation $a_1x_0 + (a_2 - a_0)x_1 - a_1x_2 = 0$; and $AP \cap V_P(x_2)$ is the point $[a_0 - a_2, a_1, 0]$.

8.12. The algebraic characterization of the local ring $\mathcal{O}_{P,X}$.

Let us recall the construction of the *ring of fractions of a ring* A with respect to a multiplicative subset S.

Let A be a ring and $S \subset A$ be a multiplicative subset. The following relation in $A \times S$ is an equivalence relation:

$$(a, s) \simeq (b, t) \text{ if and only if } \exists u \in S \text{ such that } u(at - bs) = 0.$$

Then the quotient $A \times S/_{\simeq}$ is denoted $S^{-1}A$ or A_S and $[(a, s)]$ is denoted $\frac{a}{s}$. A_S becomes a commutative ring with unit with operations $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and $\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$ (check that they are well–defined). With these operations, A_S is called the ring of fractions of A with respect to S, or the *localization* of A in S.

There is a natural homomorphism $j : A \to S^{-1}A$ such that $j(a) = \frac{a}{1}$, which makes $S^{-1}A$ an A–algebra. Note that j is the zero map if and only if $0 \in S$. More precisely if $0 \in S$ then $S^{-1}A$ is the zero ring: this case will always be excluded in what follows. Moreover j is injective if and only if every element in S is not a zero divisor. In this case $j(A)$ will be identified with A.

Examples.

1. Let A be an integral domain and set $S = A \setminus \{0\}$. Then $A_S = Q(A)$: the quotient field of A.

2. If $\mathcal{P} \subset A$ is a prime ideal, then $S = A \setminus \mathcal{P}$ is a multiplicative set and A_S is denoted $A_{\mathcal{P}}$ and called the localization of A at $\mathcal{P}$.

3. If $f \in A$, then the multiplicative set generated by f is

$$S = \{1, f, f^2, \dots, f^n, \dots\} :$$

A_S is denoted A_f.

4. If $S = \{x \in A \mid x \text{ is regular}\}$, then A_S is called the total ring of fractions of A: it is the maximum ring in which A can be canonically embedded.

It is easy to verify that the ring A_S enjoys the following *universal property*:
(i) if $s \in S$, then $j(s)$ is invertible;
(ii) if B is a ring with a given homomorphism $f : A \to B$ such that if $s \in S$, then $f(s)$ is invertible, then f factorizes through A_S, i.e. there exists a unique homomorphism $\overline{f}$ such that $\overline{f} \circ j = f$.

We will see now the relations between ideals of A_S and ideals of A.

If $\alpha \subset A$ is an ideal, then $\alpha A_S = \{\frac{a}{s} \mid a \in \alpha\}$ is called the *extension of* α in A_S and denoted also α^e. It is an ideal, precisely the ideal generated by the set $\{\frac{a}{1} \mid a \in \alpha\}$.

If $\beta \subset A_S$ is an ideal, then $j^{-1}(\beta) =: \beta^c$ is called the contraction of β and is clearly an ideal.

We have:

8.13. Proposition.

1. $\forall \alpha \subset A : \alpha^{ec} \supset \alpha$;
2. $\forall \beta \subset A_S : \beta = \beta^{ce}$;
3. α^e is proper if and only if $\alpha \cap S = \emptyset$;
4. $\alpha^{ec} = \{x \in A \mid \exists s \in S$ such that $sx \in \alpha\}$.

Proof.

1. and 2. are straightforward.

3. if $1 = \frac{a}{s} \in \alpha^e$, then there exists $u \in S$ such that $u(s - a) = 0$, i.e. $us = ua \in S \cap \alpha$. Conversely, if $s \in S \cap \alpha$ then $1 = \frac{s}{s} \in \alpha^e$.

4.
$$\alpha^{ec} = \{x \in A \mid j(x) = \frac{x}{1} \in \alpha^e\} =$$

$$= \{x \in A \mid \exists a \in \alpha, t \in S \text{ such that } \frac{x}{1} = \frac{a}{t}\} =$$

$$= \{x \in A \mid \exists a \in \alpha, t, u \in S \text{ such that } u(xt - a) = 0\}.$$

Hence, if $x \in \alpha^{ec}$, then: $(ut)x = ua \in \alpha$. Conversely: if there exists $s \in S$ such that $sx = a \in \alpha$, then $\frac{x}{1} = \frac{a}{s}$, i.e. $j(x) \in \alpha^e$. $\square$

If α is an ideal of A such that $\alpha = \alpha^{ec}$, α is called *saturated* with S. For example, if $\mathcal{P}$ is a prime ideal and $S \cap \mathcal{P} = \emptyset$, then $\mathcal{P}$ is saturated and $\mathcal{P}^e$ is prime. Conversely, if $\mathcal{Q} \subset A_S$ is a prime ideal, then $\mathcal{Q}^c$ is prime in A.

Therefore: *there is a bijection between the set of prime ideals of A_S and the set of prime ideals of A not intersecting S. In particular, if $S = A \setminus \mathcal{P}$, $\mathcal{P}$ prime, the prime ideals of $A_{\mathcal{P}}$ correspond bijectively to the prime ideals of A contained in $\mathcal{P}$, hence $A_{\mathcal{P}}$ is a local ring with maximal ideal $\mathcal{P}^e$, denoted $\mathcal{P}A_{\mathcal{P}}$, and residue field $A_{\mathcal{P}}/\mathcal{P}A_{\mathcal{P}}$.* Moreover $\dim A_{\mathcal{P}} = \mathrm{ht}\mathcal{P}$.

In particular we get the characterization of $\mathcal{O}_{P,X}$. Let $X \subset \mathbb{A}^n$ be an affine variety, let P be a point of X and $I(P) \subset K[x_1, \dots, x_n]$ be the ideal of P. Let $I_X(P) := I(P)/I(X)$ be the ideal of $K[X]$ formed by regular functions on X vanishing at P. Then we can construct the localization

$$\mathcal{O}(X)_{I_X(P)} = \{\frac{f}{g} | f, g \in \mathcal{O}(X), g(P) \neq 0\} \subset K(X):$$

it is canonically identified with $\mathcal{O}_{P,X}$. In particular: $\dim \mathcal{O}_{P,X} = \mathrm{ht}\ I_X(P) = \dim \mathcal{O}(X) = \dim X$.

There is a bijection between prime ideals of $\mathcal{O}_{P,X}$ and prime ideals of $\mathcal{O}(X)$ contained in $I_X(P)$; they also correspond to prime ideals of $K[x_1, \dots, x_n]$ contained in $I(P)$ and containing $I(X)$.

If X is affine, it is possible to define the local ring $\mathcal{O}_{P,X}$ also if X is reducible, simply as localization of $K[X]$ at the maximal ideal $I_X(P)$. The natural map j from $K[X]$ to $\mathcal{O}_{P,X}$ is injective if and only if $K[X] \setminus I_X(P)$ does not contain any zero divisor. A non-zero function f is a zero divisor in $K[X]$ if there exists a non-zero g such that $fg = 0$, i.e. $X = V(f) \cup V(g)$ is an expression of X as union of proper closed subsets. For j to be injective it is required that every zero divisor f belongs to $I_X(P)$, which means that all the irreducible components of X pass through P.

Exercises to §8.

1. Prove that the affine varieties and the open subsets of affine varieties are quasi–projective.

2. Let $X = \{P, Q\}$ be the union of two points in an affine space over K. Prove that $\mathcal{O}(X)$ is isomorphic to $K \times K$.

9. Regular and rational maps.

In the following K is an algebraically closed field.

a) Regular maps.

Let X, Y be quasi–projective varieties (or more generally locally closed sets). Let $\phi : X \to Y$ be a map.

9.1. Definition. ϕ is a *regular map* or a *morphism* if
(i) ϕ is continuous;
(ii) ϕ preserves regular functions, i.e. for all $U \subset Y$ (U open and non–empty) and for all $f \in \mathcal{O}(U)$, then $f \circ \phi \in \mathcal{O}(\phi^{-1}(U))$:

$$\begin{array}{ccccc} X & \xrightarrow{\phi} & Y & & \\ \uparrow & & \uparrow & & \\ \phi^{-1}(U) & \xrightarrow{\phi|} & U & \xrightarrow{f} & K \end{array}$$

Note that:
a) for all X the identity map $1_X : X \to X$ is regular;
b) for all X, Y, Z and regular maps $X \xrightarrow{\phi} Y$, $Y \xrightarrow{\psi} Z$, the composite map $\psi \circ \phi$ is regular.

An *isomorphism* of varieties is a regular map which possesses regular inverse, i.e. a regular $\phi : X \to Y$ such that there exists a regular $\psi : Y \to X$ verifying the conditions $\psi \circ \phi = 1_X$ and $\phi \circ \psi = 1_Y$. In this case X and Y are said to be isomorphic, and we write: $X \simeq Y$.

If $\phi : X \to Y$ is regular, there is a natural K–homomorphism $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, called the *comorphism associated to* ϕ, defined by: $f \to \phi^*(f) := f \circ \phi$.

The construction of the comorphism is functorial, which means that:
a) $1_X^* = 1_{\mathcal{O}(X)}$;
b) $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.

This implies that, if $X \simeq Y$, then $\mathcal{O}(X) \simeq \mathcal{O}(Y)$. In fact, if $\phi : X \to Y$ is an isomorphism and ψ is its inverse, then $\phi \circ \psi = 1_Y$, so $(\phi \circ \psi)^* = \psi^* \circ \phi^* = (1_Y)^* = 1_{\mathcal{O}(Y)}$ and similarly $\psi \circ \phi = 1_X$ implies $\phi^* \circ \psi^* = 1_{\mathcal{O}(X)}$.

9.2. Examples.
1) The homeomorphism $\phi_i : U_i \to \mathbb{A}^n$ of Proposition 3.2 is an isomorphism.

2) There exist homeomorphisms which are not isomorphisms. Let $Y = V(x^3 - y^2) \subset \mathbb{A}^2$. We have seen (see Exercise 7.2) that $K[X] \not\simeq K[\mathbb{A}^1]$, hence Y is not isomorphic to the affine line. Nevertheless, the following map is regular, bijective and also a homeomorphism (see Exercise 7.1):
$\phi : \mathbb{A}^1 \to Y$ such that $t \to (t^2, t^3)$;

$\phi^{-1} : Y \to \mathbb{A}^1$ is defined by $(x, y) \to \begin{cases} \frac{y}{x} \text{ if } x \neq 0 \\ 0 \text{ if } (x, y) = (0, 0). \end{cases}$
Note that ϕ^{-1} is not regular at the point $(0, 0)$.

9.3. Proposition. *Let $\phi : X \to Y \subset \mathbb{A}^n$ be a map. Then ϕ is regular if and only if $\phi_i := t_i \circ \phi$ is a regular function on X, for all $i = 1, \dots, n$, where $t_1, \dots, t_n$ are the coordinate functions on Y.*

Proof. If ϕ is regular, then $\phi_i = \phi^*(t_i)$ is regular by definition.

Conversely, assume that ϕ_i is a regular function on X for all i. Let $Z \subset Y$ be a closed subset and we have to prove that $\phi^{-1}(Z)$ is closed in X. Since any closed subset of $\mathbb{A}^n$ is an intersection of hypersurfaces, it is enough to consider $\phi^{-1}(Y \cap V(F)$ with $F \in K[x_1, \dots, x_n]$:

$$\phi^{-1}(V(F) \cap Y) = \{P \in X | F(\phi(P)) = F(\phi_1, \dots, \phi_n)(P) = 0\} = V(F(\phi_1, \dots, \phi_n)).$$

But note that $F(\phi_1, \dots, \phi_n) \in \mathcal{O}(X)$: it is the composition of F with the regular functions $\phi_1, \dots, \phi_n$. Hence $\phi^{-1}(V(F) \cap Y)$ is closed, so we can conclude that ϕ is continuous. If $U \subset Y$ and $f \in \mathcal{O}(U)$, for any point P of U choose an open neighbourhood U_P such that $f = F_P/G_P$ on U_P.

So $f \circ \phi = F_P(\phi_1, \dots, \phi_n)/G_P(\phi_1, \dots, \phi_n)$ on $\phi^{-1}(U_P)$, hence it is regular on each $\phi^{-1}(U_P)$ and by consequence on $\phi^{-1}(U)$.

□

If $\phi : X \to Y$ is a regular map and $Y \subset \mathbb{A}^n$, by Proposition 9.2. we can represent ϕ in the form $\phi = (\phi_1, \dots, \phi_n)$, where $\phi_1, \dots, \phi_n \in \mathcal{O}(X)$ and $\phi_i = \phi^*(t_i)$. $\phi_1, \dots, \phi_n$ are not arbitrary in $\mathcal{O}(X)$ but such that Im $\phi \subset Y$. If Y is closed in $\mathbb{A}^n$, let us recall that $t_1, \dots, t_n$ generate $\mathcal{O}(Y)$, hence $\phi_1, \dots, \phi_n$ generate $\phi^*(\mathcal{O}(Y))$ as K-algebra. This observation is the key for the following important result.

9.4. Theorem. *Let X be a locally closed algebraic set and Y be an affine algebraic set. Let $Hom(X, Y)$ denote the set of regular maps from X to Y and $Hom(\mathcal{O}(Y), \mathcal{O}(X))$ denote the set of K– homomorphisms from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$.*

Then the map $Hom(X, Y) \to Hom(\mathcal{O}(Y), \mathcal{O}(X))$, such that $\phi : X \to Y$ goes to $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, is bijective.

Proof. Let $Y \subset \mathbb{A}^n$ and let $t_1, \dots, t_n$ be the coordinate functions on Y, so $\mathcal{O}(Y) = K[t_1, \dots, t_n]$. Let $u : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a K–homomorphism: we want to define a morphism $u^\sharp : X \to Y$ whose associated comorphism is u. By the remark above, if $u^\sharp$ exists, its components have to be $u(t_1), \dots, u(t_n)$. So we define

$$\begin{array}{rccc} u^\sharp : & X & \to & \mathbb{A}^n \\ & P & \to & (u(t_1)(P)), \dots, u(t_n)(P)). \end{array}$$

This is a morphism by Proposition 9.3. We claim that $u^\sharp(X) \subset Y$. Let $F \in I(Y)$ and $P \in X$: then

$$\begin{aligned}(F(u^\sharp(P)) &= F(u(t_1)(P), \dots, u(t_n)(P)) = \\ &= F(u(t_1), \dots, u(t_n))(P) = \\ &= u(F((t_1, \dots, t_n))(P) \text{ because } u \text{ is } K\text{-homomorphism} = \\ &= u(0)(P) = \\ &= 0(P) = 0.\end{aligned}$$

So $u^\sharp$ is a regular map from X to Y.

We consider now $(u^\sharp)^* : \mathcal{O}(Y) \to \mathcal{O}(X)$: it takes a function f to $f \circ u^\sharp = f(u(t_1), \dots, u(t_n)) = u(f)$, so $(u^\sharp)^* = u$. Conversely, if $\phi : X \to Y$ is regular, then $(\phi^*)^\sharp$ takes P to $(\phi^*(t_1)(P), \dots, \phi^*(t_n)(P)) = (\phi_1(P), \dots, \phi_n(P))$, so $(\phi^*)^\sharp = \phi$. □

Note that, by definition, $1^\sharp_{\mathcal{O}(X)} = 1_X$, for all affine X; moreover $(v \circ u)^\sharp = u^\sharp \circ v^\sharp$ for all $u : \mathcal{O}(Z) \to \mathcal{O}(Y)$, $v : \mathcal{O}(Y) \to \mathcal{O}(X)$, K–homomorphisms of affine algebraic sets: this means that also this construction is functorial.

The previous results can be rephrased using the language of categories. We introduce a category $\mathcal{C}$ whose objects are the affine algebraic sets over a fixed algebraically closed field K and the morphisms are the regular maps. We consider also a second category $\mathcal{C}'$ with objects the K-algebras and morphisms the K-homomorphisms. Then there is a contravariant functor that operates on the objects sending X to $\mathcal{O}(X) = K[X]$, and on the morphisms sending ϕ to the associated comorphisms ϕ^*.

If we restrict the class of objects of $\mathcal{C}'$ taking only the finitely generated reduced K-algebras (a full subcategory of the previous one), then this functor becomes an equivalence of categories. Indeed the construction of the comorphism establishes a bijection between the Hom sets $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $\mathrm{Hom}_{\mathcal{C}'}(\mathcal{O}(Y), \mathcal{O}(X))$. Moreover, for any finitely generated K-algebra A, there exists an affine algebraic set X such that A is K-isomorphic to $\mathcal{O}(X)$. To see this, we choose a finite set of generators of A, such that $A = K[\xi_1, \dots, \xi_n]$. Then we can consider the surjective K-homomorphism Ψ from the polynomial ring $K[x_1, \dots, x_n]$ to A sending x_i to ξ_i for any i. In view of the fundamental theorem of homomorphism, it follows that $A \simeq K[x_1, \dots, x_n]/\ker\Psi$. The assumption that A is reduced then implies that $X := V(\ker\Psi) \subset \mathbb{A}^n$ is an affine algebraic set with $I(X) = \ker\Psi$ and $A \simeq \mathcal{O}(X)$.

We note that changing system of generators for A changes the homomorphism Ψ, and by consequence also the algebraic set X, up to isomorphism. For instance let A be a polynomial ring in one variable t: if we choose only t as system of generators, we get $X = \mathbb{A}^1$, but if we choose t, t^2, t^3 we get the affine skew cubic in $\mathbb{A}^3$.

As a consequence of the previous discussion we have the following:

9.5. Corollary. *Let X, Y be affine algebraic sets. Then $X \simeq Y$ if and only if $\mathcal{O}(X) \simeq \mathcal{O}(Y)$.* $\square$

If X and Y are quasi–projective varieties and $\phi : X \to Y$ is regular, it is not always possible to define a comorphism $K(Y) \to K(X)$. If f is a rational function on Y with $\mathrm{dom} f = U$, it can happen that $\phi(X) \cap \mathrm{dom} f = \emptyset$, in which case $f \circ \phi$ does not exist. Nevertheless, if we assume that ϕ is dominant, i.e. $\overline{\phi(X)} = Y$, then certainly $\phi(X) \cap U \neq \emptyset$, hence $\langle \phi^{-1}(U), f \circ \phi \rangle \in K(X)$. We obtain a K–homomorphism, which is necessarily injective, $K(Y) \to K(X)$, also denoted by ϕ^*. Note that in this case, we have: $\dim X \geq \dim Y$. As above, it is possible to check that, if $X \simeq Y$, then $K(X) \simeq K(Y)$, hence $\dim X = \dim Y$. Moreover, if $P \in X$ and $Q = \phi(P)$, then ϕ^* induces a map $\mathcal{O}_{Q,Y} \to \mathcal{O}_{P,X}$, such that $\phi^* \mathcal{M}_{Q,Y} \subset \mathcal{M}_{P,X}$. Also in this case, if ϕ is an isomorphism, then $\mathcal{O}_{Q,Y} \simeq \mathcal{O}_{P,X}$.

We will see now how to express in practice a regular map when the target is contained in a projective space. Let $X \subset \mathbb{P}^n$ be a quasi–projective variety and $\phi : X \to \mathbb{P}^m$ be a map.

9.6. Proposition. *ϕ is a morphism if and only if, for any $P \in X$, there exist an open neighbourhood U_P of P and $n+1$ homogeneous polynomials $F_0, \dots, F_m$ of the same degree, in $K[x_0, x_1, \dots, x_n]$, such that, if $Q \in U_P$, then $\phi(Q) = [F_0(Q), \dots, F_m(Q)]$. In particular, for any $Q \in U_P$, there exists an index i such that $F_i(Q) \neq 0$.*

Proof. "$\Rightarrow$" Let $P \in X$, $Q = \phi(P)$ and assume that $Q \in U_0$. Then $U := \phi^{-1}(U_0)$ is an open neighbourhood of P and we can consider the restriction $\phi|_U : U \to U_0$, which is regular. Possibly after restricting U, using non–homogeneous coordinates on U_0, we can assume that $\phi|_U = (F_1/G_1, \dots, F_m/G_m)$, where (F_1, G_1), $\dots$, (F_m, G_m) are pairs of homogeneous polynomials of the same degree such that $V_P(G_i) \cap U = \emptyset$ for all index i. We can reduce the fractions F_i/G_i to a common denominator F_0, so that $\deg F_0 = \deg F_1 = \ldots = \deg F_m$ and $\phi|_U = (F_1/F_0, \dots, F_m/F_0) = [F_0, F_1, \dots, F_m]$, with $F_0(Q) \neq 0$ for $Q \in U$.

"$\Leftarrow$" Possibly after restricting U_P, we can assume $F_i(Q) \neq 0$ for all $Q \in U_P$ and suitable i. Let $i = 0$: then $\phi|_{U_P} : U_P \to U_0$ operates as follows: $\phi|_{U_P}(Q) = (F_1(Q)/F_0(Q), \dots, F_m(Q)/F_0(Q))$, so it is a morphism by Proposition 9.3. From this remark, one deduces that also ϕ is a morphism. $\square$

9.7. Examples.

1. Let $X \subset \mathbb{P}^2$, $X = V_P(x_1^2 + x_2^2 - x_0^2)$, the projective closure of the unitary circle. We define $\phi : X \to \mathbb{P}^1$ by

$$[x_0, x_1, x_2] \to \begin{cases} [x_0 - x_2, x_1] \text{ if } (x_0 - x_2, x_1) \neq (0,0); \\ [x_1, x_0 + x_2] \text{ if } (x_1, x_0 + x_2) \neq (0,0). \end{cases}$$

ϕ is well–defined because on X $x_1^2 = (x_0 - x_2)(x_0 + x_2)$. Moreover

$$(x_1, x_0 - x_2) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,1]\},$$

$$(x_0 + x_2, x_1) \neq (0,0) \Leftrightarrow [x_0, x_1, x_2] \in X \setminus \{[1,0,-1]\}.$$

The map ϕ is the natural extension of the rational function $f : X \setminus \{[1,0,1]\} \to K$ such that $[x_0, x_1, x_2] \to x_1/(x_0 - x_2)$ (Example 8.9, 2). Now the point $P[1,0,1]$, the centre of the stereographic projection, goes to the point at infinity of the line $V_P(x_2)$.

By geometric reasons ϕ is invertible and $\phi^{-1} : \mathbb{P}^1 \to X$ takes $[\lambda, \mu]$ to $[\lambda^2 + \mu^2, 2\lambda\mu, \lambda^2 - \mu^2]$ (note the connection with the Pitagorean triples!).

Indeed: the line through P and $[\lambda, \mu, 0]$ has equation: $\mu x_0 - \lambda x_1 - \mu x_2 = 0$. Its intersections with X are represented by the system:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ x_1^2 + x_2^2 - x_0^2 = 0 \end{cases}$$

Assuming $\mu \neq 0$ this system is equivalent to the following:

$$\begin{cases} \mu x_0 - \lambda x_1 - \mu x_2 = 0 \\ \mu^2 x_0^2 = \mu^2(x_1^2 + x_2^2) = (\lambda x_1 + \mu x_2)^2. \end{cases}$$

Therefore, either $x_1 = 0$ and $x_0 = x_2$, or $\begin{cases} (\mu^2 - \lambda^2)x_1 - 2\lambda\mu x_2 = 0 \\ \mu x_0 = \lambda x_1 + \mu x_2 \end{cases}$, which gives the required expression.

2. *Affine transformations.*

Let $A = (a_{ij})$ be a $n \times n$–matrix with entries in K, let $B = (b_1, \dots, b_n) \in \mathbb{A}^n$ be a point. The map $\tau_A : \mathbb{A}^n \to \mathbb{A}^n$ defined by $(x_1, \dots, x_n) \to (y_1, \dots, y_n)$, such that

$$\{y_i = \sum_j a_{ij} x_j + b_i, i = 1, \dots, n,$$

is a regular map called an affine transformation of $\mathbb{A}^n$. In matrix notation τ_A is $Y = AX + B$. If A is of rank n, then τ_A is said non–degenerate and is an isomorphism: the inverse map τ_A^{-1} is represented by $X = A^{-1}Y - A^{-1}B$. More in general, an affine transformation from $\mathbb{A}^n$ to $\mathbb{A}^m$ is a map represented in matrix form by $Y = AX + B$, where A is a $m \times n$ matrix and $B \in \mathbb{A}^m$. It is injective if and only if $\mathrm{rk} A = n$ and surjective if and only if $\mathrm{rk} A = m$.

The isomorphisms of an algebraic set X in itself are called automorphisms of X: they form a group for the usual composition of maps, denoted $Aut\ X$. If $X = \mathbb{A}^n$, the non–degenerate affine transformations form a subgroup of $Aut\ \mathbb{A}^n$.

If $n = 1$ and the characteristic of K is 0, then $Aut\ \mathbb{A}^1$ coincides with this subgroup. In fact, let $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ be an automorphism: it is represented by a polynomial $F(x)$ such that there exists $G(x)$ satisfying the condition $G(F(t)) = t$

for all $t \in \mathbb{A}^1$, i.e. $G(F(x)) = x$ in the polynomial ring $K[x]$. Then, taking derivatives, we get $G'(F(x))F'(x) = 1$, which implies $F'(t) \neq 0$ for all $t \in K$, so $F'(x)$ is a non–zero constant. Hence, F is linear and G is linear too.

If $n \geq 2$, then $Aut\ \mathbb{A}^n$ is not completely described. There exist non–linear automorphisms of degree d, for all d. For example, for $n = 2$: let $\phi : \mathbb{A}^2 \to \mathbb{A}^2$ be given by $(x, y) \to (x, y + P(x)$, where P is any polynomial of $K[x]$. Then $\phi^{-1} : (x', y') \to (x', y' - P(x'))$. A very important open problem is the Jacobian conjecture, stating that, in characteristic zero, a regular map $\phi : \mathbb{A}^n \to \mathbb{A}^n$ is an automorphism if and only if the Jacobian determinant $\mid J(\phi) \mid$ is a non-zero constant.

3. *Projective transformations.*

Let A be a $(n+1)\times(n+1)$–matrix with entries in K. Let $P[x_0, \dots, x_n] \in \mathbb{P}^n$: then $[a_{00}x_0 + \dots + a_{0n}x_n, \dots, a_{n0}x_0 + \dots + a_{nn}x_n]$ is a point of $\mathbb{P}^n$ if and only if it is different from $[0, \dots, 0]$. So A defines a regular map $\tau : \mathbb{P}^n \to \mathbb{P}^n$ if and only if $\mathrm{rk}A = n+1$. If $\mathrm{rk}A = r < n+1$, then A defines a regular map whose domain is the quasi–projective variety $\mathbb{P}^n \setminus \mathbb{P}(kerA)$. If $\mathrm{rk}A = n + 1$, then τ is an isomorphism, called a projective transformation. Note that the matrices λA, $\lambda \in K^*$, all define the same projective transformation. So $PGL(n + 1, K) := GL(n + 1, K)/K^*$ acts on $\mathbb{P}^n$ as the group of projective transformations.

If $X, Y \subset \mathbb{P}^n$, they are called projectively equivalent if there exists a projective transformation $\tau : \mathbb{P}^n \to \mathbb{P}^n$ such that $\tau(X) = Y$.

9.8. Theorem. *Fundamental theorem on projective transformations.*

Let two $(n+2)$–tuples of points of $\mathbb{P}^n$ in general position be fixed: $P_0, \dots, P_{n+1}$ and $Q_0, \dots, Q_{n+1}$. Then there exists one isomorphic projective transformation τ of $\mathbb{P}^n$ in itself, such that $\tau(P_i) = Q_i$ for all index i.

Proof. Put $P_i = [v_i]$, $Q_i = [w_i]$, $i = 0, \dots, n+1$. So $\{v_0, \dots, v_n\}$ and $\{w_0, \dots, w_n\}$ are two bases of K^{n+1}, hence there exist scalars $\lambda_0, \dots, \lambda_n, \mu_0, \dots, \mu_n$ such that

$$v_{n+1} = \lambda_0 v_0 + \dots + \lambda_n v_n, \quad w_{n+1} = \mu_0 w_0 + \dots + \mu_n w_n,$$

where the coefficients are all different from 0, because of the general position assumption. We replace v_i with $\lambda_i v_i$ and w_i with $\mu_i w_i$ and get two new bases, so there exists a unique automorphism of K^{n+1} transforming the first basis in the second one and, by consequence, also v_{n+1} in w_{n+1}. This automorphism induces the required projective transformation on $\mathbb{P}^n$. $\square$

An immediate consequence of the above theorem is that projective subspaces of the same dimension are projectively equivalent. Also two subsets of $\mathbb{P}^n$ formed both by k points in general position are projectively equivalent if $k \leq n + 2$. If $k > n + 2$, this is no longer true, already in the case of four points on a projective line. The problem of describing the classes of projective equivalence of k–tuples of points of $\mathbb{P}^n$, for $k > n+2$, is one the first problems of the classical invariant theory. The solution in the case $k=4$, $n=1$ is given by the notion of *cross–ratio*.

4. Let $X \subset \mathbb{A}^n$ be an affine variety, then $X_F = X \setminus V(F)$ is isomorphic to a closed subset of $\mathbb{A}^{n+1}$, i.e. to $Y = V(x_{n+1}F - 1, G_1, \dots, G_r)$, where $I(X) = \langle G_1, \dots, G_r\rangle$. Indeed, the following regular maps are inverse each other:

$\phi : X_F \to Y$ such that $(x_1, \dots, x_n) \to (x_1, \dots, x_n, 1/F(x_1, \dots, x_n))$,

$\psi : Y \to X_F$ such that $(x_1, \dots, x_n, x_{n+1}) \to (x_1, \dots, x_n)$.

Hence, X_F is a quasi–projective variety contained in $\mathbb{A}^n$, not closed in $\mathbb{A}^n$, but isomorphic to a closed subset of another affine space.

From now on, the term *affine variety* will denote a quasi–projective variety isomorphic to some affine closed set.

If X is an affine variety and precisely $X \simeq Y$, with $Y \subset \mathbb{A}^n$ closed, then $\mathcal{O}(X) \simeq \mathcal{O}(Y) = K[t_1, \dots, t_n]$ is a finitely generated K–algebra. In particular, if K is algebraically closed and α is an ideal strictly contained in $\mathcal{O}(X)$, then $V(\alpha) \subset X$ is non–empty, by the relative form of the Nullstellensatz. From this observation, we can deduce that the quasi–projective variety of next example is not affine.

5. $\mathbb{A}^2 \setminus \{(0,0)\}$ is not affine.

Set $X = \mathbb{A}^2 \setminus \{(0,0)\}$: first of all we will prove that $\mathcal{O}(X) \simeq K[x,y] = \mathcal{O}(\mathbb{A}^2)$, i.e. any regular function on X can be extended to a regular function on the whole plane.

Indeed: let $f \in \mathcal{O}(X)$: if $P \neq Q$ are points of X, then there exist polynomials F, G, F', G' such that $f = F/G$ on a neighbourhood U_P of P and $f = F'/G'$ on a neighbourhood U_Q of Q. So $F'G = FG'$ on $U_P \cap U_Q \neq \emptyset$, which is open also in $\mathbb{A}^2$, hence dense. Therefore $F'G = FG'$ in $K[x,y]$. We can clearly assume that F and G are coprime and similarly for F' and G'. So by the unique factorization property, it follows that $F' = F$ and $G' = G$. In particular f admits a unique representation as F/G on X and $G(P) \neq 0$ for all $P \in X$. Hence G has no zeroes on $\mathbb{A}^2$, so $G = c \in K^*$ and $f \in \mathcal{O}(X)$.

Now, the ideal $\langle x, y\rangle$ has no zeroes in X and is proper: this proves that X is not affine.

We have exploited the fact that a polynomial in more than one variables has infinitely many zeroes, a fact that allows to generalise the previous observation.

On the other hand, the following property holds:

9.9. Proposition. *Let $X \subset \mathbb{P}^n$ be quasi–projective. Then X admits an open covering by affine varieties.*

Proof. Let $X = X_0 \cup \ldots \cup X_n$ be the open covering of X where $X_i = U_i \cap X$ $= \{P \in X | P[a_0, \dots, a_n], a_i \neq 0\}$. So, fixed P, there exists an index i such that

$P \in X_i$. We can assume that $P \in X_0$: X_0 is open in some affine variety Y of $\mathbb{A}^n$ (identified with U_0); set $X_0 = Y \setminus Y'$, where Y, Y' are both closed. Since $P \notin Y'$, there exists F such that $F(P) \neq 0$ and $V(F) \supset Y'$. So $P \in Y \setminus V(F) \subset Y \setminus Y'$ and $Y \setminus V(F)$ is an affine open neighbourhood of P in $Y \setminus Y' = X_0 \subset X$.

□

6. *The Veronese maps.*

Let n, d be positive integers; put $N(n,d) = \binom{n+d}{d} - 1$. Note that $\binom{n+d}{d}$ is equal to the number of (monic) monomials of degree d in the variables $x_0, \ldots, x_n$, that is equal to the number of $n+1$–tuples $(i_0, \ldots, i_n)$ such that $i_0 + \ldots + i_n = d$, $i_j \geq 0$. Then in $\mathbb{P}^{N(n,d)}$ we can use coordinates $\{v_{i_0 \ldots i_n}\}$, where $i_0, \ldots, i_n \geq 0$ and $i_0 + \ldots + i_n = d$. For example: if $n = 2$, $d = 2$, then $N(2,2) = \binom{4}{2} - 1 = 5$. In $\mathbb{P}^5$ we can use coordinates $v_{200}, v_{110}, v_{101}, v_{020}, v_{011}, v_{002}$.

For all n, d we define the map $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$ such that $[x_0, \ldots, x_n] \to [v_{d00\ldots0}, v_{d-1,10\ldots0}, \ldots, v_{0\ldots00d}]$ where $v_{i_0\ldots i_n} = x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n}$: $v_{n,d}$ is clearly a morphism, its image is denoted $V_{n,d}$ and called *the Veronese variety* of type (n,d). It is in fact the projective variety of equations:

$$(*)\{v_{i_0\ldots i_n} v_{j_0\ldots j_n} - v_{h_0\ldots h_n} v_{k_0\ldots k_n}, \forall i_0 + j_0 = h_0 + k_0, i_1 + j_1 = h_1 + k_1, \ldots$$

We prove this statement in the particular case $n = d = 2$; the general case is similar.

First of all, it is clear that the points of $v_{n,d}(\mathbb{P}^n)$ satisfy the system $(*)$. Conversely, assume that $P[v_{200}, v_{110}, \ldots] \in \mathbb{P}^5$ satisfies the equations $(*)$, which become:

$$\begin{cases} v_{200}v_{020} = v_{110}^2 \\ v_{200}v_{002} = v_{101}^2 \\ v_{002}v_{020} = v_{011}^2 \\ v_{200}v_{011} = v_{110}v_{101} \\ v_{020}v_{101} = v_{110}v_{011} \\ v_{110}v_{002} = v_{011}v_{101} \end{cases}$$

Then, at least one of the coordinates $v_{200}, v_{020}, v_{002}$ is different from 0.

Therefore, if $v_{200} \neq 0$, then $P = v_{2,2}([v_{200}, v_{110}, v_{101}])$; if $v_{020} \neq 0$, then $P = v_{2,2}([v_{110}, v_{020}, v_{011}])$; if $v_{002} \neq 0$, then $P = v_{2,2}([v_{101}, v_{011}, v_{002}])$. Note that, if two of these three coordinates are different from 0, then the points of $\mathbb{P}^2$ found in this way have proportional coordinates, so they coincide.

We have also proved in this way that $v_{2,2}$ is an isomorphism between $\mathbb{P}^2$ and $V_{2,2}$, called the Veronese surface of $\mathbb{P}^5$. The same happens in the general case.

If $n = 1$, $v_{1,d} : \mathbb{P}^1 \to \mathbb{P}^d$ takes $[x_0, x_1]$ to $[x_0^d, x_0^{d-1}x_1, \ldots, x_1^d]$: the image is called the *rational normal curve* of degree d, it is isomorphic to $\mathbb{P}^1$. If $d = 3$, we find the skew cubic.

Let now $X \subset \mathbb{P}^n$ be a hypersurface of degree d: $X = V_P(F)$, with

$$F = \sum_{i_0 + \ldots + i_n = d} a_{i_0\ldots i_n} x_0^{i_0} \ldots x_n^{i_n}.$$

Then $v_{n,d}(X) \simeq X$: it is the set of points

$$\{v_{i_0\dots i_n} \in \mathbb{P}^{N(n,d)} | \sum_{i_0+\dots+i_n=d} a_{i_0\dots i_n} v_{i_0\dots i_n} = 0 \text{ and } [v_{i_0\dots i_n}] \in V_{n,d}\}.$$

It coincides with $V_{n,d} \cap H$, where H is a hyperplane of $\mathbb{P}^{N(n,d)}$: a hyperplane section of the Veronese variety. This is called the linearisation process, allowing to "transform" a hypersurface in a hyperplane, modulo the Veronese isomorphism.

The Veronese surface V of $\mathbb{P}^5$ enjoys a lot of interesting properties. Most of them follow from its property of being covered by a 2-dimensional family of conics, which are precisely the images via $v_{2,2}$ of the lines of the plane.

To see this, we'll use as coordinates in $\mathbb{P}^5$ $w_{00}, w_{01}, w_{02}, w_{11}, w_{12}, w_{22}$, so that $v_{2,2}$ sends $[x_0, x_1, x_2]$ to the point of coordinates $w_{ij} = x_i x_j$. With this choice of coordinates, the equations of V are obtained by annihilating the 2×2 minors of the symmetric matrix:

$$M = \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix}$$

Let ℓ be a line of $\mathbb{P}^2$ of equation $b_0x_0 + b_1x_1 + b_2x_2 = 0$. Its image is the set of points of $\mathbb{P}^5$ with coordinates $w_{ij} = x_i x_j$, such that there exists a non-zero triple $[x_0, x_1, x_2]$ with $b_0x_0 + b_1x_1 + b_2x_2 = 0$. But this last equation is equivalent to the system:

$$\begin{cases} b_0x_0^2 + b_1x_0x_1 + b_2x_0x_2 = 0 \\ b_0x_0x_1 + b_1x_1^2 + b_2x_1x_2 = 0 \\ b_0x_0x_2 + b_1x_1x_2 + b_2x_2^2 = 0 \end{cases}$$

It represents the intersection of V with the plane

$$(*) \begin{cases} b_0w_{00} + b_1w_{01} + b_2w_{02} = 0 \\ b_0w_{01} + b_1w_{11} + b_2w_{12} = 0 \\ b_0w_{02} + b_1w_{12} + b_2w_{22} = 0 \end{cases},$$

so $v_{2,2}(\ell)$ is a plane curve. Its degree is the number of points in its intersection with a general hyperplane in $\mathbb{P}^5$: this corresponds to the intersection in $\mathbb{P}^2$ of ℓ with a conic (a hypersurface of degree 2). Therefore $v_{2,2}(\ell)$ is a conic.

So the isomorphism $v_{2,2}$ transforms the geometry of the lines in the plane in the geometry of the conics on the Veronese surface. In particular, given two distinct points on V, there is exactly one conic contained in V and passing through them.

From this observation it is easy to deduce that the *secant lines* of V, i.e. the lines meeting V at two points, are precisely the lines of the planes generated by the conics contained in V, so that the (closure of the) union of these secant lines coincides with the union of the planes of the conics of V. This union results to be the cubic hypersurface defined by the equation

$$\det M = \det \begin{pmatrix} w_{00} & w_{01} & w_{02} \\ w_{01} & w_{11} & w_{12} \\ w_{02} & w_{12} & w_{22} \end{pmatrix} = 0.$$

Indeed a point of $\mathbb{P}^5$, of coordinates $[w_{ij}]$ belongs to the plane of a conic contained in V if and only if there exists a non-zero triple $[b_0, b_1, b_2]$ which is solution of the homogeneous system (*).

b) Rational maps

Let X, Y be quasi–projective varieties.

9.10. Definition. The *rational maps* from X to Y are the germs of regular maps from open subsets of X to Y, i.e. equivalence classes of pairs (U, ϕ), where $U \neq \emptyset$ is open in X and $\phi : U \to Y$ is regular, with respect to the relation: $(U, \phi) \sim (V, \psi)$ if and only if $\phi|_{U \cap V} = \psi|_{U \cap V}$. The following Lemma guarantees that the above defined relation satisfies the transitive property.

9.11. Lemma. *Let $\phi, \psi : X \to Y \subset \mathbb{P}^n$ be regular maps between quasi-projective varieties. If $\phi|_U = \psi|_U$ for $U \subset X$ open and non–empty, then $\phi = \psi$.*

Proof. Let $P \in X$ and consider $\phi(P)$, $\psi(P) \in Y$. There exists a hyperplane H such that $\phi(P) \notin H$ and $\psi(P) \notin H$ (otherwise the dual projective space $\check{\mathbb{P}}^n$ would be the union of its two hyperplanes consisting of hyperplanes of $\mathbb{P}^n$ passing through $\phi(P)$ and $\psi(P)$). Up to a projective transformation, we can assume that $H = V_P(x_0)$, so $\phi(P), \psi(P) \in U_0$. Set $V = \phi^{-1}(U_0) \cap \psi^{-1}(U_0)$: an open neighbourhood of P. Consider the restrictions of ϕ and ψ from V to $Y \cap U_0$: they are regular maps which coincide on $V \cap U$, hence their coordinates ϕ_i, ψ_i, $i = 1, \dots, n$, coincide on $V \cap U$, hence on V. So $\phi_i|_V = \psi_i|_V$. In particular $\phi(P) = \psi(P)$. $\square$

A rational map from X to Y will be denoted $\phi : X \dashrightarrow Y$. As for rational functions, the domain of definition of ϕ, dom ϕ, is the maximum open subset of X such that ϕ is regular at the points of dom ϕ.

The following proposition follows from the characterization of rational functions on affine varieties.

9.12. Proposition. *Let X, Y be affine algebraic sets, with Y closed in $\mathbb{A}^n$. Then $\phi : X \dashrightarrow Y$ is a rational map if and only if $\phi = (\phi_1, \dots, \phi_n)$, where $\phi_1, \dots, \phi_n \in K(X)$.* $\square$

If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$, then a rational map $X \dashrightarrow Y$ is assigned by giving $m+1$

homogeneous polynomials of $K[x_0, x_1, \dots, x_n]$ of the same degree, $F_0, \dots, F_m$, such that *at least one* of them is not identically zero on X.

A rational map $\phi : X \dashrightarrow Y$ is called *dominant* if the image of X via ϕ is dense in X, i.e. if $\overline{\phi(U)} = X$, where $U = \text{dom } \phi$. If $\phi : X \dashrightarrow Y$ is dominant and $\psi : Y \dashrightarrow Z$ is any rational map, then dom $\psi \cap \text{Im}\phi \neq \emptyset$, so we can define $\psi \circ \phi : X \dashrightarrow Z$: it is the germ of the map $\psi \circ \phi$, regular on $\phi^{-1}(\text{dom } \psi \cap \text{Im}\phi)$.

9.13. Definition. A *birational map* from X to Y is a rational map $\phi : X \dashrightarrow Y$ such that ϕ is dominant and there exists $\psi : Y \dashrightarrow X$, a dominant rational map, such that $\psi \circ \phi = 1_X$ and $\phi \circ \psi = 1_Y$ as rational maps. In this case, X and Y are called *birationally equivalent* or simply *birational*.

If $\phi : X \dashrightarrow Y$ is a dominant rational map, then we can define the comorphism $\phi^* : K(Y) \to K(X)$ in the usual way: it is an injective K–homomorphism.

9.14. Proposition. *Let X, Y be quasi–projective varieties, $u : K(Y) \to K(X)$ be a K–homomorphism. Then there exists a rational map $\phi : X \dashrightarrow Y$ such that $\phi^* = u$.*

Proof. Y is covered by open affine varieties Y_α, $\alpha \in I$ (by Proposition 9.9): for all index α, $K(Y) \simeq K(Y_\alpha)$ (Prop. 8.8) and $K(Y_\alpha) \simeq K(t_1, \dots, t_n)$, where $t_1, \dots, t_n$ can be interpreted as coordinate functions on Y_α. Then $u(t_1), \dots, u(t_n) \in K(X)$ and there exists $U \subset X$, non–empty open subset such that $u(t_1), \dots, u(t_n)$ are all regular on U. So $u(K[t_1, \dots, t_n]) \subset \mathcal{O}(U)$ and we can consider the regular map $u^\sharp : U \to Y_\alpha \hookrightarrow Y$. The germ of $u^\sharp$ gives a rational map $X \dashrightarrow Y$. It is possible to check that this rational map does not depend on the choice of Y_α and U. $\square$

9.15. Theorem. *Let X, Y be quasi–projective varieties. The following are equivalent:*

(i) X is birational to Y;

(ii) $K(X) \simeq K(Y)$;

(iii) there exist non–empty open subsets $U \subset X$ and $V \subset Y$ such that $U \simeq V$.

Proof.

(i) $\Leftrightarrow$ (ii) via the construction of the comorphism ϕ^* associated to ϕ and of $u^\sharp$, associated to $u : K(Y) \to K(X)$. One checks that both constructions are functorial.

(i) $\Rightarrow$ (iii) Let $\phi : X \dashrightarrow Y$, $\psi : Y \dashrightarrow X$ be inverse each other. Put $U' = \text{dom } \phi$ and $V' = \text{dom } \psi$. By assumption, $\psi \circ \phi$ is defined on $\phi^{-1}(V')$ and coincides with 1_X there. Similarly, $\psi \circ \phi$ is defined on $\psi^{-1}(U')$ and equal to 1_Y. Then ϕ and ψ establish an isomorphism between the corresponding sets $U := \phi^{-1}(\psi^{-1}(U'))$ and $V := \psi^{-1}(\phi^{-1}(V'))$.

(iii) $\Rightarrow$ (ii) $U \simeq V$ implies $K(U) \simeq K(V)$; but $K(U) \simeq K(X)$ and $K(V) \simeq K(Y)$ (Prop.8.8), so $K(X) \simeq K(Y)$ by transitivity. $\square$

9.16. Corollary. *If X is birational to Y, then* $\dim X = \dim Y$. □

9.17. Examples.

a) The cuspidal cubic $Y = V(x^3 - y^2) \subset \mathbb{A}^2$.

We have seen that Y is not isomorphic to $\mathbb{A}^1$, but in fact Y and $\mathbb{A}^1$ are birational. Indeed, the regular map $\phi : \mathbb{A}^1 \to Y$, $t \to (t^2, t^3)$, admits a rational inverse $\psi : Y \dashrightarrow \mathbb{A}^1$, $(x, y) \to \frac{y}{x}$. ψ is regular on $Y \setminus \{(0,0)\}$, ψ is dominant and $\psi \circ \phi = 1_{\mathbb{A}^1}$, $\phi \circ \psi = 1_Y$ as rational maps. In particular, $\phi^* : K(Y) \to K(X)$ is a field isomorphism. Recall that $K[Y] = K[t_1, t_2]$, with $t_1^2 = t_2^3$, so $K(Y) = K(t_1, t_2) = K(t_2/t_1)$, because $t_1 = (t_2/t_1)^2 = t_2^2/t_1^2 = t_1^3/t_1^2$ and $t_2 = (t_2/t_1)^3 = t_2^3/t_1^3 = t_2^3/t_2^2$, so $K(Y)$ is generated by a unique transcendental element. Notice that ϕ and ψ establish isomorphisms between $\mathbb{A}^1 \setminus \{0\}$ and $Y \setminus \{(0,0)\}$.

b)*Rational maps from $\mathbb{P}^1$ to $\mathbb{P}^n$.*

Let $\phi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^n$ be rational: on some open $U \subset \mathbb{P}^1$,

$$\phi([x_0, x_1]) = [F_0(x_0, x_1), \dots, F_n(x_0, x_1)],$$

with $F_0, \dots, F_n$ homogeneous of the same degree, without non–trivial common factors. Assume that $F_i(P) = 0$ for a certain index i, with $P = [a_0, a_1]$. Then $F_i \in I_h(P) = \langle a_1x_0 - a_0x_1 \rangle$, i.e. $a_1x_0 - a_0x_1$ is a factor of F_i. This remark implies that $\forall\ Q \in \mathbb{P}^1$ there exists $i \in \{0, \dots, n\}$ such that $F_i(Q) \neq 0$, because otherwise $F_0, \dots, F_n$ would have a common factor of degree 1. Hence we conclude that ϕ is regular.

We have obtained that any rational map from $\mathbb{P}^1$ is in fact regular.

c) *Projections.*

Let $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be given in matrix form by $Y = AX$, where A is a $(m+1) \times (n+1)$-matrix, with entries in K. Then ϕ is a rational map, regular on $\mathbb{P}^n \setminus \mathbb{P}(\mathrm{Ker}A)$. Put $\Lambda := \mathbb{P}(\mathrm{Ker}A)$. If $A = (a_{ij})$, this means that Λ has cartesian equations

$$\begin{cases} a_{00}x_0 + \ldots + a_{0n}x_n = 0 \\ a_{10}x_0 + \ldots + a_{1n}x_n = 0 \\ \dots \\ a_{m0}x_0 + \ldots + a_{mn}x_n = 0 \end{cases}$$

The map ϕ has a geometric interpretation: it can be seen as the *projection of centre* Λ to a complementar linear space. First of all, we can assume that rk $A = m + 1$, otherwise we replace $\mathbb{P}^m$ with $\mathbb{P}(\mathrm{Im}\ A)$; hence $\dim \Lambda = n - (m + 1)$.

Consider first the case $\Lambda : x_0 = \ldots = x_m = 0$; we identify $\mathbb{P}^m$ with the subspace of $\mathbb{P}^n$ of equations $x_{m+1} = \ldots = x_n = 0$, so Λ and $\mathbb{P}^m$ are complementar subspaces, i.e. $\Lambda \cap \mathbb{P}^m = \emptyset$ and the linear span of Λ and $\mathbb{P}^m$ is $\mathbb{P}^n$. Then, for $Q \in \mathbb{P}^n \setminus \Lambda$, $\phi(Q) = [x_0, \dots, x_m, 0, \dots, 0]$: it is the intersection of $\mathbb{P}^m$ with the linear span of Λ and Q. In fact, if $Q[a_0, \dots, a_n]$ then $\overline{\Lambda Q}$ has equations

$$\{a_ix_j - a_jx_i = 0, i, j = 0, \dots, m \quad \text{(check!)}$$

so $\overline{\Lambda Q} \cap \mathbb{P}^m$ has coordinates $[a_0, \dots, a_m, 0, \dots, 0]$.

In the general case, if $\Lambda = V_P(L_0, \dots, L_m)$, with $L_0, \dots, L_m$ linearly independent forms, we can identify $\mathbb{P}^m$ with $V_P(L_{m+1}, \dots, L_n)$, where $L_0, \dots, L_m$, $L_{m+1}, \dots, L_n$ is a basis of $(K^{n+1})^*$. Then $L_0, \dots, L_m$ can be interpreted as coordinate functions on $\mathbb{P}^m$.

If $m = n - 1$, then Λ is a point P and ϕ, often denoted π_P, is the projection from P to a hyperplane not containing P.

d) *Rational and unirational varieties.*

A quasi–projective variety X is called *rational* if it is birational to a projective space $\mathbb{P}^n$, or equivalently to $\mathbb{A}^n$. Indeed, in view of Thereom 9.15 (*iii*), $\mathbb{P}^n$ and $\mathbb{A}^n$ are birationally equivalent.

By Theorem 9.15, X is rational if and only if $K(X) \simeq K(\mathbb{P}^n) = K(x_1, \dots, x_n)$ for some n, i.e. $K(X)$ is an extension of K generated by a transcendence basis (a purely transcendental extension of K). In an equivalent way, X is rational if there exists a rational map $\phi : \mathbb{P}^n \dashrightarrow X$ which is dominant and is an isomorphism if restricted to a suitable open subset $U \subset \mathbb{P}^n$. Hence X admits a *birational parametrization* by polynomials in n parameters.

A weaker notion is that of *unirational* variety: X is unirational if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$ i.e. if $K(X)$ is contained in the quotient field of a polynomial ring. Hence X can be parametrised by polynomials, but not necessarily generically one–to–one.

It is clear that, if X is rational, then it is unirational. The converse implication has been an important open problem, up to 1971, when it has been solved in the negative, for varieties of dimension ≥ 3 (Clemens–Griffiths and Iskovskih–Manin). Nevertheless rationality and unirationality are equivalent for curves (Theorem of Lüroth, 1880) and for surfaces if $\mathrm{char} K = 0$ (Theorem of Castelnuovo, 1894).

As an example of rational variety with an explicit rational parametrization constructed geometrically, let us consider the following quadric of maximal rank in $\mathbb{P}^3$: $X = V_P(x_0x_3 - x_1x_2)$, an irreducible hypersurface of degree 2. Let $\pi_P : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection of centre $P[1, 0, 0, 0]$, such that $\pi_P([y_0, y_1, y_2, y_3]) = [y_1, y_2, y_3]$. The restriction of π_P to X is a rational map $\tilde{\pi}_P : X \dashrightarrow \mathbb{P}^2$, regular on $X \setminus \{P\}$. $\tilde{\pi}_P$ has a rational inverse: indeed consider the rational map $\psi : \mathbb{P}^2 \dashrightarrow X$, $[y_1, y_2, y_3] \to [y_1y_2, y_1y_3, y_2y_3, y_3^2]$. The equation of X is satisfied by the points of $\psi(\mathbb{P}^2)$: $(y_1y_2)y_3^2 = (y_1y_3)(y_2y_3)$. ψ is regular on $\mathbb{P}^2 \setminus V_P(y_1y_2, y_3)$. Let us compose ψ and $\tilde{\pi}_P$:

$$[y_0, \dots, y_3] \in X \xrightarrow{\pi_P} [y_1, y_2, y_3] \xrightarrow{\psi} [y_1y_2, y_1y_3, y_2y_3, y_3^2];$$

$y_1y_2 = y_0y_3$ implies $\psi \circ \pi_P = 1_X$. In the opposite order:

$$[y_1, y_2, y_3] \xrightarrow{\psi} [y_1y_2, y_1y_3, y_2y_3, y_3^2] \xrightarrow{\pi_P} [y_1y_3, y_2y_3, y_3^2] = [y_1, y_2, y_3].$$

So X is birational to $\mathbb{P}^2$ hence it is a rational surface.

Note that if we consider a projection π_P whose centre P is not on the quadric, we get a regular $2:1$ map to the plane, certainly not birational.

e) *A birational non–regular map from* $\mathbb{P}^2$ *to* $\mathbb{P}^2$.

The following rational map is called the *standard quadratic map*:

$$Q : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad [x_0, x_1, x_2] \to [x_1x_2, x_0x_2, x_0x_1].$$

Q is regular on $U := \mathbb{P}^2 \setminus \{A, B, C\}$, where $A[1,0,0]$, $B[0,1,0]$, $C[0,0,1]$ are the fundamental points (see Fig. 2)

Let a be the line through B and C: $a = V_P(x_0)$, and similarly $b = V_P(x_1)$, $c = V_P(x_2)$. Then $Q(a) = A$, $Q(b) = B$, $Q(c) = C$. Outside these three lines Q is an isomorphism. Precisely, put $U' = \mathbb{P}^2 \setminus \{a \cup b \cup c\}$; then $Q : U' \to \mathbb{P}^2$ is regular, the image is U' and $Q^{-1} : U' \to U'$ coincides with Q. Indeed,

$$[x_0, x_1, x_2] \overset{Q}{\to} [x_1x_2, x_0x_2, x_0x_1] \overset{Q}{\to} [x_0^2x_1x_2, x_0, x_1^2x_2, x_0x_1x_2^2].$$

So $Q \circ Q = 1_{\mathbb{P}^2}$ as rational map, hence Q is birational and $Q = Q^{-1}$.

– Fig. 2 –

The set of the birational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a group, called the *Cremona group*. At the end of XIX century, Max Noether proved that the Cremona group is generated by $PGL(3, K)$ and by the single standard quadratic map above. The analogous groups for $\mathbb{P}^n$, $n \geq 3$, are much more complicated and a complete description is still unknown.

We conclude this section with a theorem illustrating an application of the linearisation procedure. We shall use the following notation: given a homogeneous polynomial $F \in K[x_0, x_1, \dots, x_n]$, $D(F) := \mathbb{P}^n \setminus V_P(F)$.

9.16. Theorem. *Let $W \subset \mathbb{P}^n$ be a closed projective variety. Let F be a homogeneous polynomial of degree d in $K[x_0, x_1, \dots, x_n]$ such that $W \not\subseteq V_P(F)$. Then $W \cap D(F)$ is an affine variety.*

Proof. The assumption $W \not\subseteq V_P(F)$ is equivalent to $W \cap D(F) \neq \emptyset$. Let us consider the d-tuple Veronese embedding $v_{n,d} : \mathbb{P}^n \to \mathbb{P}^{N(n,d)}$, with $N(n,d) = \binom{n+d}{d} - 1$, that gives the isomorphism $\mathbb{P}^n \simeq V_{n,d}$. In this isomorphism the hypersurface $V_P(F)$ corresponds to a hyperplane section $V_{n,d} \cap H$, for a suitable hyperplane H in $\mathbb{P}^{N(n,d)}$. Therefore we have $W \cap D(F) \simeq v_{n,d}(W \cap D(F)) = v_{n,d}(W) \setminus H = v_{n,d}(W) \cap (\mathbb{P}^{N(n,d)} \setminus H)$. There exists a projective isomorphism $\tau : \mathbb{P}^{N(n,d)} \to \mathbb{P}^{N(n,d)}$ such that $\tau(H) = H_0$, the fundamental hyperplane of equation $x_0 = 0$. Therefore, denoting $X := v_{n,d}(W)$, we get $X \cap (\mathbb{P}^{N(n,d)} \setminus H) \simeq \tau(X) \cap (\mathbb{P}^{N(n,d)} \setminus H_0) = \tau(X) \cap U_0$, which proves the theorem. □

As a consequence of Theorem 9.16, we get that the open subsets of the form $W \cap D(F)$ form a topology basis of affine varieties for W.

Exercises to §9.

1. Let $\phi : \mathbb{A}^1 \to \mathbb{A}^n$ be the map defined by $t \to (t, t^2, \dots, t^n)$.

a) Prove that ϕ is regular and describe $\phi(\mathbb{A}^1)$;

b) prove that $\phi : \mathbb{A}^1 \to \phi(\mathbb{A}^1)$ is an isomorphism;

c) give a description of ϕ^* and ϕ^{-1*}.

2. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be defined by: $(x, y) \to (x, xy)$.

a) Describe $f(\mathbb{A}^2)$ and prove that it is not locally closed in $\mathbb{A}^2$.

b) Prove that $f(\mathbb{A}^2)$ is a constructible set in the Zariski topology of $\mathbb{A}^2$ (i.e. a finite union of locally closed sets).

3. Prove that the Veronese variety $V_{n,d}$ is not contained in any hyperplane of $\mathbb{P}^{N(n,d)}$.

4. Let $GL_n(K)$ be the set of invertible $n \times n$ matrices with entries in K. Prove that $GL_n(K)$ can be given the structure of an affine variety.

5. Show the unicity of the projective transformation τ of Theorem 9.8.

6. Let $\phi : X \to Y$ be a regular map and ϕ^* its comorphism. Prove that the kernel of ϕ^* is the ideal of $\phi(X)$ in $\mathcal{O}(Y)$. In the affine case, deduce that ϕ is dominant if and only if ϕ^* is injective.

7. Prove that $\mathcal{O}(X_F)$ is isomorphic to $\mathcal{O}(X)_f$, where X is an affine algebraic variety, F a polynomial and f the function on X defined by F.

10. Products of quasi–projective varieties, tensors and Grassmannians.

a) Products

Let $\mathbb{P}^n$, $\mathbb{P}^m$ be projective spaces over the same field K. The cartesian product $\mathbb{P}^n \times \mathbb{P}^m$ is simply a set: we want to define an injective map from $\mathbb{P}^n \times \mathbb{P}^m$ to a suitable projective space, so that the image is a projective variety, which will be identified with our product.

Let $N = (n+1)(m+1)-1$ and define $\sigma : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ in the following way: $\sigma([x_0,\dots,x_n],[y_0,\dots,y_m]) = [x_0y_0, x_0y_1,\dots,x_iy_j,\dots,x_ny_m]$. Using coordinates w_{ij}, $i = 0,\dots,n$, $j = 0,\dots,m$, in $\mathbb{P}^N$, σ is defined by

$$\{w_{ij} = x_iy_j, i = 0,\dots,n, j = 0,\dots,m.$$

It is easy to observe that σ is a well–defined map.

Let $\Sigma_{n,m}$ (or simply Σ) denote the image $\sigma(\mathbb{P}^n \times \mathbb{P}^m)$.

10.1. Proposition. *σ is injective and $\Sigma_{n,m}$ is a closed subset of $\mathbb{P}^N$.*

Proof. If $\sigma([x],[y]) = \sigma([x'],[y'])$, then there exists $\lambda \neq 0$ such that $x'_iy'_j = \lambda x_iy_j$ for all i,j. In particular, if $x_h \neq 0$, $y_k \neq 0$, then also $x'_h \neq 0$, $y'_k \neq 0$, and for all i $x'_i = \lambda \frac{y_k}{y'_k} x_i$, so $[x_0,\dots,x_n] = [x'_0,\dots,x'_n]$. Similarly for the second point.
To prove the second assertion, I claim: $\Sigma_{n,m}$ is the closed set of equations:

$$(*)\{w_{ij}w_{hk} = w_{ik}w_{hj},\ i,h = 0,\dots,n; j,k = 0\dots,m.$$

It is clear that if $[w_{ij}] \in \Sigma$, then it satisfies (*). Conversely, assume that $[w_{ij}]$ satisfies (*) and that $w_{\alpha\beta} \neq 0$. Then

$$\begin{aligned}[w_{00},\dots,w_{ij},\dots,w_{nm}] &= [w_{00}w_{\alpha\beta},\dots,w_{ij}w_{\alpha\beta},\dots,w_{nm}w_{\alpha\beta}] = \\ &= [w_{0\beta}w_{\alpha 0},\dots,w_{i\beta}w_{\alpha j},\dots,w_{n\beta}w_{\alpha m}] = \\ &= \sigma([w_{0\beta},\dots,w_{n\beta}],[w_{\alpha 0},\dots,w_{\alpha m}]).\end{aligned}$$

□

σ is called the Segre map and $\Sigma_{n,m}$ the Segre variety or biprojective space. Note that Σ is covered by the affine open subsets $\Sigma^{ij} = \Sigma \cap W_{ij}$, where $W_{ij} = \mathbb{P}^N \setminus V_P(w_{ij})$. Moreover $\Sigma^{ij} = \sigma(U_i \times V_j)$, where $U_i \times V_j$ is naturally identified with $\mathbb{A}^{n+m}$.

10.2. Proposition. *$\sigma|_{U_i\times V_j} : U_i \times V_j = \mathbb{A}^{n+m} \to \Sigma^{ij}$ is an isomorphism of varieties.*

Proof. Assume by simplicity $i = j = 0$. Choose non–homogeneous coordinates on U_0: $u_i = x_i/x_0$ and on V_0: $v_j = y_j/y_0$. So $u_1,\dots u_n, v_1,\dots,v_m$ are coordinates on

$U_0 \times V_0$. Take non–homogeneous coordinates also on W_{00}: $z_{ij} = w_{ij}/w_{00}$. Using these coordinates we have:

$$\begin{array}{l} \sigma|_{U_i \times V_j} :(u_1, \dots u_n, v_1, \dots, v_m) \to (v_1, \dots, v_m, u_1, u_1v_1, \dots, u_1v_m, \dots, u_nv_m) \\ \qquad\qquad || \\ \qquad ([1, u_1, \dots, u_n], [1, v_1, \dots, v_m]) \end{array}$$

i.e. $\sigma(u_1, \dots, v_m) = (z_{01}, \dots, z_{nm})$, where

$$\begin{cases} z_{i0} = u_i, & \text{if } i = 1, \dots, n; \\ z_{0j} = v_j, & \text{if } j = 1, \dots, m; \\ z_{ij} = u_iv_j = z_{i0}z_{0j} & \text{otherwise.} \end{cases}$$

Hence $\sigma|_{U_0 \times V_0}$ is regular.
The inverse map takes $(z_{01}, \dots, z_{nm})$ to $(z_{10}, \dots, z_{n0}, z_{01}, \dots, z_{0m})$, so it is also regular. $\square$

10.3. Corollary. *$\mathbb{P}^n \times \mathbb{P}^m$ is irreducible and birational to $\mathbb{P}^{n+m}$.*

Proof. The first assertion follows from Ex.5, Ch.6, considering the covering of Σ by the open subsets Σ^{ij}. Indeed, $\Sigma^{ij} \cap \Sigma^{hk} = \sigma((U_i \times V_j) \cap (U_h \times V_k)) = \sigma((U_i \cap U_h) \times (V_j \cap V_k))$, and $U_i \cap U_h \neq \emptyset \neq V_j \cap V_k$.

For the second assertion, by Theorem 9.15, it is enough to note that $\Sigma_{n,m}$ and $\mathbb{P}^{n+m}$ contain isomorphic open subsets, i.e. Σ^{ij} and $\mathbb{A}^{n+m}$. $\square$

From now on, we shall identify $\mathbb{P}^n \times \mathbb{P}^m$ with $\Sigma_{n,m}$. If $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are any quasi–projective varieties, then $X \times Y$ will be automatically identified with $\sigma(X \times Y) \subset \Sigma$.

10.4. Proposition. *If X and Y are projective varieties (resp. quasi–projective varieties), then $X \times Y$ is projective (resp. quasi–projective).*

Proof.

$$\begin{aligned} \sigma(X \times Y) &= \bigcup_{i,j} (\sigma(X \times Y) \cap \Sigma^{ij}) = \\ &= \bigcup_{i,j} (\sigma(X \times Y) \cap (U_i \times V_j)) = \\ &= \bigcup_{i,j} (\sigma((X \cap U_i) \times (Y \cap V_j))). \end{aligned}$$

If X and Y are projective varieties, then $X \cap U_i$ is closed in U_i and $Y \cap V_j$ is closed in V_j, so their product is closed in $U_i \times V_j$; since $\sigma|_{U_i \times V_j}$ is an isomorphism, also $\sigma(X \times Y) \cap \Sigma^{ij}$ is closed in Σ^{ij}, so $\sigma(X \times Y)$ is closed in Σ, by Lemma 8.3.

If X, Y are quasi–projective, the proof is similar: $X \cap U_i$ is locally closed in U_i and $Y \cap V_j$ is locally closed in V_j, so $X \cap U_i = Z \setminus Z'$, $Y \cap V_j = W \setminus W'$, with

Z, Z', W, W' closed. Therefore $(Z \setminus Z') \times (W \setminus W') = Z \times W \setminus ((Z' \times W) \cup (Z \times W'))$, which is locally closed.

As for the irreducibility, see Exercise 10.1. □

10.5. Example. $\mathbb{P}^1 \times \mathbb{P}^1$
$\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is given by $\{w_{ij} = x_i y_j,\ i = 0, 1,\ j = 0, 1$. Σ has only one non–trivial equation: $w_{00}w_{11} - w_{01}w_{10}$, hence Σ is a quadric. The equation of Σ can be written as

$$(*) \qquad \begin{vmatrix} w_{00} & w_{01} \\ w_{10} & w_{11} \end{vmatrix} = 0.$$

Σ contains two families of special closed subsets parametrised by $\mathbb{P}^1$, i.e.

$$\{\sigma(P \times \mathbb{P}^1)\}_{P \in \mathbb{P}^1} \quad \text{and} \quad \{\sigma(\mathbb{P}^1 \times Q)\}_{Q \in \mathbb{P}^1}.$$

If $P[a_0, a_1]$, then $\sigma(P \times \mathbb{P}^1)$ is given by the equations:

$$\begin{cases} w_{00} = a_0 y_0 \\ w_{01} = a_0 y_1 \\ w_{10} = a_1 y_0 \\ w_{11} = a_1 y_1 \end{cases}$$

hence it is a line. Cartesian equations of $\sigma(P \times \mathbb{P}^1)$ are:

$$\begin{cases} a_1 w_{00} - a_0 w_{10} = 0 \\ a_1 w_{01} - a_0 w_{11} = 0; \end{cases}$$

they express the proportionality of the rows of the matrix (*) with coefficients $[a_1, -a_0]$. Similarly, $\sigma(\mathbb{P}^1 \times Q)$ is the line of equations

$$\begin{cases} a_1 w_{00} - a_0 w_{01} = 0 \\ a_1 w_{10} - a_0 w_{11} = 0. \end{cases}$$

Hence Σ contains two families of lines, called the rulings of Σ: two lines of the same ruling are clearly disjoint while two lines of different rulings intersect at one point $(\sigma(P, Q))$. Conversely, through any point of Σ there pass two lines, one for each ruling. Note that Σ is exactly the quadric surface of Example 9.17, d) and that the projection of centre $[1, 0, 0, 0]$ realizes an explicit birational map between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$.

b) Tensors

The product of projective spaces has a coordinate-free description in terms of tensors. Precisely, let $\mathbb{P}^n = \mathbb{P}(V)$ and $\mathbb{P}^m = \mathbb{P}(W)$. The tensor product $V \otimes W$ of the vector spaces V, W is constructed as follows: let $K(V \times W)$ be the K-vector space with basis $V \times W$ obtained as the set of formal finite linear combinations of type $\Sigma_i a_i(v_i, w_i)$ with $a_i \in K$. Let U be the vector subspace generated by all elements of the form:

$(v, w) + (v', w) - (v + v', w)$,
$(v, w) + (v, w') - (v, w + w')$,
$(\lambda v, w) - \lambda(v, w)$,
$(v, \lambda w) - (\lambda(v, w)$,

with $v, v' \in V$, $w, w' \in W$, $\lambda \in K$. The tensor product is by definition the quotient $V \otimes W := K(V \times W)/U$. The class of a pair (v, w) is denoted $v \otimes w$, and called a decomposable tensor. So $V \otimes W$ is generated by the decomposable tensors; more precisely, a general element $\omega \in V \otimes W$ is of the form $\Sigma_{i=1}^k v_i \otimes w_i$, with $v_i \in V$, $w_i \in W$. The minimum k such that an expression of this type exists is called the tensor rank of ω.

There is a natural bilinear map $\otimes : V \times W \to V \otimes W$, such that $(v, w) \to v \otimes w$. It enjoys the following universal property: for any K-vector space Z with a bilinear map $f : V \times W \to Z$, there exists a unique linear map $\bar{f} : V \otimes W \to Z$ such that f factorizes in the form $f = \bar{f} \circ \otimes$.

If $\dim V = n$, $\dim W = m$, and bases $\mathcal{B} = (e_1, \ldots, e_n), \mathcal{B}' = (e'_1, \ldots, e'_m)$ are given, then $(e_1 \otimes e'_1, \ldots, e_i \otimes e'_j, \ldots e_n \otimes e'_m)$ is a basis of $V \otimes W$: therefore $\dim V \otimes W = nm$.

If $v = x_1 e_1 + \ldots x_n e_n$, $w = y_1 e'_1 + \ldots y_m e'_m$, then $v \otimes w = \Sigma x_i y_j e_i \otimes e'_j$. So, passing to the projective spaces, the map $\otimes$ defines precisely the Segre map $\sigma : \mathbb{P}(V) \times \mathbb{P}(W) \to \mathbb{P}(V \otimes W)$, $([v], [w]) \to [v \otimes w]$. Indeed in coordinates we have $([x_0, \ldots, x_n], [y_0, \ldots, y_m]) \to [w_{00}, \ldots, w_{nm}]$, with $w_{ij} = x_i y_j$. The image of $\otimes$ is the set of decomposable tensors, or rank one tensors.

The tensor product $V \otimes W$ has the same dimension, and is therefore isomorphic to the vector space of $n \times m$ matrices. The coordinates w_{ij} can be interpreted as the entries of such a $n \times m$ matrix. The equations of the Segre variety $\Sigma_{n,m}$ are the 2×2 minors of the matrix, therefore $\Sigma_{n,m}$ can be interpreted as the set of matrices of rank one.

The construction of the tensor product can be iterated, to construct $V_1 \otimes V_2 \otimes \ldots \otimes V_r$. The following properties can easily be proved:

1. $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$;
2. $V \otimes W \simeq W \otimes V$;
3. $V^* \otimes W \simeq Hom(V, W)$, where $f \otimes w \to (V \to W : v \to f(v)w)$.

Also the Veronese morphism has a coordinate free description, in terms of symmetric tensors. Given a vector space V, for any $d \geq 0$ the d-th symmetric power of V, $S^d V$ or $Sym^d V$, is constructed as follows. We consider the tensor product of d copies of V: $V \otimes \ldots \otimes V = V^{\otimes d}$, and we consider its subvector space U generated by tensors of the form $v_1 \otimes \ldots v_d - v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}$, where σ varies in the symmetric group on d elements $\mathcal{S}_d$. Then by definition $S^d V := V^{\otimes d}/U$.

The equivalence class $[v_1 \otimes \ldots \otimes v_d]$ is denoted as a product $v_1 \ldots v_d$.

There is a natural multilinear and symmetric map $V \times \ldots \times V = V^d \to S^dV$, such that $(v_1, \ldots, v_d) \to v_1 \ldots v_d$, which enjoys the universal property. S^dV is generated by the products $v_1 \ldots v_d$.

S^dV can also be interpreted as a subspace of $V^{\otimes d}$, by considering the following map, that is an isomorphism to the image:

$$S^dV \to V^{\otimes d}, \quad v_1 \ldots v_d \to \Sigma_{\sigma \in \mathcal{S}_d} \frac{1}{d!} v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(d)}.$$

If $\mathcal{B} = (e_1, \ldots, e_n)$ is a basis of V, then it is easy to check that a basis of S^dV is formed by the monomials of degree d in $e_1, \ldots, e_n$; therefore $\dim S^dV = \binom{n+d-1}{d}$.

For instance, in S^2V the product v_1v_2 can be identified with $\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1)$.

The symmetric algebra of V is $SV := \oplus_{d \geq 0} S^dV = K \oplus V \oplus S^2V \oplus \ldots$. An inner product can be naturally defined to give it the structure of a K-algebra, which results to be isomorphic to the polynomial ring in n variables, where $n = \dim V$.

If $char K = 0$ the Veronese morphism can be interpreted in the following way:

$$v_{n,d} : \mathbb{P}(V) \to \mathbb{P}(S^dV), \; [v] = [x_0e_0 + \ldots x_ne_n] \to [v^d] = [(x_0e_0 + \ldots + x_ne_n)^d].$$

Moreover S^2V can be interpreted as space of the symmetric $d \times d$ matrices, and the Veronese variety $V_{n,2}$ as the subset of the symmetric matrices of rank one.

In a similar way it is possible to define the exterior powers of the vector space V. One defines the d-th exterior power $\wedge^dV$ as the quotient $V^{\otimes d}/\Lambda$, where Λ is generated by the tensors of the form $v_1 \otimes \ldots \otimes v_i \otimes \ldots \otimes v_j \otimes \ldots \otimes v_d$, with $v_i = v_j$ for some $i \neq j$. The following notation is used: $[v_1 \otimes \ldots \otimes v_d] = v_1 \wedge \ldots \wedge v_d$.

There is a natural multilinear alternating map $V \times \ldots \times V = V^d \to \wedge^dV$, that enjoys the universal property. Given a basis of V as before, a basis of $\wedge^dV$ is formed by the tensors $e_{i_1} \wedge \ldots \wedge e_{i_d}$, with $1 \leq i_1 < \ldots < i_d \leq n$. Therefore $\dim \wedge^dV = \binom{n}{d}$. The exterior algebra of V is the following direct sum: $\wedge V = \oplus_{d \geq 0} \wedge^d V = K \oplus V \oplus \wedge^2V \oplus \ldots$. To define an inner product that gives it the structure of algebra we can proceed as follows.

Step 1. Fixed $v_1, \ldots, v_q \in V$, define $f : V^d \to \wedge^{d+p}V$ posing $f(x_1, \ldots, x_d) = x_1 \wedge \ldots \wedge x_d \wedge v_1 \wedge \ldots \wedge v_q$. Since f results to be multilinear and alternating, by the universal property we get a factorization of f through $\wedge^dV$, which gives a linear map $\bar{f} : \wedge^dV \to \wedge^{d+p}V$, extending f. For any $\omega \in \wedge^dV$, we denote $\bar{f}(\omega)$ by $\omega \wedge v_1 \wedge \ldots \wedge v_d$.

Step 2. Fixed $\omega \in \wedge^dV$, consider the map $g : V^p \to \wedge^{d+p}V$ such that $g(y_1, \ldots, y_p) = \omega \wedge y_1 \wedge \ldots \wedge y_p$: it is multilinear and alternating, therefore it factorizes through $\wedge^pV$ and we get a linear map $\bar{g} : \wedge^pV \to \wedge^{d+p}V$, extending g. We denote $\bar{g}(\sigma) := \omega \wedge \sigma$.

Step 3. For any $d, p \geq 0$ we have got a map $\wedge : \wedge^d V \times \wedge^p V \to \wedge^{d+p} V$, that results to be bilinear, and extends to an inner product $\wedge : (\wedge V) \times (\wedge V) \to \wedge V$, which gives $\wedge V$ the required structure of algebra.

10.6. Proposition. *Let V be a vector space of dimension n.*

(i) Vectors $v_1, \dots, v_p \in V$ are linearly dependent if and only if $v_1 \wedge \dots \wedge v_p = 0$.

(ii) Let $v \in V$ be a non-zero vector, and $\omega \in \wedge^p V$. Then $\omega \wedge v = 0$ if and only if there exists $\Phi \in \wedge^{p-1} V$ such that $\omega = \Phi \wedge v$. In this case we say that v divides ω.

Proof. The proof of (i) is standard. If $\omega = \Phi \wedge v$, then $\omega \wedge v = (\Phi \wedge v) \wedge v = \Phi \wedge (v \wedge v) = 0$. Conversely, if $\omega \wedge v = 0$, $v \neq 0$, we choose a basis of V, $\mathcal{B} = (e_1, \dots, e_n)$ with $e_1 = v$. Write $\omega = \Sigma_{i_1 < \dots < i_p} a_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$. Then $0 = \omega \wedge e_1 = \Sigma_{i_1 < \dots < i_p} (+-) a_{i_1 \dots i_p} e_1 \wedge e_{i_1} \wedge \dots \wedge e_{i_p}$. If $i_1 = 1$, the corresponding summand does not appear in this sum, so it remains a linear combination of linearly independent tensors, which implies $a_{i_1 \dots i_p} = 0$ every time $i_1 > 1$. Therefore $\omega = e_1 \wedge \Phi$ for a suitable Φ.

□

10.7. Proposition. *Let $\omega \neq 0$ be an element of $\wedge^p V$. Then ω is totally decomposable if and only if the subspace of V: $W = \{v \in V \mid v \text{ divides } \omega\}$ has dimension p.*

Proof. If $\omega = x_1 \wedge \dots \wedge x_p \neq 0$, then $x_1, \dots, x_p$ are linearly independent and belong to W. So we can extend them to a basis of V adding vectors $x_{p+1}, \dots, x_n$. If $v \in W$, $v = \alpha_1 x_1 + \dots + \alpha_n x_n$, and v divides ω, then $\omega \wedge v = 0$, i.e. $x_1 \wedge \dots \wedge x_n \wedge (\alpha_1 x_1 + \dots + \alpha_n x_n) = 0$. This implies $\alpha_{p+1} x_1 \wedge \dots \wedge x_p \wedge x_{p+1} + \dots + \alpha_n x_1 \wedge \dots \wedge x_p \wedge x_n$, therefore $\alpha_{p+1} = \dots = \alpha_n = 0$, so $v \in \langle x_1, \dots, x_n \rangle$.

Conversely, if $(x_1, \dots, x_p)$ is a basis of W, we can complete it to a basis of V and write $\omega = \Sigma a_{i_1 \dots i_p} x_{i_1} \wedge \dots \wedge x_{i_p}$. But x_1 divides ω, so $\omega \wedge x_1 = 0$. Replacing ω with its explicit expression, we obtain that $a_{i_1 \dots i_p} = 0$ if $1 \notin \{i_1, \dots, i_p\}$. Repeating this argument for $x_2, \dots, x_p$, it remains $\omega = a_{1 \dots p} x_1 \wedge \dots \wedge x_p$. □

With explicit computations, one can prove the following proposition.

10.8. Proposition. *Let V be a vector space with $\dim V = n$. Let $\mathcal{B} = (e_1, \dots, e_n)$ be a basis of V and $v_1, \dots, v_n$ be any vectors. Then $v_1 \wedge \dots \wedge v_n = \det(A) e_1 \wedge \dots \wedge e_n$, where A is the matrix of the coordinates of the vectors $v_1, \dots, v_n$ with respect to $\mathcal{B}$.*

10.9. Corollary. *Let $v_1, \dots, v_p \in V$, with $v_j = \Sigma a_{ij} e_j$, $j = 1, \dots, p$. Then $v_1 \wedge \dots \wedge v_p = \Sigma_{i_1 < \dots < i_p} a_{i_1 \dots i_p} e_{i_1} \wedge \dots \wedge e_{i_p}$, with $a_{i_1 \dots i_p} = \det(A_{i_1 \dots i_p})$, the determinant of the $p \times p$ submatrix of A containing the columns of indices $i_1, \dots, i_p$.*

c) Grassmannians

Let V be a vector space of dimension n, and r be a positive integer, $1 \leq r \leq n$. The Grassmannian $G(r,V)$ is the set of the subspaces of V of dimension r. It can be denoted also $G(r,n)$.

There is a natural bijection between $G(r,V)$ and the set of the projective subspaces of $\mathbb{P}(V)$ of dimension $r-1$, denoted $\mathbb{G}(r-1,\mathbb{P}(V))$ or $\mathbb{G}(r-1,n-1)$. Let $W \in G(r,V)$; if $(w_1,\dots,w_r)$ and $(x_1,\dots,x_r)$ are two bases of W, then $w_1 \wedge \dots \wedge w_r = \lambda x_1 \wedge x_r$, where $\lambda \in K$ is the determinant of the matrix of the change of basis. Therefore W uniquely determines an element of $\wedge^r V$ up to proportionality. This allows to define a map, called the Plücker map, $\psi : G(r,V) \to \mathbb{P}(\wedge^r V)$, such that $\psi(W) = [w_1 \wedge \dots w_r]$.

10.10. Proposition. *The Plücker map is injective.*

Proof. Assume $\psi(W) = \psi(W')$, where W, W' are subspaces of V of dimension r with bases $(x_1,\dots,x_r)$ and $(y_1,\dots,y_r)$. So there exists $\lambda \neq 0$ in K such that $x_1 \wedge \dots \wedge x_r = \lambda y_1 \wedge \dots \wedge y_r$. This implies $x_1 \wedge \dots \wedge x_r \wedge y_i = 0$ for all i, so y_i is linearly dependent from $x_1,\dots,x_r$, so $y_i \in W$. Therefore $W' \subset W$. The reverse inclusion is similar. $\square$

In coordinates, $\psi(W)$ is given by the minors of maximal order r of the matrix of the coordinates of the vectors of a basis of W, with respect to a fixed basis of V.

10.11. Examples.

(i) $r = n-1$: $\wedge^{n-1}V$ has dimension n, so it is isomorphic to the dual vector space V^*, associating to $e_1 \wedge \dots \wedge \hat{e}_k \wedge \dots \wedge e_n$ the linear form e_k^* of the dual basis. In this case the Plücker map is surjective, so $G(n-1,n) \simeq V^*$.

(ii) $n = 4, r = 2$: $G(2,4)$ or $\mathbb{G}(1,3)$, the Grassmannian of lines of $\mathbb{P}^3$. In this case $\psi : \mathbb{G}(1,3) \to \mathbb{P}(\wedge^2 V) \simeq \mathbb{P}^5$. Let (e_0,e_1,e_2,e_3) be a basis of V. If ℓ is the line of $\mathbb{P}^3$ obtained by projectivisation of a subspace $L \subset V$ of dimension 2, let $L = \langle x,y \rangle$; then $\psi(\ell) = [x \wedge y]$. Its Plücker coordinates are denoted $p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23}$, and $p_{ij} = x_i y_j - x_j y_i$, the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}.$$

This time ψ is not surjective; its image is formed by the totally decomposable tensors. They satisfy the equation of degree 2: $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$, which represents a quadric of maximal rank in $\mathbb{P}^5$, called the Klein quadric. The fact that this equation is satisfied can be seen by considering the 4×4 matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}:$$

its determinant is precisely the above equation.

For instance the line of equations $x_2 = x_3 = 0$, obtained projectivising the subspace $\langle e_0, e_1\rangle$, has Plücker coordinates $[1,0,0,0,0,0]$.

In general we can prove the following theorem.

10.12. Theorem. *The image of the Plücker map is a closed subset in* $\mathbb{P}(\wedge^r V)$.

Proof. The image of the Plücker map is the set of the proportionality classes of totally decomposable tensors. By Proposition 10.7, a tensor $\omega \in \wedge^r V$ is totally decomposable if and only if the subspace $W = \{v \in V \mid v \text{ divides } \omega\}$ has dimension r. We consider the linear map $\Phi : V \to \wedge^{r+1} V$, such that $\Phi(v) = \omega \wedge v$. The kernel of Φ is equal to W. So ω is totally decomposable if and only if the rank of Φ is $n-r$. Fixed a basis $\mathcal{B} = (e_1, \dots, e_n)$ of V, we write $\omega = \Sigma_{i_1<\dots<i_r} a_{i_1\dots i_r} e_{i_1} \wedge \dots \wedge e_{i_r}$. We then consider the basis of $\wedge^{r+1}V$ associated to $\mathcal{B}$ and we construct the matrix A of Φ with respect to these bases: its minors of order $n-p+1$ are equations of the image of ψ, and they are polynomials in the coordinates $a_{i_1\dots i_r}$ of ω. □

From now on we shall identify the Grassmannian with the projective algebraic set that is its image in the Plücker map. The equations obtained in Theorem 10.12 are nevertheless not generators for the ideal of the Grassmannian. For instance, in the case $n = 4, r = 2$, let $\omega = p_{01}e_0 \wedge e_1 + p_{02}e_0 \wedge e_2 + \dots$. Then:

$\Phi(e_0) = \omega \wedge e_0 = p_{12}e_0 \wedge e_1 \wedge e_2 + p_{13}e_0 \wedge e_1 \wedge e_3 + p_{23}e_0 \wedge e_2 \wedge e_3$;

$\Phi(e_1) = \omega \wedge e_1 = -p_{02}e_0 \wedge e_1 \wedge e_2 - p_{03}e_0 \wedge e_1 \wedge e_3 + p_{23}e_1 \wedge e_2 \wedge e_3$;

$\Phi(e_2) = \omega \wedge e_2 = p_{01}e_0 \wedge e_1 \wedge e_2 - p_{03}e_0 \wedge e_2 \wedge e_3 + p_{13}e_1 \wedge e_2 \wedge e_3$;

$\Phi(e_3) = \omega \wedge e_3 = p_{01}e_0 \wedge e_1 \wedge e_3 + p_{02}e_0 \wedge e_2 \wedge e_3 + p_{12}e_1 \wedge e_2 \wedge e_3$.

So the matrix is

$$\begin{pmatrix} p_{12} & -p_{02} & p_{01} & 0 \\ p_{13} & -p_{03} & 0 & p_{01} \\ p_{23} & 0 & -p_{03} & p_{02} \\ 0 & p_{23} & p_{13} & p_{12} \end{pmatrix}.$$

Its 3×3 minors are equations defining $\mathbb{G}(1,3)$, but the radical of the ideal generated by these minors is in fact $(p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12})$.

To find equations for the Grassmannian and to prove that it is irreducible, it is convenient to give an explicit open covering with affine open subsets. In $\mathbb{P}(\wedge^r V)$, let $U_{i_1\dots i_r}$ be the affine open subset where the Plücker coordinate $p_{i_1\dots i_r} \neq 0$. For semplicity assume $i_1 = 1, i_2 = 2, \dots, i_r = r$, and put $U = U_{1\dots r}$. If $W \in G(r,n) \cap U$, and $w_1, \dots, w_r$ is a basis of W, then the first minor of the matrix M, of the coordinates of $w_1, \dots, w_r$ with respect to a fixed basis of V, is non-degenerate. So we can choose a new basis of W such that M is of the form

$$M = \begin{pmatrix} 1 & 0 & \dots & 0 & \alpha_{1,r+1} & \dots & \alpha_{1,n} \\ 0 & 1 & \dots & 0 & \alpha_{2,r+1} & \dots & \alpha_{2,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & \alpha_{r,r+1} & \dots & \alpha_{r,n} \end{pmatrix}.$$

Conversely, any matrix of this form defines a subspace $W \in G(r,n) \cap U$. So there is a bijection between $G(r,n) \cap U$ and $K^{r(n-r)}$, i.e. the affine space of dimension $r(n-r)$. The coordinates of W result to be equal to 1 and all minors of all orders of the submatrix of the last $n-r$ columns of M. Therefore they are expressed as polynomials in the $r(n-r)$ coordinates elements of M. This shows that $G(r,n) \cap U$ is an affine rational subvariety of U. By homogenising the equations obtained in this way, one gets equations for $G(r,n)$.

In the case $n = 4, r = 2$, the matrix M becomes

$$M = \begin{pmatrix} 1 & 0 & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \end{pmatrix}.$$

One gets $1 = p_{01}, \alpha_{23} = p_{02}, \alpha_{24} = p_{03}, -\alpha_{13} = p_{12}, -\alpha_{14} = p_{13}, \alpha_{13}\alpha_{24} - \alpha_{23}\alpha_{14} = p_{23}$. If we make the substitutions and homogenise the last equation with respect to p_{01}, we find the equation of the Klein quadric.

We remark that $G(r,n) \cap U_{i_1 \dots i_r}$ is the set of the subspaces W which are complementar to the subspace of equations $x_{i_1} = \ldots = x_{i_r} = 0$.

Concluding, the projective algebraic set $G(r,n)$ has an affine open covering with irreducible varieties isomorphic to $\mathbb{A}^{r(n-r)}$, and it is easy to check that they have two by two non-empty intersection. Using Ex. 5 of §6, we deduce that $G(r,n)$ is a projective variety, of dimension $r(n-r)$, and it is rational.

In the special case $n \geq 4, r = 2$, using the Plücker coordinates $[\ldots, p_{ij}, \ldots]$, the equations of the Grassmannian $G(2,n)$ are of the form $p_{ij}p_{hk} - p_{ih}p_{jk} + p_{ik}p_{jh} = 0$, for any $i < j < h < k$.

Also in the case of $G(2,n)$, as for $\mathbb{P}^n \times \mathbb{P}^m$ and $V_{n,2}$, there is an interpretation in terms of matrices. Given a tensor in $\wedge^2 V$ with coordinates $[p_{ij}]$, we can consider the skew-symmetric $n \times n$ matrix whose term of position i, j is indeed p_{ij}, with the conditions $p_{ii} = 0$ and $p_{ji} = -p_{ij}$. In this way we can construct an isomorphism between $\wedge^2 V$ and the vector space of skew-symmetric matrices of order n.

From ${}^tA = -A$, it follows $\det(A) = (-1)^n \det(A)$. If n is odd, this implies $\det(A) = 0$. If n is even, one can prove that $\det(A)$ is a square. For instance if $n = 2$, and $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, then $\det(A) = a^2$.

If $n = 4$, and $P = \begin{pmatrix} 0 & p_{12} & p_{13} & p_{14} \\ -p_{12} & 0 & p_{23} & p_{24} \\ -p_{13} & -p_{23} & 0 & p_{34} \\ -p_{14} & -p_{24} & -p_{34} & 0 \end{pmatrix}$, then $\det(P) = (p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23})^2$.

In general, for a skew-symmetric matrix A of even order $2n$, one defines the pfaffian of A, $pf(A)$, in one of the following equivalent ways:

(i) by recursion: if $n = 1$, $pf \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a^2$; if $n > 1$, one defines $pf(A) =$

$\Sigma_{i=2}^{2n}(-1)^i a_{1i} Pf(A_{1i})$, where A_{1i} is the matrix obtained from A removing the rows and the columns of indices 1 and i. Then one verifies that $pf(A)^2 = \det(A)$.

(ii) Given the matrix A, one considers the tensor $\omega = \Sigma_{i,j=1}^{2n} a_{ij} e_i \wedge e_j \in K^{2n}$. Then one defines the pfaffian as the unique constant such that $pf(A) e_1 \wedge \ldots \wedge e_{2n} = \frac{1}{n!}\omega \wedge \ldots \wedge \omega$.

For a skew-symmetric matrix of odd order, one defines the pfaffian to be 0.

10.13. Proposition. *A 2-tensor $\omega \in \wedge^2 V$ is totally decomposable if and only if $\omega \wedge \omega = 0$.*

Proof. If ω is decomposable, the conclusion easily follows. Conversely, if $\omega = \Sigma_{i,j=1}^{2n} a_{ij} e_i \wedge e_j$ and $\omega \wedge \omega = 0$, then the pfaffians of the principal minors of order 4 of the matrix A corresponding to ω are all 0, therefore from definition (ii) it follows that the pfaffians of the principal minors of all orders are 0, and also $\det(A) = 0$. In conclusion A has rank 2. Then one checks that ω is the $\wedge$ product of two vectors corresponding to two linearly independent rows of A. For instance, if $a_{12} \neq 0$, then $\omega = (a_{12}e_2 + \ldots + a_{1n}e_n) \wedge (-a_{12}e_1 + a_{23}e_3 + \ldots + a_{2n}e_n)$. $\square$

The equations of $G(2, n)$ are the pfaffians of the principal minors of order 4 of the matrix P. They are all zero if and only if the rank of P is 2. Therefore the points of the Grassmannian $G(2, n)$, for any n, can be interpreted as skew-symmetric matrices of order n and rank 2.

The subvarieties of the Grassmannian $\mathbb{G}(r, n)$ correspond to subvarieties of $\mathbb{P}^n$ covered by linear spaces of dimension r. Conversely, any subvariety of $\mathbb{P}^n$ covered by linear spaces of dimension r gives rise to a subvariety of the Grassmannian.

10.14. Examples. 1. Pencils of lines. A pencil of lines in $\mathbb{P}^n$ is the set of lines passing through a fixed point O and contained in a 2-plane π such that $O \in \pi$. Assume that O has coordinates $[y_0, \ldots, y_n]$, and fix two points $A, B \in \pi$, different from O. Let $A = [a_0, \ldots, a_n]$, $B[b_0, \ldots, b_n]$. Then a general line of the pencil is generated by O and by a point of coordinates $[\ldots, \lambda a_i + \mu b_i, \ldots]$. Therefore the Plücker coordinates of a general line of the pencil are $p_{ij} = y_i(\lambda a_j + \mu b_j) - y_j(\lambda a_i + \mu b_i) = \lambda q_{ij} + \mu q'_{ij}$, where q_{ij}, q'_{ij} are the Plücker coordinates of the lines OA and OB respectively. So the lines of the pencil are represented in the Grassmannian by the points of a line. Conversely one can check that any line contained in a Grassmannian of lines represents the lines of a pencil.

2. Lines a smooth quadric surface. Let $\Sigma : x_0x_3 - x_1x_2 = \det\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix} = 0$ be the Segre quadric in $\mathbb{P}^3$. A line of the first ruling of Σ is characterised by a constant ratio of the rows of the matrix $\begin{pmatrix} x_0 & x_1 \\ x_2 & x_3 \end{pmatrix}$. Therefore it can be generated by two points with coordinates $[x_0, x_1, 0, 0]$, $[0, 0, x_0, x_1]$. The Plücker coordinates of such a line are $[0, x_0^2, x_0x_1, x_0x_1, x_1^2, 0]$. This describes a conic contained in

$\mathbb{G}(1,3)$. Similarly, the lines of the second ruling describe the points of another conic, indeed the coordinates are $[x_0^2, 0, x_0x_2, -x_0x_2, 0, x_2^2]$. These two conics are disjoint and contained in disjoint planes.

3. One can prove that $\mathbb{G}(1,3)$ contains two families of planes, and no linear space of dimension > 2. The planes of one family correspond to stars of lines in $\mathbb{P}^3$ (lines of $\mathbb{P}^3$ through a fixed point), while the planes of the second family correspond to the lines contained in the planes of $\mathbb{P}^3$. The geometry of the lines in $\mathbb{P}^3$ translates to give a decription of the geometry of the planes contained in $\mathbb{G}(1,3)$. Since on an algebraically closed field of characteristic $\neq 2$ two quadric hypersurfaces are projectively equivalent if and only if they have the same rank, one obtains a description of the geometry of all quadrics of maximal rank in $\mathbb{P}^5$.

Exercises to §10.

1. Using Ex. 5 of §6, prove that, if $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$ are irreducible projective varieties, then $X \times Y$ is irreducible.

2. (*) Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^n$. Show that $X \cap Y \simeq (X \times Y) \cap \Delta_{\mathbb{A}^n}$, where $\Delta_{\mathbb{A}^n}$ is the diagonal subvariety.

3. Let L, M, N be the following lines in $\mathbb{P}^3$:

$$L : x_0 = x_1 = 0, M : x_2 = x_3 = 0, N : x_0 - x_2 = x_1 - x_3 = 0.$$

Let X be the union of lines meeting L, M and N: write equations for X and describe it: is it a projective variety? If yes, of what dimension and degree?

4. Let X, Y be quasi–projective varieties, identify $X \times Y$ with its image via the Segre map. Check that the two projection maps $X \times Y \xrightarrow{p_1} X$, $X \times Y \xrightarrow{p_2} Y$ are regular. (Hint: use the open covering of the Segre variety by the Σ^{ij}'s.)

11. The dimension of an intersection.

Our aim in this section is to prove the following theorem:

11.1. Theorem. *Let K be an algebraically closed field. Let $X, Y \subset \mathbb{P}^n$ be quasi–projective varieties. Assume that $X \cap Y \neq \emptyset$. Then if Z is any irreducible component of $X \cap Y$, then* $\dim Z \geq \dim X + \dim Y - n$.

The proof uses in an essential way the Krull's principal ideal theorem (see for instance Atiyah–MacDonald [1]).

The proof of Theorem 11.1 will be divided in three steps. Note first that we can assume that $X \cap Y$ intersects $U_0 \simeq \mathbb{A}^n$, so, possibly after restricting X and Y, we may work with closed subsets of the affine space. Put $r = \dim X$, $s = \dim Y$.

Step 1. Assume that $X = V(F)$ is an irreducible hypersurface, with F irreducible polynomial of $K[x_1, \ldots, x_n]$. The irreducible components of $X \cap Y$ correspond, by the Nullstellensatz, to the minimal prime ideals containing $I(X \cap Y)$

in $K[x_1, \dots, x_n]$. Let me recall that $I(X \cap Y) = \sqrt{I(X) + I(Y)} = \sqrt{\langle I(Y), F \rangle}$. So those prime ideals are the minimal ones over $\langle I(Y), F \rangle$. They correspond bijectively to minimal prime ideals containing $\langle f \rangle$ in $\mathcal{O}(Y)$, where f is the regular function on Y defined by F. We distinguish two cases:

- if $Y \subset X = V(F)$, then $f = 0$ and $Y \cap X = Y$; $s = \dim Y > r + s - n = (n-1) + s - n$. So the theorem is true.

- if $Y \not\subset X$, then $f \neq 0$, moreover f is not invertible, otherwise $X \cap Y = \emptyset$: hence the minimal prime ideals over $\langle f \rangle$ in $\mathcal{O}(Y)$ have all height one, so for all Z, irreducible component of $X \cap Y$, $\dim Z = \dim Y - 1 = r + s - n$ (Theorem 7.7).

Step 2. Assume that $I(X)$ is generated by $n - r$ polynomials (where $n - r$ is the codimension of X): $I(X) = \langle F_1, \dots, F_{n-r} \rangle$. Then we can argue by induction on $n - r$: we first intersect Y with $V(F_1)$, whose irreducible components are all hypersurfaces, and apply Step 1: all irreducible components of $Y \cap V(F_1)$ have dimension either s or $s - 1$. Then we intersect each of these components with $V(F_2)$, and so on. We conclude that every irreducible component Z has $\dim Z \geq \dim Y - (n - r) = r + s - n$.

Step 3. We use the isomorphism $\psi : X \cap Y \simeq (X \times Y) \cap \Delta_{\mathbb{A}^n}$ (see Ex.2, §10). Note that $X \times Y$ is irreducible by Proposition 6.11. ψ preserves the irreducible components and their dimensions, so we consider instead of X and Y, the algebraic sets $X \times Y$ and $\Delta_{\mathbb{A}^n}$, contained in $\mathbb{A}^{2n}$. We have $\dim X \times Y = r + s$ (Proposition 7.10). $\Delta_{\mathbb{A}^n}$ is a linear subspace of $\mathbb{A}^{2n}$, so it satisfies the assumption of Step 2; indeed it has dimension n in $\mathbb{A}^{2n}$ and is defined by n linear equations. Hence, for all Z we have: $\dim Z \geq (r + s) + n - 2n = r + s - n$. $\square$

The above theorem can be seen as a generalization of the Grassmann relation for linear subspaces. It is not an existence theorem, because it says nothing about $X \cap Y$ being non–empty. But for projective varieties, the following more precise version of the theorem holds:

11.2. Theorem. *Let $X, Y \subset \mathbb{P}^n$ be projective varieties of dimensions r, s. If $r + s - n \geq 0$, then $X \cap Y \neq \emptyset$.*

Proof. Let $C(X)$, $C(Y)$ be the affine cones associated to X and Y. Then $C(X) \cap C(Y)$ is certainly non–empty, because it contains the origin $O(0, 0, \dots, 0)$. Assume we know that $C(X)$ has dimension $r + 1$ and $C(Y)$ has dimension $s + 1$: then by Theorem 11.1 all irreducible components Z of $C(X) \cap C(Y)$ have dimension $\geq (r+1) + (s+1) - (n+1) = r + s - n + 1 \geq 1$, hence Z contains points different from O. These points give rise to points of $\mathbb{P}^n$ belonging to $X \cap Y$. It remains to show:

11.3. Proposition. *Let $Y \subset \mathbb{P}^n$ be a projective variety.*

Then $\dim Y = \dim C(Y) - 1$. *If $S(Y)$ denotes the homogeneous coordinate ring, hence also* $\dim Y = \dim S(Y) - 1$.

Proof. Let $p : \mathbb{A}^{n+1} \setminus \{O\} \to \mathbb{P}^n$ be the canonical morphism. Let us recall that $C(Y) = p^{-1}(Y) \cup \{O\}$. Assume that $Y_0 := Y \cap U_0 \neq \emptyset$ and consider also $C(Y_0) = p^{-1}(Y_0) \cup \{O\}$. Then we have:

$$C(Y_0) = \{(\lambda, \lambda a_1, \dots, \lambda a_n) \mid \lambda \in K, (a_1, \dots, a_n) \in Y_0\}.$$

So we can define a birational map between $C(Y_0)$ and $Y_0 \times \mathbb{A}^1$ as follows:

$$(y_0, y_1, \dots, y_n) \in C(Y_0) \to ((y_1/y_0, \dots, y_n/y_0), y_0) \in Y_0 \times \mathbb{A}^1,$$

$$((a_1, \dots, a_n), \lambda) \in Y_0 \times \mathbb{A}^1 \to (\lambda, \lambda a_1, \dots, \lambda a_n) \in C(Y_0).$$

Therefore $\dim C(Y_0) = \dim(Y_0 \times \mathbb{A}^1) = \dim Y_0 + 1$. To conclude, it is enough to remark that $\dim Y = \dim Y_0$ and $\dim C(Y) = \dim C(Y_0) = \dim S(Y)$. $\square$

We observe that also $C(Y)$ and $Y \times \mathbb{P}^1$ are birationally equivalent.

11.4. Corollaries.
1. If $X, Y \subset \mathbb{P}^2$ are projective curves over an algebraically closed field, then $X \cap Y \neq \emptyset$.
2. $\mathbb{P}^1 \times \mathbb{P}^1$ is not isomorphic to $\mathbb{P}^2$.

Proof. 1. is a straightforward application of Theorem 11.2. To prove 2., assume by contradiction that $\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ is an isomorphism. If L, L' are skew lines on $\mathbb{P}^1 \times \mathbb{P}^1$, then $\phi(L)$, $\phi(L')$ are rational disjoint curves of $\mathbb{P}^2$, which contradicts 1.

If $X, Y \subset \mathbb{P}^n$ are varieties of dimensions r, s, then $r + s - n$ is called the *expected dimension* of $X \cap Y$. If all irreducible components Z of $X \cap Y$ have the expected dimension, then we say that the intersection $X \cap Y$ is *proper* or that *X and Y intersect properly.*

For example, two plane projective curves X, Y intersect properly if they don't have any common irreducible component. In this case, it is possible to predict the number of points of intersections. Precisely, it is possible to associate to every point $P \in X \cap Y$ a number $i(P)$, called the *multiplicity of intersection of X and Y at P*, in such a way that $\sum_{P \in X \cap Y} i(P) = dd'$, where d is the degree of X and d' is the degree of Y. This result is known as Theorem of Bézout, and is the first result of the branch of algebraic geometry called Intersection Theory. For a proof of the Theorem of Bézout, see for instance the classical book of Walker [8], or the book of Fulton on Algebraic Curves [5].

Let X be a closed subvariety of $\mathbb{P}^n$ (resp. of $\mathbb{A}^n$) of codimension r. X is called a *complete intersection* if $I_h(X)$ (resp. $I(X)$) is generated by r polynomials.

Hence, if X is a complete intersection of codimension r, then X is certainly the intersection of r hypersurfaces. Conversely, if X is intersection of r hypersurfaces, then, by Theorem 11.1, using induction, we deduce that $\dim X \geq n - r$; even assuming equality, we cannot conclude that X is a complete intersection, but simply that $I(X)$ is the radical of an ideal generated by r polynomials.

11.5. Example. Let $X \subset \mathbb{P}^3$ be the skew cubic. The homogeneous ideal of X is generated by the three polynomials F_1, F_2, F_3, the 2×2–minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix},$$

which are linearly independent polynomials of degree 2. Note that $I_h(X)$ does not contain any linear polynomial, because X is not contained in any hyperplane, and that the homogeneous component of minimal degree 2 of $I_h(X)$ is a vector space of dimension 3. Hence $I_h(X)$ cannot be generated by two polynomials, i.e. X is not a complete intersection.

Nevertheless, X is the intersection of the surfaces $V_P(F)$, $V_P(G)$, where

$$F = F_1 = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix} \text{ and } G = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_0 \end{vmatrix}.$$

Clearly $F, G \in I_h(X)$ so $X \subset V_P(F) \cap V_P(G)$. Conversely, observe that $G = x_0F - x_3(x_0x_3 - x_1x_2) + x_2(x_1x_3 - x_2^2)$. If $P[x_0, \dots, x_3] \in V_P(F) \cap V_P(G)$, then P is a zero of $x_0x_3^2 - 2x_1x_2x_3 + x_2^3$, and therefore also of

$$x_2(x_0x_3^2 - 2x_1x_2x_3 + x_2^3) = x_1^2x_3^2 - 2x_1x_2^2x_3 + x_2^4 = (x_1x_3 - x_2^2)^2 = F_3^2.$$

Hence P is a zero also of $F_3 = x_1x_3 - x_2^2$. So P annihilates $x_3(x_0x_3 - x_1x_2) = x_3F_2$. If P satisfies the equation $x_3 = 0$, then it satisfies also $x_2 = 0$ and $x_1 = 0$, therefore $P = [1, 0, 0, 0] \in X$. If $x_3 \neq 0$, then $P \in V_P(F_1, F_2, F_3) = X$.

The geometric description of this phenomenon is that the skew cubic X is the set-theoretic intersection of a quadric and a cubic, which are tangent along X, so their intersection is X counted with multiplicity 2.

This example motivates the following definition: X is a *set–theoretic complete intersection* if $\operatorname{codim} X = r$ and the ideal of X is the radical of an ideal generated by r polynomials. It is an open problem if all irreducible curves of $\mathbb{P}^3$ are set–theoretic complete intersections. For more details, see [4].

Exercises to §11.

1. Let $X \subset \mathbb{P}^2$ be the union of three points not lying on a line. Prove that the homogeneous ideal of X cannot be generated by two polynomials.

12. Complete varieties.

We work over an algebraically closed field K.

12.1. Definition. Let X be a quasi–projective variety. X is *complete* if, for any quasi–projective variety Y, the natural projection on the second factor $p_2 : X \times Y \to Y$ is a closed map. (Note that both projections p_1, p_2 are morphisms: see Exercise 4 to §10.)

Example. The affine line $\mathbb{A}^1$ is not complete: let $X = Y = \mathbb{A}^1$, $p_2 : \mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2 \to \mathbb{A}^1$ is the map such that $(x_1, x_2) \to x_2$. Then $Z := V(x_1x_2 - 1)$ is closed in $\mathbb{A}^2$ but $p_2(Z) = \mathbb{A}^1 \setminus \{O\}$ is not closed.

12.2. Proposition. *(i) If $f : X \to Y$ is a regular map and X is complete, then $f(X)$ is a closed complete subvariety of Y.*

(ii) If X is complete, then all closed subvarieties of X are complete.

Proof. (i) Let $\Gamma_f \subset X \times Y$ be the graph of f: $\Gamma_f = \{(x, f(x)) \mid x \in X\}$. It is clear that $f(X) = p_2(\Gamma_f)$, so to prove that $f(X)$ is closed it is enough to check that Γ_f is closed in $X \times Y$. Let us consider the diagonal of Y: $\Delta_Y = \{(y, y) \mid y \in Y\} \subset Y \times Y$. If $Y \subset \mathbb{P}^n$, then $\Delta_Y = \Delta_{\mathbb{P}^n} \cap (Y \times Y)$, so it is closed because $\Delta_{\mathbb{P}^n}$ is the closed subset defined in $\Sigma_{n,n}$ by the equations $w_{ij} - w_{ji} = 0$, $i, j = 0, \dots, n$. There is a natural map $f \times 1_Y : X \times Y \to Y \times Y$, $(x, y) \to (f(x), y)$, such that $(f \times 1_Y)^{-1}(\Delta_Y) = \Gamma_f$. It is easy to see that $f \times 1_Y$ is regular, so Γ_f is closed, so also $f(X)$ is closed.

Let now Z be any variety and consider $p_2 : f(X) \times Z \to Z$ and the regular map $f \times 1_Z : X \times Z \to f(X) \times Z$. There is a commutative diagram:

$$\begin{array}{ccc} X \times Z & \xrightarrow{p_2'} & Z \\ \downarrow {\scriptstyle f \times 1_Z} & \nearrow & {\scriptstyle p_2} \\ f(X) \times Z & & \end{array}$$

If $T \subset f(X) \times Z$, then $(f \times 1_Z)^{-1}(T)$ is closed and $p_2(T) = p_2'((f \times 1_Z)^{-1}(T))$ is closed because X is complete. We conclude that $f(X)$ is complete.

(ii) Let $T \subset X$ be a closed subvariety and Y be any variety. We have to prove that $p_2 : T \times Y \to Y$ is closed. If $Z \subset T \times Y$ is closed, then Z is closed also in $X \times Y$, hence $p_2(Z)$ is closed because X is complete. □

12.3. Corollaries.

1. If X is a complete variety, then $\mathcal{O}(X) \simeq K$.

2. If X is an affine complete variety, then X is a point.

Proof. 1. If $f \in \mathcal{O}(X)$, f can be interpreted as a regular map $f : X \to \mathbb{A}^1$. By Proposition 12.2, (i), $f(X)$ is a closed complete subvariety of $\mathbb{A}^1$, which is not complete. Hence $f(X)$ has dimension < 1 and is irreducible, hence it is a point, so $f \in K$.

2. By 1., $\mathcal{O}(X) \simeq K$. But $\mathcal{O}(X) \simeq K[x_1, \dots, x_n]/I(X)$, hence $I(X)$ is maximal. By the Nullstellensatz, X is a point. □

12.4. Theorem. *Let X be a projective variety. Then X is complete.*

Proof. (sketch, see Šafarevič [7].)

1. It is enough to prove that $p_2 : \mathbb{P}^n \times \mathbb{A}^m \to \mathbb{A}^m$ is closed, for all n, m. This can be observed by using the local character of closedness and the affine open coverings of quasi–projective varieties.

2. If $x_0, \dots, x_n$ are homogeneous coordinates on $\mathbb{P}^n$ and $y_1, \dots, y_m$ are coordinates on $\mathbb{A}^m$, then any closed subvariety of $\mathbb{P}^n \times \mathbb{A}^m$ can be characterised as the set of common zeroes of a set of polynomials in the variables $x_0, \dots, x_n, y_1, \dots, y_m$, homogeneous in the first group of variables $x_0, \dots, x_n$.

3. Let $Z \subset \mathbb{P}^n \times \mathbb{A}^m$ be closed. Then Z is the set of solutions of a system of equations

$$\{G_i(x_0, \dots, x_n; y_1, \dots, y_m) = 0, i = 1, \dots, t$$

where G_i is homogeneous in the x's. A point $P(\overline{y}_1, \dots, \overline{y}_m)$ is in $p_2(Z)$ if and only if the system

$$\{G_i(x_0, \dots, x_n; \overline{y}_0, \dots, \overline{y}_m) = 0, i = 1, \dots, t$$

has a solution in $\mathbb{P}^n$, i.e. if the ideal of $K[x_0, \dots, x_n]$ generated by $G_1(x; \overline{y})$,..., $G_t(x; \overline{y})$ has at least one zero in $\mathbb{P}^n$. Hence

$$\begin{aligned} p_2(Z) &= \{(\overline{y}_1, \dots, \overline{y}_m) | \ \forall\ d \geq 1\ \langle G_1(x; \overline{y}), \dots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \dots, x_n]_d\} = \\ &= \bigcap_{d \geq 1} \{(\overline{y}_1, \dots, \overline{y}_m) | \ \langle G_1(x; \overline{y}), \dots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \dots, x_n]_d\}. \end{aligned}$$

Let $\{M_\alpha\}_{\alpha=1,\dots,\binom{n+d}{d}}$ be the set of the monomials of degree d in $K[x_0, \dots, x_n]$; let $d_i = \deg G_i(x; \overline{y})$; let $\{N_i^\beta\}$ be the set of the monomials of degree $d - d_i$; let finally $T_d = \{(\overline{y}_1, \dots, \overline{y}_m) | \ \langle G_1(x; \overline{y}), \dots, G_t(x; \overline{y})\rangle \not\supset K[x_0, \dots, x_n]_d\}$.

Then $P(\overline{y}_1, \dots, \overline{y}_m) \notin T_d$ if and only if $M_\alpha = \sum_i G_i(x; \overline{y}) F_{i,\alpha}(x_0, \dots, x_n)$, for all α and for suitable polynomials $F_{i,\alpha}$ homogeneous of degree $d - d_i$. So $P \notin T_d$ if and only if, for all index α, M_α is a linear combination of the polynomials $\{G_i(x; \overline{y}) N_i^\beta\}$, i.e. the matrix A of the coefficients of the polynomials $G_i(x; \overline{y}) N_i^\beta$ with respect to the basis $\{M_\alpha\}$ has maximal rank $\binom{n+d}{d}$. So T_d is the set of zeroes of the minors of a fixed order of the matrix A, hence it is closed. $\square$

12.5. Corollary. *Let X be a projective variety. Then $\mathcal{O}(X) \simeq K$.*

12.6. Corollary. *Let X be a projective variety, $\phi : X \to Y \subset \mathbb{P}^n$ be any regular map. Then $\phi(X)$ is a projective variety. In particular, if $X \simeq Y$, then Y is projective.*

In algebraic terms, Theorem 12.4 can be seen as a result in Elimination Theory. Indeed it can be reformulated by saying that, given a system of algebraic equations in two sets of variables, $x_0, \dots, x_n$ and $y_1, \dots, y_m$, homogeneous in the first ones, it is possible to find another system of algebraic equations only in $y_1, \dots, y_m$, such that $\bar{y}_1, \dots, \bar{y}_m$ is a solution of the second system if and only if there exist $\bar{x}_0, \dots, \bar{x}_n$, that, together with $\bar{y}_1, \dots, \bar{y}_m$, are a solution of the first system. In other words, it is possible to eliminate a set of homogeneous variables from any system of algebraic equations.

12.7. Example. Let $S = K[x_0, \dots, x_n]$. Let $d \geq 1$ be an integer number and consider S_d, the vector space of homogeneous polynomials of degree d. As an application of Theorem 12.4, we shall prove that the set of (proportionality classes of) reducible polynomials is a projective algebraic set in $\mathbb{P}(S_d)$.

We denote by $X \subset \mathbb{P}(S_d)$ the set of reducible polynomials. For any integer k, $0 < k < d$, let $X_k \subseteq X$ be the set of polynomials of the form F_1F_2 with $\deg F_1 = k, \deg F_2 = d-k$. Then $X = \bigcup_{k=1}^{d-1} X_k$. Let $f_k : \mathbb{P}(S_k) \times \mathbb{P}(S_{d-k}) \to \mathbb{P}(S_d)$ be the multiplication of polynomials, i.e. $f_k([F_1], [F_2]) = [F_1F_2]$. f_k is clearly a regular map, and its image is $X_k = X_{d-k}$. Since the domain is a projective variety, and precisely a Segre variety, it follows from Theorem 12.4 that X_k is also projective.

In the special case $d = 2$, the quadratic polynomials, the equations of $X = X_1$ are the minors of order 3 of the matrix associated to the quadric.

13. The fibres of a morphism.

In this section we will see the notion of finite morphism, the Theorem on the dimension of the fibres of a morphism, and an application about the existence of lines on a hypersurface of given degree in a projective space.

I'm only giving a sketch. For the proofs, see Atiyah-MacDonald [1] for the algebraic part, and Šafarevič, [7] for the geometric part.

Let $A \subseteq B$ be rings, A subring of B. B is called an integral extension of B if any $b \in B$ is integral over A.

13.1. Theorem. *Let $x \in B$, let $A[x] \subseteq B$ be the A-algebra generated by x.*

The following are equivalent:

1) x is integral over A;

2) $A[x]$ is a finite A-module;

3) there exists a subring $C \subset B$, with $A[x] \subset C$, such that C is a finite A-module.

Proof. Atiyah-MacDonald.

13.2. Corollaries. *Let $A \subseteq B$.*

1. Let $b_1, \dots, b_n \in B$ be integral over A. Then $A[b_1, \dots, b_n]$ is a finite A-module.

2. Let $C = \{b \in B \mid b$ integral over $A\}$: it is a subring of B containing A, called the integral closure of A in B. If $C = A$, then A is called integrally closed in B.

3. Transitivity: Let $A \subset B \subset C$. If B is integral extension of A and C is integral extension of B, then C is integral extension of A.

4. Let C be the integral closure of A in B. Then C is integrally closed in B.

5. Assume that A and B are both integral domains, and B is integral extension of A. Then A is a field if and only if B is a field.

6. Property of Lying Over - LO: let B be integral extension of A. If $\mathcal{P} \subset \mathcal{A}$ is a prime ideal, then there exists a prime ideal $\mathcal{Q}$ of B such that $\mathcal{P} = \mathcal{Q} \cap \mathcal{A}$.

We give now the geometric interpretation of the previous notions.

Let $f : X \to Y$ be a dominant morphism of affine varieties. Then the comorphism $f^* : K[Y] \to K[X]$ is injective: we will identify $K[Y]$ with its image $f^*K[Y] \subset K[X]$.

13.3. Definition. f is a finite morphism if $K[X]$ is an integral extension of $K[Y]$.

Finite morphisms enjoy the following properties, which are consequences of Corollaries 13.2.

13.4. Proposition.

1. *The composition of finite morphisms is a finite morphism;*
2. *let $y \in Y$, then $f^{-1}(y)$ is a finite set;*
3. *Finite morphisms are surjective, i.e. $f^{-1}(y)$ is non-empty for any $y \in Y$;*
4. *Finite morphisms are closed maps.*

An example of non-finite morphism is the projection $V(xy-1) \to \mathbb{A}^1$. Instead the projection $p_2 : V(y - x^2) \to \mathbb{A}^1$ is finite.

One can prove that being a finite morphism is a local property, in the following sense: let $f : X \to Y$ be a morphism of affine varieties. Then f is finite if and only if any $y \in Y$ has an affine open neighbourhood V, such that $U := f^{-1}(V)$ is affine, and the restriction $f |: U \to V$ is a finite morphism. This property allows to give the definition of finite morphisms between arbitrary varieties. They always have the property that all the fibres are finite, where we call fibres of a morphism the inverse images of the points of the codomain.

13.5. Examples.

1. Let $X \subset \mathbb{P}^n$ be a closed algebraic set, let $\Lambda \subset \mathbb{P}^n$ be a linear subspace of dimension d such that $X \cap \Lambda = \emptyset$. Then the restriction of the projection $\pi_\Lambda : X \to \mathbb{P}^{n-d-1}$ defines a finite morphism from X to $\pi_\Lambda(X)$.
2. Let $X \subset \mathbb{P}^n$ be a closed algebraic set and $F_0, \dots, F_r$ be homogeneous polynomials of the same degree without any common zero on X. Then $\phi : X \to \mathbb{P}^r$ defined by the polynomials $F_0, \dots, F_r$ is a finite morphism to the image.
3. Geometric interpretation of the Normalisation Lemma: Let $X \subset \mathbb{A}^n$ be an affine variety of dimension d. Then there exists a finite morphism $X \to \mathbb{A}^d$. Moreover it can be taken to be a projection.

For general morphism, the following theorem gives informations about the behaviour of the fibres.

13.6. Theorem on the dimension of the fibres.

Let $f : X \to Y$ be a surjective morphism of algebraic sets. Let $n = \dim X$, $m = \dim Y$. Then:

1. $n \geq m$;

2. for any $y \in Y$, and for any irreducible component F of $f^{-1}(y)$, $\dim F \geq n - m$;

3. there exists a non-empty open subset $U \subset Y$, such that $\dim f^{-1}(y) = n - m$ for any $y \in U$.

As a consequence of this theorem, it is possible to prove the following useful proposition:

13.7. Proposition. *Let $f : X \to Y$ be a surjective morphism of projective algebraic sets. Assume that Y is irreducible and that all fibres of f are ireducible and of the same dimension, then also X is irreducible.*

As an application, we will study the existence of lines on hypersurfaces of fixed degree. Let $S = K[x_0, \ldots, x_n]$, let $d \geq 1$ be an integer number, then $\mathbb{P}(S_d)$ is a projective space of dimension $N = \binom{n+d}{d} - 1$, parametrising the hypersurfaces of degree d in $\mathbb{P}^n$. Among them there are reducible and even non-reduced hypersurfaces (i.e. those corresponding to non square-free polynomials). Let us introduce the *incidence correspondence* line-hypersurface as follows. We consider the product variety $\mathbb{G}(1, n) \times \mathbb{P}(S_d)$, whose points are the pairs $(\ell, [F])$, where ℓ is a line in $\mathbb{P}^n$ and $F \in S_d$, that we can identify with the hypersurface $V_P(F)$. The incidence variety in $\mathbb{G}(1, n) \times \mathbb{P}(S_d)$ is $\Gamma_d := \{(\ell, [F]) \mid \ell \subset V_P(F)\}$.

13.8. Proposition. *Γ_d is a projective algebraic set, i.e. it is the set of zeroes of a set of bihomogeneous polynomials in the Plücker coordinates p_{ij} and in the coefficients $a_{i_0 \ldots i_n}$ of F.*

Proof. Let $P = (p_{ij})$ be the skew-symmetric matrix, whose elements are the coordinates of a line ℓ: it has rank two and from Proposition 10.13 it follows that each non-zero row of P contains the coordinates of a point of ℓ. So the rows of P are a system of generators of a vector plane W, such that $\ell = \mathbb{P}(W)$. Hence the coordinates of any point of ℓ are linear combinations of the rows of P, of the form $(x_0 = \Sigma_i \lambda_i p_{0i}, \ldots, x_n = \Sigma_i \lambda_i p_{ni})$. A line ℓ is contained in $V_P(F)$ if and only if the equation $F(\Sigma_i \lambda_i p_{0i}, \ldots, \Sigma_i \lambda_i p_{ni}) = 0$ is an identity in $\lambda_0, \ldots, \lambda_n$. Therefore, Γ_d is the set of common zeroes of the coefficients of the monomials of degree d in $\lambda_0, \ldots, \lambda_n$: they are homogeneous of degree 1 in the coefficients of F and of degree d in the p_{ij}'s. $\square$

13.8. Example. Let $n = d = 3$, $F = x_0^3 - x_1x_2x_3 \in S_3$. We put

$$\begin{cases} x_0 = \lambda_1 p_{01} + \lambda_2 p_{02} + \lambda_3 p_{03} \\ x_1 = -\lambda_0 p_{01} + \lambda_2 p_{12} + \lambda_3 p_{13} \\ x_2 = -\lambda_0 p_{02} - \lambda_1 p_{12} + \lambda_3 p_{23} \\ x_3 = -\lambda_0 p_{03} - \lambda_1 p_{13} - \lambda_2 p_{23} \end{cases}$$

then we replace in F, and we get the identity $(\lambda_1 p_{01} + \lambda_2 p_{02} + \lambda_3 p_{03})^3 - (-\lambda_0 p_{01} + \lambda_2 p_{12} + \lambda_3 p_{13})(-\lambda_0 p_{02} - \lambda_1 p_{12} + \lambda_3 p_{23})(-\lambda_0 p_{03} - \lambda_1 p_{13} - \lambda_2 p_{23}) = 0$. By equating to zero the coefficients of the 20 monomials of degree 3 in $\lambda_0, \ldots, \lambda_3$ we get the equations representing the lines contained in $V_P(F)$.

As a matter of fact, for this particular surface finding the lines contained in it is particularly simple. Indeed, we can distinguish the lines contained in the hyperplane "at infinity" from the lines which are projective closure of a line in $\mathbb{A}^3$. The first ones are contained in $x_0 = 0$, and it is clear that there are only three of them: $x_0 = x_1 = 0, x_0 = x_2 = 0, x_0 = x_3 = 0$. To find the others we dehomogenise F and get the equation $x_1x_2x_3 - 1 = 0$, and consider the parametrisation of a general line in $\mathbb{A}^3$: $x_i = a_it + b_i$, $i = 1, 2, 3$. By substituting, we immediately see that there are no solutions. We conclude that the surface contains only three lines.

We consider now the restrictions to Γ_d of the two projections, and we get $\phi_1 : \Gamma_d \to \mathbb{G}(1, n)$, $\phi_2 : \Gamma_d \to \mathbb{P}(S_d)$. We have:

1. $\phi_1(\Gamma_d) = \mathbb{G}(1, n)$, because any line ℓ is contained in some hypersurface of degree d; up to a change of coordinates, we can assume that $\ell : x_0 = x_1 = \ldots = x_{n-2} = 0$. So $\ell \subset V_P(F)$ if and only if $F(0, \ldots, 0, x_{n-1}, x_n) \equiv 0$, if and only if the coefficients of the monomials containing only x_{n-1}, x_n vanish, i.e. F is of the form $x_0G_0 + \ldots + x_{n-2}G_{n-2}$. So $\phi_1^{-1}(\ell)$ is a linear subspace of dimension $N - (d + 1)$. In particular we have that the fibres of ϕ_1 are all irreducible and of the same dimension. By applying Theorem 13.6, we obtain that Γ_d is irreducible of dimension $\dim \mathbb{G}(1, n) + \dim \phi_1^{-1}(\ell) = 2(n - 1) + N - (d + 1)$.

2. Consider now $\phi_2 : \Gamma_d \to \mathbb{P}(S_d) = \mathbb{P}^N$. If $\dim \Gamma_d > N$, then ϕ_2 cannot be surjective. This happens if

$$2(n - 1) + N - (d + 1) < N \text{ if and only if } d > 2n - 3.$$

We have proved the following theorem.

13.9. Theorem. *If $d > 2n - 3$, there is an open non-empty subset $U \subset \mathbb{P}(S_d)$, such that if $[F] \in U$ then the hypersurface $V_P(F)$ does not contain any line; shortly, a "general" hypersurface of degree d in $\mathbb{P}^n$ does not contain any line. The hypersurfaces containing a line form a proper closed subset in $\mathbb{P}(S_d)$.*

13.10. Example. Let $n = 3$, the case of surfaces in $\mathbb{P}^3$. Theorem 13.9 says that a general surface of degree ≥ 4 does not contain lines. Let us analyse the cases $d = 1, 2, 3$.

• $d = 1$: the surface is a plane, the lines in a plane form a $\mathbb{P}^2$.

• $d = 2$: the surface is a quadric, any quadric contains lines, and precisely, if its rank is 4, it contains two families of dimension 1 parametrised by two conics in $\mathbb{G}(1,3)$; if the rank is 3, the quadric is a cone, and it contains a family of dimension 1 of lines, parametrised by a conic in $\mathbb{G}(1,3)$. In both cases of rank 3, 4 the fibres of ϕ_2 have dimension 1. If the rank is 2 or 1, the quadric is a pair of distinct planes or one plane with multiplicity 2, and the fibre of ϕ_2 has dimension 2.

• $d = 3$: in this case $N = 19 = \dim \Gamma_d$. Two cases can occur: either ϕ_2 is surjective, and a general fibre has dimension 0, or it is not surjective, so if a cubic surface contains a line, it contains by consequence infinitely many lines. But in Example 13.8 we have seen an explicit example of a cubic surface containing finitely many lines; this shows that the first possibility occurs, i.e. a general cubic surface contains finitely many lines.

It is a classical fact that a general cubic contains exactly 27 lines, whose configuration is completely described (see for instance Hartshorne [3]). In particular, among these 27 lines there are many pairs of skew lines. It is a nice application of the theory we have developed so far to prove that such a cubic surface is rational.

13.11. Theorem. *Let $S \subset \mathbb{P}^3$ be a cubic surface containing two skew lines. Then S is rational.*

Proof. Let ℓ, ℓ' be two skew lines contained in S. For any point $P \in \mathbb{P}^3$, $P \notin \ell \cup \ell'$, there is exactly one line r_P passing through P and meeting both ℓ and ℓ': r_P is the intersection of the two planes passing through P and containing ℓ and ℓ' respectively. So we can consider the rational map $f : \mathbb{P}^3 \dashrightarrow \ell \times \ell' \simeq \mathbb{P}^1 \times \mathbb{P}^1$, such that $f(P)$ is the pair of points of intersection of r_P with ℓ and ℓ'. We consider now the restriction $\bar{f}$ of f to S, and we get a birational map. Indeed, for any pair of points $x \in \ell$ and $x' \in \ell'$, the line joining x and x', if not contained in S, meets S in a third point. Since not all lines meeting ℓ and ℓ' can be contained in S, this defines the rational inverse of $\bar{f}$. Therefore S is birational to $\mathbb{P}^1 \times \mathbb{P}^1$, that is birational to $\mathbb{P}^2$. By transitivity we conclude that S is rational. $\square$

14. The tangent space.

We define the tangent space $T_{X,p}$ at a point P of an *affine* variety X as the union of the lines passing through P and "touching" X at P. Then we will find a "local" characterization of $T_{X,p}$, only depending on the local ring $\mathcal{O}_{X,p}$: this will allow to define the tangent space at a point of any quasi–projective variety.

Assume first that $X \subset \mathbb{A}^n$ is closed and $P = O = (0, \dots, 0)$. Let L be a line through P: if $A(a_1, \dots, a_n)$ is another point of L, then a general point of L has coordinates $(ta_1, \dots, ta_n)$, $t \in K$. If $I(X) = (F_1, \dots, F_m)$, then the intersection

$X \cap L$ is determined by the following system of equations in the indeterminate t:

$$F_1(ta_1, \dots, ta_n) = \dots = F_m(ta_1, \dots, ta_n) = 0.$$

The solutions of this system of equations are the roots of the greatest common divisor $G(t)$ of the polynomials $F_1(ta_1, \dots, ta_n), \dots, F_m(ta_1, \dots, ta_n)$ in $K[t]$. We may factorize $G(t)$ as $G(t) = ct^e(t-\alpha_1)^{e_1} \dots (t-\alpha_s)^{e_s}$, where $e > 0$ if and only if $P \in X \cap L$, and $\alpha_1, \dots, \alpha_s \neq 0$. The number e is by definition the **intersection multiplicity at P of X and L.** If $G(t)$ is identically zero, then $L \subset X$ and the intersection multiplicity is, by definition, $+\infty$.

Note that the polynomial $G(t)$ doesn't depend on the choice of the generators $F_1, \dots, F_m$ of $I(X)$, but only on the ideal $I(X)$ and on L.

14.1. Definition. The line L is **tangent to the variety X at P** if the intersection multiplicity of L and X at P is at least 2 (in particular if $L \subset X$). The **tangent space to X at P** is the union of the lines that are tangent to X at P; it is denoted $T_{P,X}$.

We will see now that $T_{P,X}$ is an affine subspace of $\mathbb{A}^n$. Assume that $P \in X$: then the polynomials F_i may be written in the form $F_i = L_i + G_i$, where L_i is a homogeneous linear polynomial (possibly zero) and G_i contains only terms of degree ≥ 2. Then

$$F_i(ta_1, \dots, ta_n) = tL_i(a_1, \dots, a_n) + G_i(ta_1, \dots, ta_n),$$

where the last term is divisible by t^2. So L is tangent to X at P if and only if $L_i(a_1, \dots, a_n) = 0$ for all $i = 1, \dots, m$.

Therefore a point $A(a_1, \dots, a_n)$ belongs to $T_{P,X}$ if and only if

$$L_1(a_1, \dots, a_n) = \dots = L_m(a_1, \dots, a_n) = 0.$$

This shows that $T_{P,X}$ is a linear subspace of $\mathbb{A}^n$, whose equations are the linear components of the equations defining X.

14.2. Examples.

(i) $T_{O,\mathbb{A}^n} = \mathbb{A}^n$, because $I(\mathbb{A}^n) = (0)$.

(ii) If X is a hypersurface, $I(X) = (F)$, $F = L + G$, then $T_{O,X} = V(L)$: so $T_{O,X}$ is either a hypersurface if $L \neq 0$, or the whole space $\mathbb{A}^n$ if $L = 0$. For instance, if X is the affine plane cuspidal cubic $V(x^3 - y^2) \subset \mathbb{A}^2$, $T_{O,X} = \mathbb{A}^2$.

Assume now that $P \in X$ has coordinates $(y_1, \dots, y_n)$. With a linear transformation we may translate P to the origin $(0, \dots, 0)$, taking as new coordinates functions on $\mathbb{A}^n$ $x_1 - y_1, \dots, x_n - y_n$. This corresponds to considering the K-isomorphism $K[x_1, \dots, x_n] \longrightarrow K[x_1 - y_1, \dots, x_n - y_n]$, which takes a polynomial $F(x_1, \dots, x_n)$ to its Taylor expansion

$$G(x_1 - y_1, \dots, x_n - y_n) = F(y_1, \dots, y_n) + d_P F + d_P^{(2)} F + \dots,$$

where $d_P^{(i)}F$ denotes the i^{th} differential of F at P: it is a homogeneous polynomial of degree i in the variables $x_1 - y_1, \dots, x_n - y_n$. In particular the linear term is

$$d_P F = \frac{\partial F}{\partial x_1}(P)(x_1 - y_1) + \ldots + \frac{\partial F}{\partial x_n}(P)(x_n - y_n).$$

We get that, if $I(X) = (F_1, \dots, F_m)$, then $T_{P,X}$ is the linear subspace of $\mathbb{A}^n$ defined by the equations

$$d_P F_1 = \ldots = d_P F_m = 0.$$

The affine space $\mathbb{A}^n$, which may identified with K^n, has a natural structure of K-vector space with origin P, so in a natural way $T_{P,X}$ is a vector subspace (with origin P). The functions $x_1 - y_1, \dots, x_n - y_n$ form a basis of the dual space $(K^n)^*$ and their restrictions generate $T^*_{P,X}$. Note moreover that $\dim T^*_{P,X} = k$ if and only if $n - k$ is the maximal number of polynomials linearly independent among $d_P F_1, \dots, d_P F_m$. If $d_P F_1, \dots, d_P F_{n-k}$ are these polynomials, then they form a base of the orthogonal $T^{\perp}_{P,X}$ of the vector space $T_{P,X}$ in $(K^n)^*$, because they vanish on $T_{P,X}$.

Let us define now the *differential of a regular function.* Let $f \in \mathcal{O}(X)$ be a regular function on X. We want to define the differential of f at P. Since X is closed in $\mathbb{A}^n$, f is induced by a polynomial $F \in K[x_1, \dots, x_n]$ as well as by all polynomials of the form $F + G$ with $G \in I(X)$. Fix $P \in X$: then $d_P(F + G) = d_P F + d_P G$ so the differentials of two polynomials inducing the same function f on X differ by the term $d_P G$ with $G \in I(X)$. By definition, $d_P G$ is zero along $T_{P,X}$, so we may define $d_p f$ as a regular function on $T_{P,X}$, the differential of f at P: it is the function on $T_{P,X}$ induced by $d_P F$. Since $d_P F$ is a linear combination of $x_1 - y_1, \dots, x_n - y_n$, $d_p f$ can also be seen as an element of $T^*_{P,X}$.

There is a natural map $d_p : \mathcal{O}(X) \to T^*_{P,X}$, which sends f to $d_p f$. Because of the rules of derivation, it is clear that $d_P(f + g) = d_P f + d_P g$ and $d_P(fg) = f(P) d_P g + g(P) d_P f$. In particular, if $c \in K$, $d_p(cf) = c d_P f$. So d_p is a linear map of K-vector spaces. We denote again by d_P the restriction of d_P to $I_X(P)$, the maximal ideal of the regular functions on X which are zero at P. Since clearly $f = f(P) + (f - f(P))$ then $d_P f = d_P(f - f(P))$, so this restriction doesn't modify the image of the map.

14.3. Proposition. *The map $d_P : I_X(P) \to T^*_{P,X}$ is surjective and its kernel is $I_X(P)^2$. Therefore $T^*_{P,X} \simeq I_X(P)/I_X(P)^2$ as K-vector spaces.*

Proof. Let $\phi \in T^*_{P,X}$ be a linear form on $T_{P,X}$. ϕ is the restriction of a linear form on K^n: $\lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n)$, with $\lambda_1, \dots, \lambda_n \in K$. Let G be the polynomial of degree 1 $\lambda_1(x_1 - y_1) + \ldots + \lambda_n(x_n - y_n)$: the function g induced by G on X is zero at P and coincides with its own differential, so d_p is surjective.

Let now $g \in I_X(P)$ such that $d_p g = 0$, g induced by a polynomial G. Note that $d_P G$ may be interpreted as a linear form on K^n which vanishes on $T_{P,X}$, hence as an element of $T_{P,X}^{\perp}$. So $d_P G = c_1 d_p F_1 + \ldots + c_m d_p F_m$ ($c_1, \ldots, c_m$ suitable elements of K). Let us consider the polynomial $G - c_1 F_1 - \ldots - c_m F_m$: since its differential at P is zero, it doesn't have any term of degree 0 or 1 in $x_1 - y_1, \ldots, x_n - y_n$, so it belongs to $I(P)^2$. Since $G - c_1 F_1 - \ldots - c_m F_m$ defines the function g on X, we conclude that $g \in I_X(P)^2$. □

14.4. Corollary. *The tangent space $T_{P,X}$ is isomorphic to $(I_X(P)/I_X(P)^2)^*$ as an abstract K-vector space.* □

14.5. Corollary. *If $\phi : X \to Y$ is an isomorphism of affine varieties and $P \in X$, then the tangent spaces $T_{P,X}$ and $T_{\phi(P),Y}$ are isomorphic.*

Proof. ϕ induces the comorphism $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, which is an isomorphism such that $\phi^* I_Y(\phi(P)) = I_X(P)$ and $\phi^* I_Y(\phi(P))^2 = I_X(P)^2$. So there is an induced homomorphism

$$I_Y(\phi(P))/I_Y(\phi(P))^2 \to I_X(P)/I_X(P)^2$$

which is an isomorphism of K-vector spaces. By dualizing we get the claim. □

The above map from $T_{P,X}$ to $T_{\phi(P),Y}$ is called the *differential of ϕ at P* and is denoted by $d_P\phi$.

Now we would like to find a "more local" characterization of $T_{P,X}$. To this end we consider the local ring of P in X: $\mathcal{O}_{P,X}$. We recall that $\mathcal{O}(X)$ has the natural map to $\mathcal{O}_{P,X}$, which is the localization $\mathcal{O}(X)_{I_X(P)}$. It is natural to extend the map $d_P : \mathcal{O}(X) \to T_{P,X}^*$ to $\mathcal{O}_{P,X}$ setting

$$d_P(\frac{f}{g}) = \frac{g(P)d_P f - f(P)d_P g}{g(P)^2}.$$

As in the proof of Proposition 14.3 one proves that the map $d_P : \mathcal{O}_{P,X} \to T_{P,X}^*$ induces an isomorphism $\mathcal{M}_{P,X}/\mathcal{M}_{P,X}^2 \to T_{P,X}^*$, where $\mathcal{M}_{P,X}$ is the maximal ideal of $\mathcal{O}_{P,X}$. So by duality we have: $T_{P,X} \simeq (\mathcal{M}_{P,X}/\mathcal{M}_{P,X}^2)^*$. This proves that the tangent space $T_{P,X}$ is a *local invariant* of P in X.

14.6. Definition. Let X be any quasi-projective variety, $P \in X$. The *Zariski tangent space* of X at P is the vector space $(\mathcal{M}_{P,X}/\mathcal{M}_{P,X}^2)^*$.

It is an abstract vector space, but if $X \subset \mathbb{A}^n$ is closed, taking the dual of the comorphism associated to the embedding of X in $\mathbb{A}^n$, we have an embedding of $T_{P,X}$ into $T_{P,\mathbb{A}^n} = \mathbb{A}^n$. If $X \subset \mathbb{P}^n$ and $P \in U_i = \mathbb{A}^n$, then $T_{P,X} \subset U_i$: its projective closure is called the *embedded tangent space* to X at P.

As we have seen the tangent space $T_{P,X}$ is invariant by isomorphism. In particular its dimension is invariant. If $X \subset \mathbb{A}^n$ is closed, $I(X) = (F_1, \dots, F_m)$, then $\dim T_{P,X} = n - r$, where r is the maximal number of linearly independent elements in the set $\{d_P F_1, \dots, d_p F_m\}$.

Since $d_P F_i = \frac{\partial F_i}{\partial x_1}(P)(x_1 - y_1) + \dots + \frac{\partial F_i}{\partial x_n}(P)(x_n - y_n)$, r is the rank of the following $m \times n$ matrix, the *Jacobian matrix of X at P*:

$$J(P) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(P) & \dots & \frac{\partial F_1}{\partial x_n}(P) \\ \dots & \dots & \dots \\ \frac{\partial F_m}{\partial x_1}(P) & \dots & \frac{\partial F_m}{\partial x_n}(P) \end{pmatrix}.$$

The *generic Jacobian matrix of X* is instead the following matrix with entries in $\mathcal{O}(X)$:

$$J = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial F_m}{\partial x_1} & \dots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}.$$

The rank of J is ρ when all minors of order $\rho + 1$ are functions identically zero on X, while some order ρ minor is different from zero at some point. Hence, for all $P \in X$ rk $J(P) \leq \rho$, and rk $J(P) < \rho$ if and only if all minors of order ρ of J vanish at P. It is then clear that there is a non-empty open subset of X where $\dim T_{P,X}$ is minimal, equal to $n - \rho$, and a proper (possibly empty) closed subset formed by the points P such that $\dim T_{P,X} > n - \rho$.

14.7. Definition. The points of an irreducible variety X for which $\dim T_{P,X} = n - \rho$ (the minimal) are called *smooth* or *non-singular* or *simple points* of X. The remaining points are called *singular* (or multiple). X is called *smooth* if all its points are smooth.

If X is quasi-projective, the same argument may be repeated for any affine open subset.

14.8. Example. Let $X \subset \mathbb{A}^n$ be the irreducible hypersurface $V(F)$. Then $J = (\frac{\partial F}{\partial x_1} \dots \frac{\partial F}{\partial x_n})$ is a row matrix. So rk $J = 0$ or 1. If rk $J = 0$, then $\frac{\partial F}{\partial x_i} = 0$ in $\mathcal{O}(X)$ for all i. So $\frac{\partial F}{\partial x_i} \in I(Y) = (F)$. Since the degree of $\frac{\partial F}{\partial x_i}$ is $\leq \deg F - 1$, it follows that $\frac{\partial F}{\partial x_i} = 0$ in the polynomial ring. If the characteristic of k is zero this means that F a constant: a contradiction. If char $K = p$, then $F \in K[x_1^p, \dots, x_n^p]$; since K is algebraically closed, then $F = G^p$ for a suitable G, but this is impossible because F is irreducible. So always rk $J = 1 = \rho$. Hence for P general in X, i.e. for P varying in a suitable non-empty open subset, $\dim T_{P,X} = n - 1$. For some particular points, the singular points of X, we can find $\dim T_{P,X} = n$, i.e. $T_{P,X} = \mathbb{A}^n$.

So in the case of a hypersurface $\dim T_{P,X} \geq \dim X$ for every point P in X, and equality holds in the smooth locus of X. In general, one reduces to the case of hypersurfaces using the following theorem:

14.9. Theorem. *Every quasi-projective irreducible variety X is birational to a hypersurface in some affine space.*

Proof. We observe that we can reduce the proof to the case in which X is affine, closed in $\mathbb{A}^n$. Let $m = \dim X$. We have to prove that the field of rational functions $K(X)$ is isomorphic to a field of the form $K(t_1, \ldots, t_{m+1})$, where $t_1, \ldots, t_{m+1}$ satisfy only one non-trivial relation $F(t_1, \ldots, t_{m+1}) = 0$, where F is an irreducible polynomial with coefficients in K. This will follow from the "Abel's primitive element Theorem" concerning extensions of fields. To state it, we need some preliminaries.

Let $K \subset L$ be an extension of fields. Let $a \in L$ be algebraic over K, and let $f_a \in K[x]$ be its minimal polynomial: it is irreducible and monic. Let E be the splitting field of f_a.

14.10. Definition. An element a, algebraic over K, is *separable* if f_a does not have any multiple root in E, i.e. if f_a and its derivative f'_a don't have any common factor of positive degree. Otherwise a is inseparable. If $K \subset L$ is an algebraic extension of fields, it is called separable if any element of L is separable.

In view of the fact that f_a is irreducible, and that the GCD of two polynomials is independent of the field where one considers the coefficients, if a is inseparable, then f'_a is the zero polynomial. If char $K = 0$, this implies that f_a is constant, which is a contradiction. So in characteristic 0, any algebraic extension is separable. If char $K = p > 0$, then $f_a \in K[x^p]$, and f_a is called an inseparable polynomial. In particular algebraic inseparable elements can exist only in positive characteristic. On the other hand, if char $(K) = p > 0$ and K is algebraically closed, is $f_a = a_0 + a_1x^p + a_2x^{2p} + \ldots + a_kx^{kp}$, then all coefficients are p-th powers in K, i.e. $a_i = b_i^p$ for suitable elements b_i; therefore $f_a = b_0^p + b_1^px^p + b_2^px^{2p} + \ldots + b_k^px^{kp} = (b_0 + b_1x + b_2x^2 + \ldots + b_kx^k)^p$, and this contradicts the irreducibility of f_a.

14.11. Abel's primitive element Theorem *Let $K \subseteq L = K(y_1, \ldots, y_m)$ be an algebraic finite extension. If L is a separable extension, then there exists $\alpha \in L$, called a primitive element of L, such that $L = K(\alpha)$ is a simple extension.*

We can now prove Theorem 14.9. The field of rational functions of X is of the form $K(X) = Q(K[X]) = K(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are the coordinate functions on X and $tr.d.K(X)/K = m$. Possibly after renumbering them, we can assume that the first m coordinate functions $t_1, \ldots, t_m$ are algebraically independent over K, and $K(X)$ is an algebraic extension of $L := K(t_1, \ldots, t_m)$. So in our situation we can apply Theorem 14.11: there exists a primitive element α

such that $K(X) = L(\alpha) = K(t_1 \ldots, t_m, \alpha)$. So there exists an irreducible polynomial $f \in L[x]$ such that $K(X) = L[x]/(f)$. Multiplying f by a suitable element of $K[t_1, \ldots, t_m]$, invertible in L, we can eliminate the denominator of f and replace f by a polynomial $g \in K[t_1, \ldots, t_m, x] \subset L[x]$. Now $K[t_1, \ldots, t_m, x]/(g)$ is contained in $L[x](g) = K(X)$, and its quotient field is again $K(X)$. But $K[t_1, \ldots, t_m, x]/(g)$ is the coordinate ring of the hypersurface $Y \subset \mathbb{A}^{m+1}$ of equation $g = 0$. Moreover X and Y are birationally equivalent. This concludes the proof. $\square$

One can show that the coordinate functions on Y, $t_1, \ldots, t_{m+1}$, can be chosen to be linear combinations of the original coordinate functions on X: this means that Y is obtained as a suitable birational projection of X.

14.12. Theorem. *The dimension of the tangent space at a non-singular point of an irreducible variety X is equal to* $\dim X$.

Proof. It is enough to prove the claim under the assumption that X is affine. Let Y be an affine hypersurface birational to X (existing by the previous theorem) and $\phi : X \dashrightarrow Y$ be a birational map. There exist open non-empty subsets $U \subset X$ and $V \subset Y$ such that $\phi : U \to Y$ is an isomorphism. The set of smooth points of Y is an open subset W of Y such that $W \cap V$ is non-empty and $\dim T_{P,Y} = \dim Y = \dim X$ for all $P \in W \cap V$. But $\phi^{-1}(W \cap V) \subset U$ is open non-empty and $\dim T_{Q,X} = \dim X$ for all $Q \in \phi^{-1}(W \cap V)$. This proves the theorem. $\square$

We would like now to study a variety X in a neighbourhood of a smooth point. We have seen that P is smooth for X if and only if $\dim T_{P,X} = \dim X$. Assume X affine: in this case the local ring of P in X is $\mathcal{O}_{P,X} \simeq \mathcal{O}(X)_{I_X(P)}$. But by Theorem 7.7 we have: $\dim \mathcal{O}_{P,X} = \mathrm{ht}\mathcal{M}_{P,X} = \mathrm{ht}I_X(P) = \dim \mathcal{O}(X) = \dim X$ and $\dim T_{P,X} = \dim_K \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$. Therefore P is smooth if and only if

$$\dim_K \mathcal{M}_{P,X}/\mathcal{M}^2_{P,X} = \dim \mathcal{O}_{P,X}$$

(the first one is a dimension as K-vector space, the second one is a Krull dimension). By the Nakayama's Lemma a base of $\mathcal{M}_{P,X}/\mathcal{M}^2_{P,X}$ corresponds bijectively to a minimal system of generators of the ideal $\mathcal{M}_{P,X}$ (observe that the residue field of $\mathcal{O}_{P,X}$ is K). Therefore P is smooth for X if and only if $\mathcal{M}_{P,X}$ is minimally generated by r elements, where $r = \dim X$, in other words if and only if $\mathcal{O}_{P,X}$ is a regular local ring.

For example, if X is a curve, P is smooth if and only if $T_{P,X}$ has dimension 1, i.e. $\mathcal{M}_{P,X}$ is principal: $\mathcal{M}_{P,X} = (t)$. This means that the equation $t = 0$ only defines the point P, i.e. P has one local equation in a suitable neighborhood of P.

Let P be a smooth point of X and $\dim X = n$. Functions $u_1, \ldots, u_n \in \mathcal{O}_{P,X}$ are called *local parameters* at P if $u_1, \ldots, u_n \in \mathcal{M}_{P,X}$ and their residues $\bar{u}_1, \ldots, \bar{u}_n$

in $\mathcal{M}_{P,X}/\mathcal{M}_{P,X}^2$ $(= T^*_{P,X})$ form a base, or equivalently if $u_1, \ldots, u_n$ is a minimal set of generators of $\mathcal{M}_{P,X}$. Recalling the isomorphism

$$d_P : \mathcal{M}_{P,X}/\mathcal{M}_{P,X}^2 \to T^*_{P,X}$$

we deduce that $u_1, \ldots, u_n$ are local parameters if and only if $d_P\bar{u}_1, \ldots, d_P\bar{u}_n$ are linearly independent forms on $T_{P,X}$ (which is a vector space of dimension n), if and only if the system of equations on $T_{P,X}$

$$d_P\bar{u}_1 = \ldots = d_P\bar{u}_n = 0$$

has only the trivial solution P (which is the origin of the vector space $T_{P,X}$.

Let $u_1, \ldots, u_n$ be local parameters at P. There exists an open affine neighborhood of P on which $u_1, \ldots, u_n$ are all regular. We replace X by this neighborhood, so we assume that X is affine and that $u_1, \ldots, u_n$ are polynomial functions on X. Let X_i be the closed subset $V(u_i)$ of X: it has codimension 1 in X, because u_i is not identically zero on X ($u_1, \ldots, u_n$ is a minimal set of generators of $\mathcal{M}_{P,X}$).

14.13. Proposition. *In this notation, P is a smooth point of X_i, for all $i = 1, \ldots, n$ and $\bigcap T_{P,X_i} = \{P\}$.*

Proof. Assume that U_i is a polynomial inducing u_i, then $X_i = V(U_i) \cap X = V(I(X) + (U_i))$. So $I(X_i) \supset I(X) + (U_i)$. By considering the linear parts of the polynomials of the previous ideal, we get: $T_{P,X_i} \subset T_{P,X} \cap V(d_PU_i)$. By the assumption on the u_i, it follows that $T_{P,X} \cap V(d_PU_1) \cap \ldots \cap V(d_PU_n) = \{P\}$. Since $\dim T_{P,X} = n$, we can deduce that $T_{P,X} \cap V(d_PU_i)$ is strictly contained in $T_{P,X}$, and $\dim T_{P,X} \cap V(d_PU_i) = n-1$. So $\dim T_{P,X_i} \le n-1 = \dim X_i$, hence P is a smooth point on X_i, equality holds and $T_{P,X_i} = T_{P,X} \cap V(d_PU_i)$. Moreover $\bigcap T_{P,X_i} = \{P\}$. $\square$

Note that $\bigcap X_i$ has no positive-dimensional component passing through P: otherwise the tangent space to T at P would be contained in T_{P,X_i} for all i, against the fact that $\bigcap T_{P,X_i} = \{P\}$.

14.13. Definition. Let X be a smooth variety. Subvarieties $Y_1, \ldots, Y_r$ of X are called *transversal at* P, $P \in \bigcap Y_i$, if the intersection of the tangent spaces T_{P,Y_i} has dimension as small as possible, i.e. if $\operatorname{codim}_{T_{P,X}}(\bigcap T_{P,Y_i}) = \sum \operatorname{codim}_X Y_i$.

Taking $T_{P,X}$ as ambient variety, one gets the relation:

$$\dim \bigcap T_{P,Y_i} \ge \sum \dim T_{P,Y_i} - (r-1)\dim T_{P,X};$$

hence

$$\operatorname{codim}_{T_{P,X}}(\bigcap T_{P,Y_i}) = \dim T_{P,X} - \dim \bigcap T_{P,Y_i} \le \sum(\dim T_{P,X} - \dim T_{P,Y_i}) =$$

$$= \sum \operatorname{codim}_{T_{P,X}}(T_{P,Y_i}) \leq \sum \operatorname{codim}_X Y_i.$$

If equality holds, P is a smooth point for Y_i for all i, moreover we get that P is a simple point for the set $\bigcap Y_i$.

For example if X is a surface and $P \in X$ is smooth, there is a nbhd U of P such that P is the transversal intersection of two curves in U, corresponding to local parameters u_1, u_2. If P is singular we need three functions u_1, u_2, u_3 to generate the maximal ideal $\mathcal{M}_{P,X}$.

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