Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA

PER IL SISTEMA TERRA

# SURFACE WAVES

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# The Wave Equation: Potentials

*On Waves Propagated along the Plane Surface of an Elastic Solid.* *By* Lord Rayleigh, D.C.L., F.R.S.

[*Read November* 12*th*, 1885.]

It is proposed to investigate the behaviour of waves upon the plane free surface of an infinite homogeneous isotropic elastic solid, their character being such that the disturbance is confined to a superficial region, of thickness comparable with the wave-length. The case is thus analogous to that of deep-water waves, only that the potential energy here depends upon elastic resilience instead of upon gravity.*

Denoting the displacements by $\alpha$, $\beta$, $\gamma$, and the dilatation by $\theta$, we have the usual equations

$$\mathbf{u} = \nabla\Phi + \nabla \times \Psi$$

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

$\mathbf{u}$ displacement

$\Phi$ scalar potential

$\Psi_i$ vector potential

$$\partial_t^2 \Phi = \alpha^2 \nabla^2 \Phi$$

$$\partial_t^2 \Psi_i = \beta^2 \nabla^2 \Psi_i$$

$\alpha$ P-wave speed

$\beta$ S-wave speed

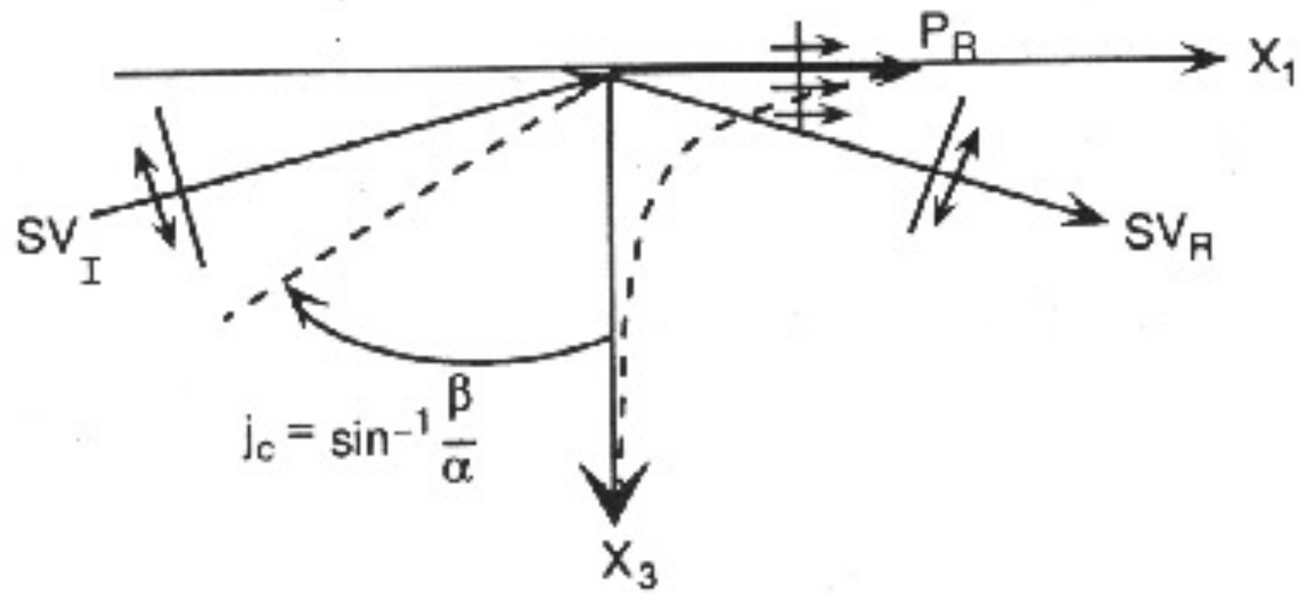
# Rayleigh Waves

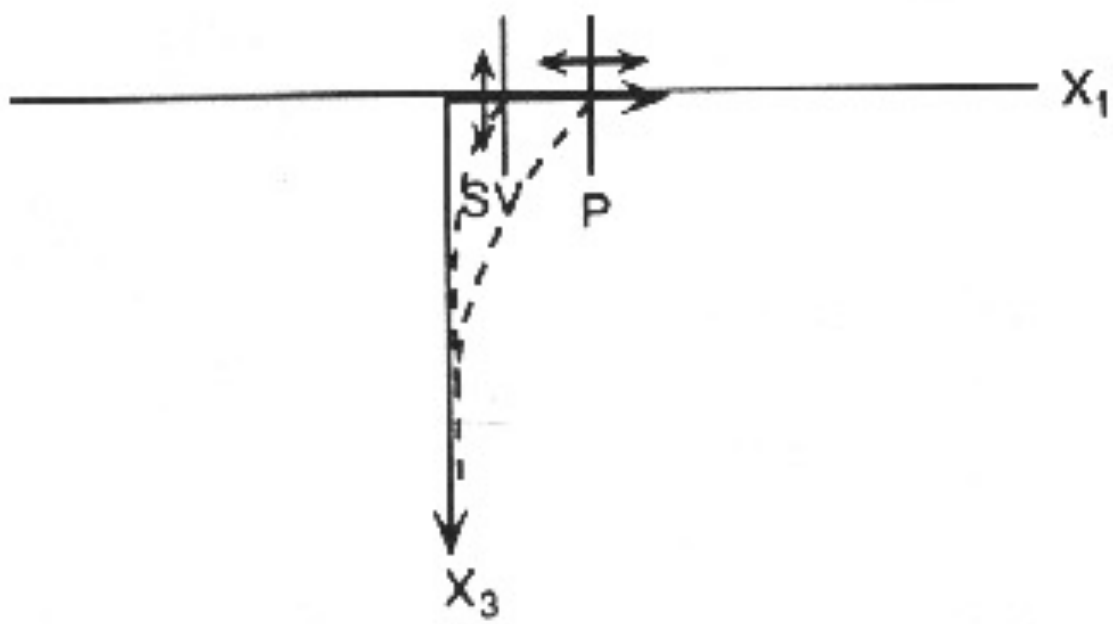
SV waves incident on a free surface: conversion and reflection

An evanescent P-wave propagates along the free surface decaying exponentially with depth.

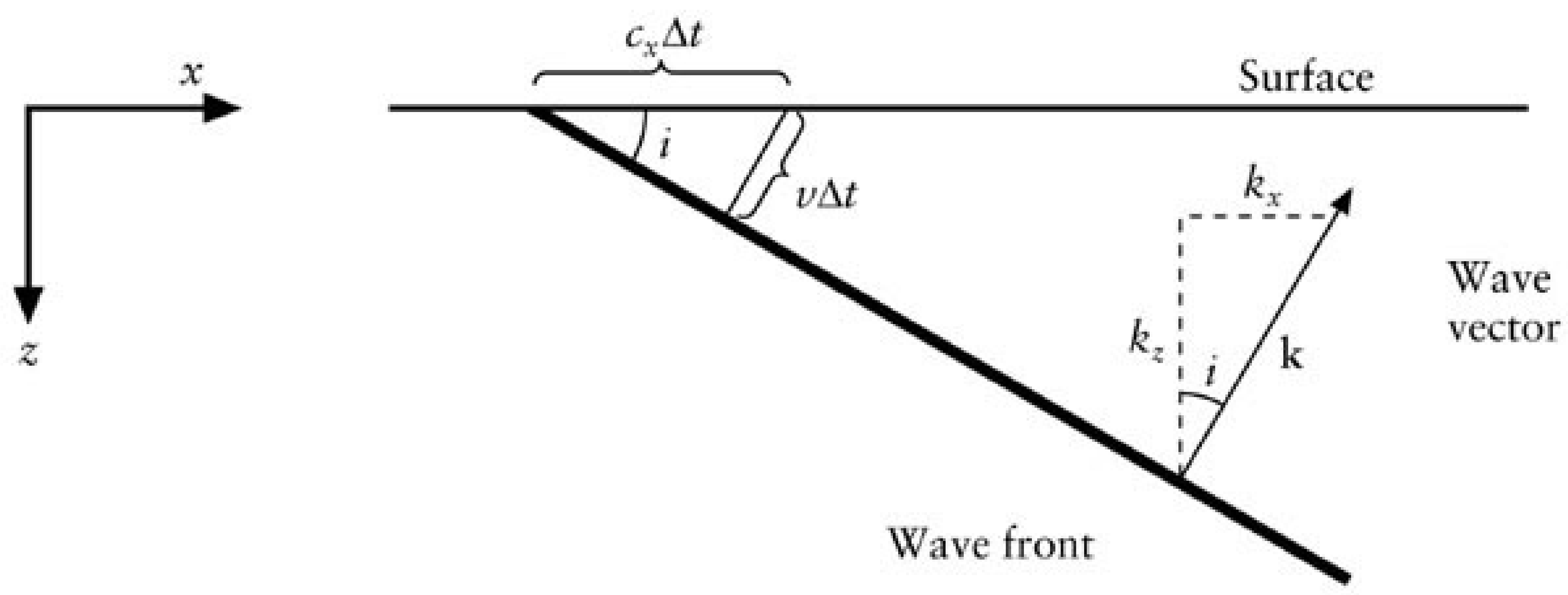
The reflected post-critically reflected SV wave is totally reflected and phase-shifted. These two wave types can only exist together, they both satisfy the free surface boundary condition:

-> Surface waves



# Apparent horizontal velocity




$$k_x = k\sin(i) = \omega\frac{\sin(i)}{\alpha} = \frac{\omega}{c}$$

$$k_z = k\cos(i) = \sqrt{k^2 - k_x^2} = \omega\sqrt{\left(\frac{1}{\alpha}\right)^2 - \left(\frac{1}{c}\right)^2} = \frac{\omega}{c}\sqrt{\left(\frac{c}{\alpha}\right)^2 - 1} = k_x r_\alpha$$

In current terminology, $k_x$ is k!

# Surface waves: Geometry

We are looking for plane waves traveling along one horizontal coordinate axis, so we can - for example - set

$$\partial_y(\cdot) = 0$$

And consider only wave motion in the x,z plane. Then

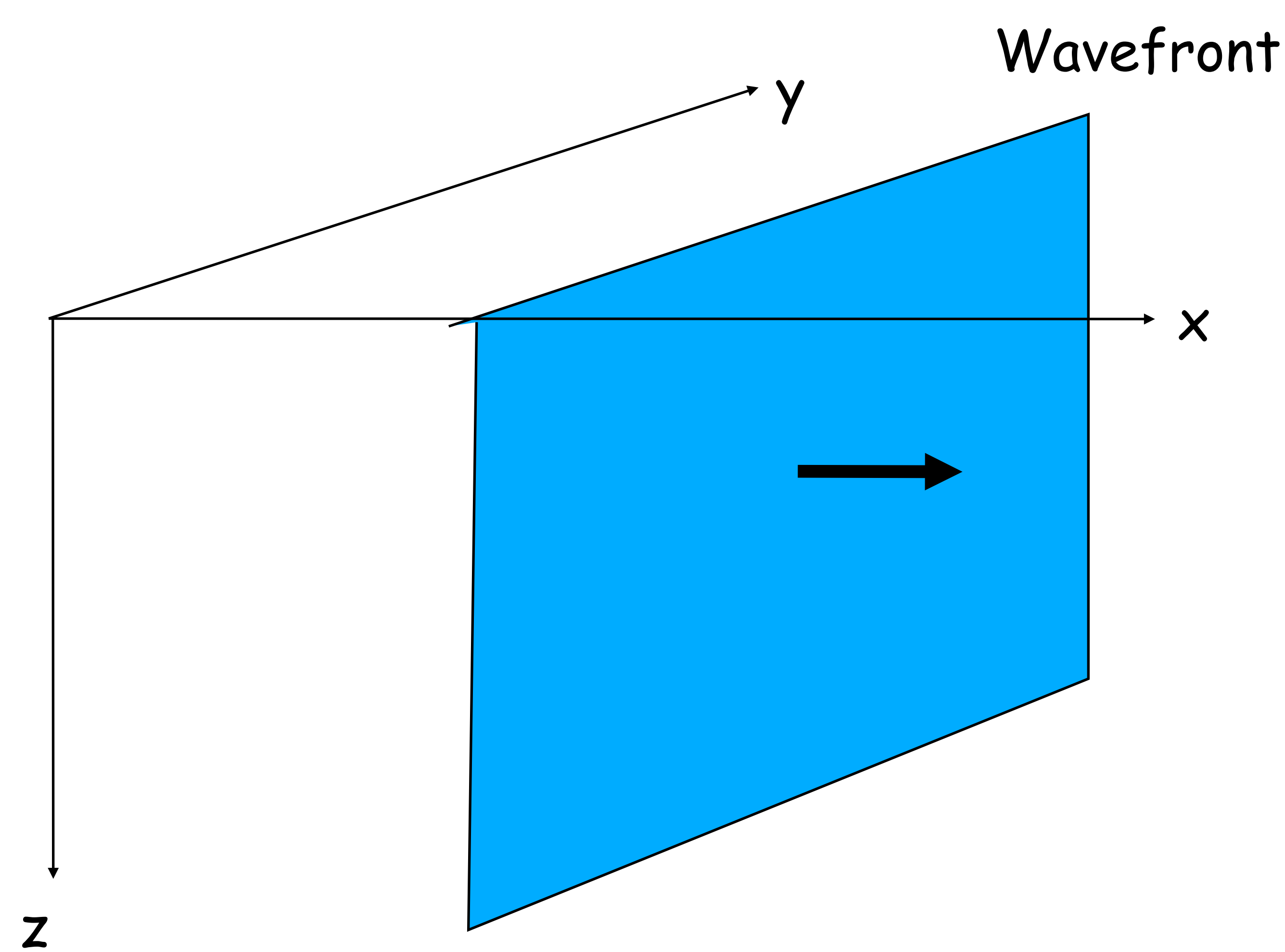
$$u_x = \partial_x \Phi - \partial_z \Psi_y$$

$$u_z = \partial_z \Phi + \partial_x \Psi_y$$

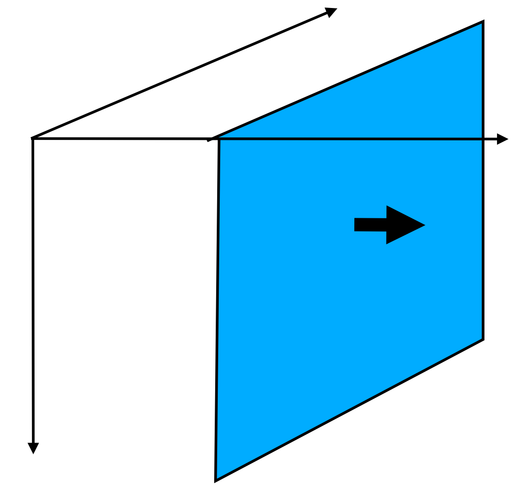
As we only require $\Psi_y$ we set $\Psi_y = \Psi$ from now on. Our trial solution is thus

$$\Phi = A\exp[ik(x \pm r_\alpha z - ct)]$$

$$\Psi = B\exp[ik(x \pm r_\beta z - ct)]$$



# Condition of existence



With that ansatz one has that, in order to desired solution exists, the coefficients

$$r_\alpha = \pm\sqrt{\frac{c^2}{\alpha^2} - 1} \qquad r_\beta = \pm\sqrt{\frac{c^2}{\beta^2} - 1}$$

have to express a decay along z, i.e.

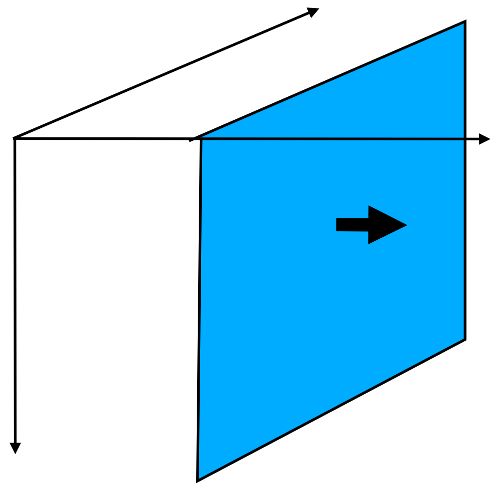
$$c < \beta < \alpha$$

to obtain

$$\Phi = A\exp\left[i(kx - \omega t) - kz\sqrt{1 - \frac{c^2}{\alpha^2}}\right] = A\exp\left(-kz\sqrt{1 - \frac{c^2}{\alpha^2}}\right)\exp[i(kx - \omega t)]$$

$$\Psi = B\exp\left[i(kx - \omega t) - kz\sqrt{1 - \frac{c^2}{\beta^2}}\right] = B\exp\left(-kz\sqrt{1 - \frac{c^2}{\beta^2}}\right)\exp[i(kx - \omega t)]$$

# Surface waves: Boundary Conditions



Analogous to the problem of finding the reflection-transmission coefficients we now have to satisfy the boundary conditions at the free surface (stress free)

$$\sigma_{zz} = 0 = \sigma_{zx}$$

In isotropic media we have

$$\sigma_{zz} = \lambda(\partial_x u_x + \partial_z u_z) + 2\mu \partial_z u_z$$
$$\sigma_{xz} = 2\mu \partial_x u_z$$

where

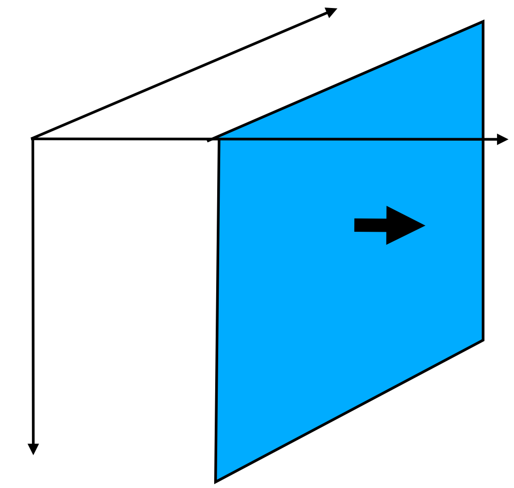
$$u_x = \partial_x \Phi - \partial_z \Psi$$
$$u_z = \partial_z \Phi + \partial_x \Psi$$

and

$$\Phi = A\exp[ik(x \pm r_\alpha z - ct)]$$

$$\Psi = B\exp[ik(x \pm r_\beta z - ct)]$$

# Rayleigh waves: solutions



This leads to the following relationship for c, the phase velocity:

$$(2 - c^2/\beta^2)^2 = 4(1 - c^2/\alpha^2)^{1/2}(1 - c^2/\beta^2)^{1/2}$$

For simplicity we take a fixed relationship between P and shear-wave velocity (Poisson's medium):

$$\alpha = \sqrt{3}\,\beta$$

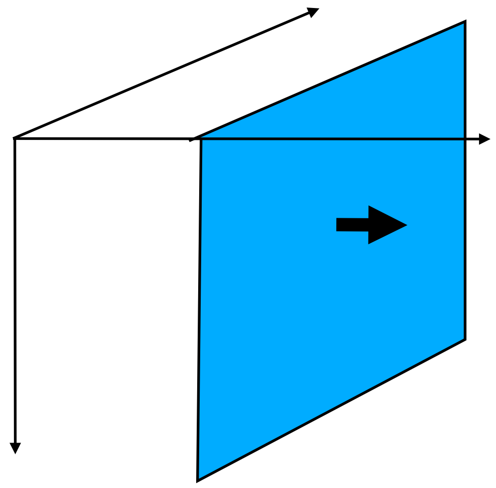
… to obtain

$$\frac{c^6}{\beta^6} - 8\frac{c^4}{\beta^4} + \frac{56}{3}\frac{c^2}{\beta^2} - 16 = 0$$

… and the only root which fulfills the condition $c<\beta$ is

$$\boxed{c \cong 0.92\,\beta}$$

# Displacement



Putting this value back into our solutions we finally obtain the displacement in the x-z plane for a plane harmonic surface wave propagating along direction x

$$u_x = C\left(e^{-0.8475kz} - 0.5773e^{-0.3933kz}\right)\sin k(x-ct)$$

$$u_z = C\left(-0.8475e^{-0.8475kz} + 1.4679e^{-0.3933kz}\right)\cos k(x-ct)$$

This development was first made by Lord Rayleigh in 1885.

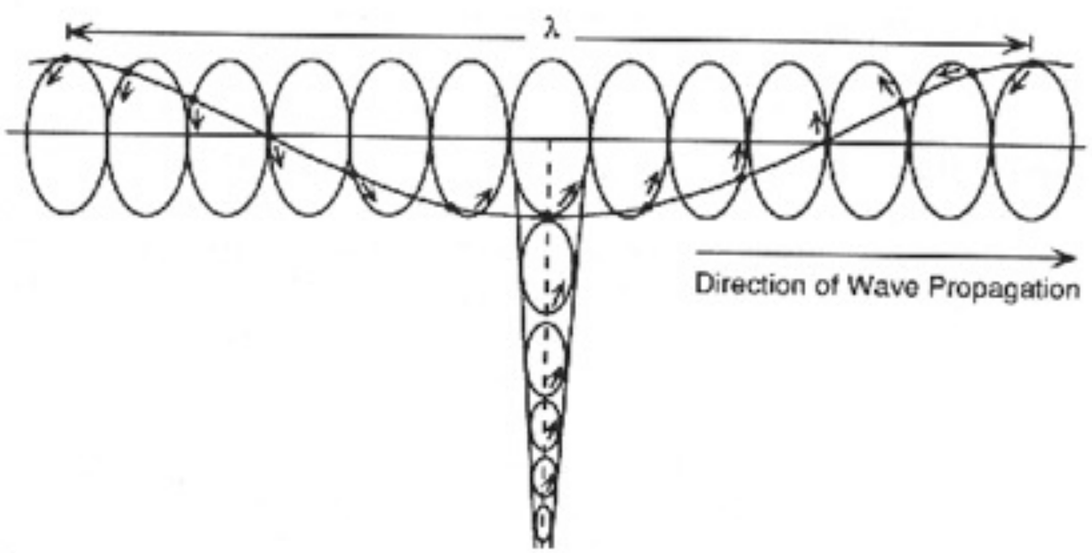
It demonstrates that YES there are solutions to the wave equation propagating along a **free surface!**

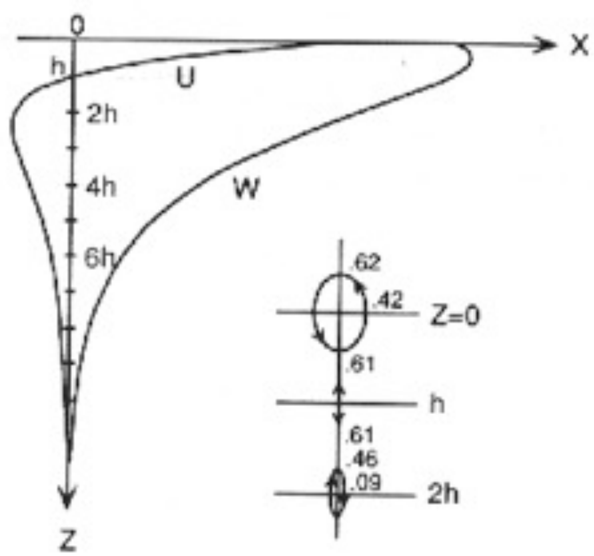
Some remarkable facts can be drawn from this particular form:

# Particle Motion (1)

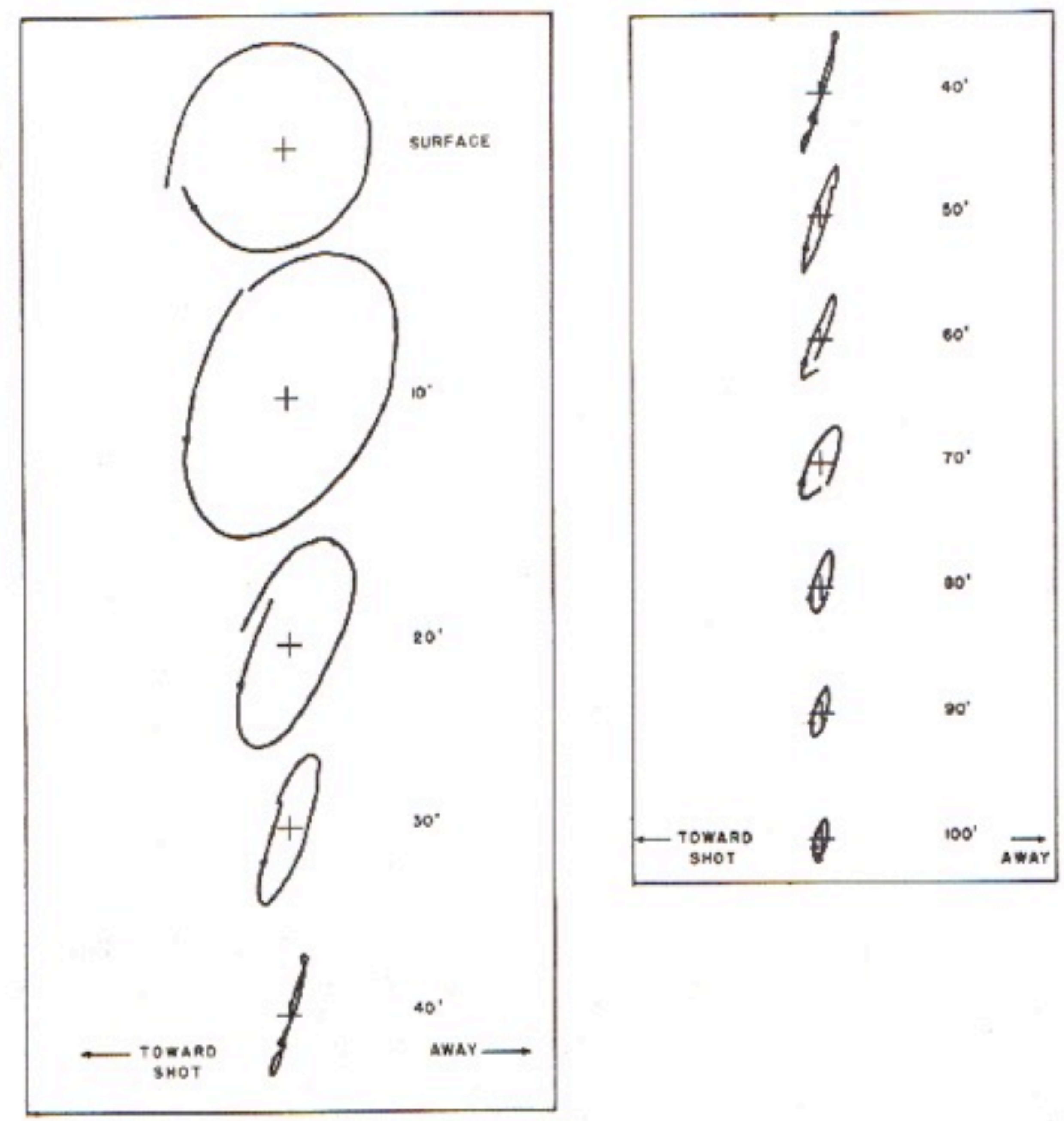
How does the particle motion look like?

theoretical




experimental



# Lamb's Problem and Rayleigh waves

Transient solution to an impulsive vertical point force at the surface of a half space is called **Lamb's problem** (after Horace Lamb, 1904).

-the two components are out of phase by π/2

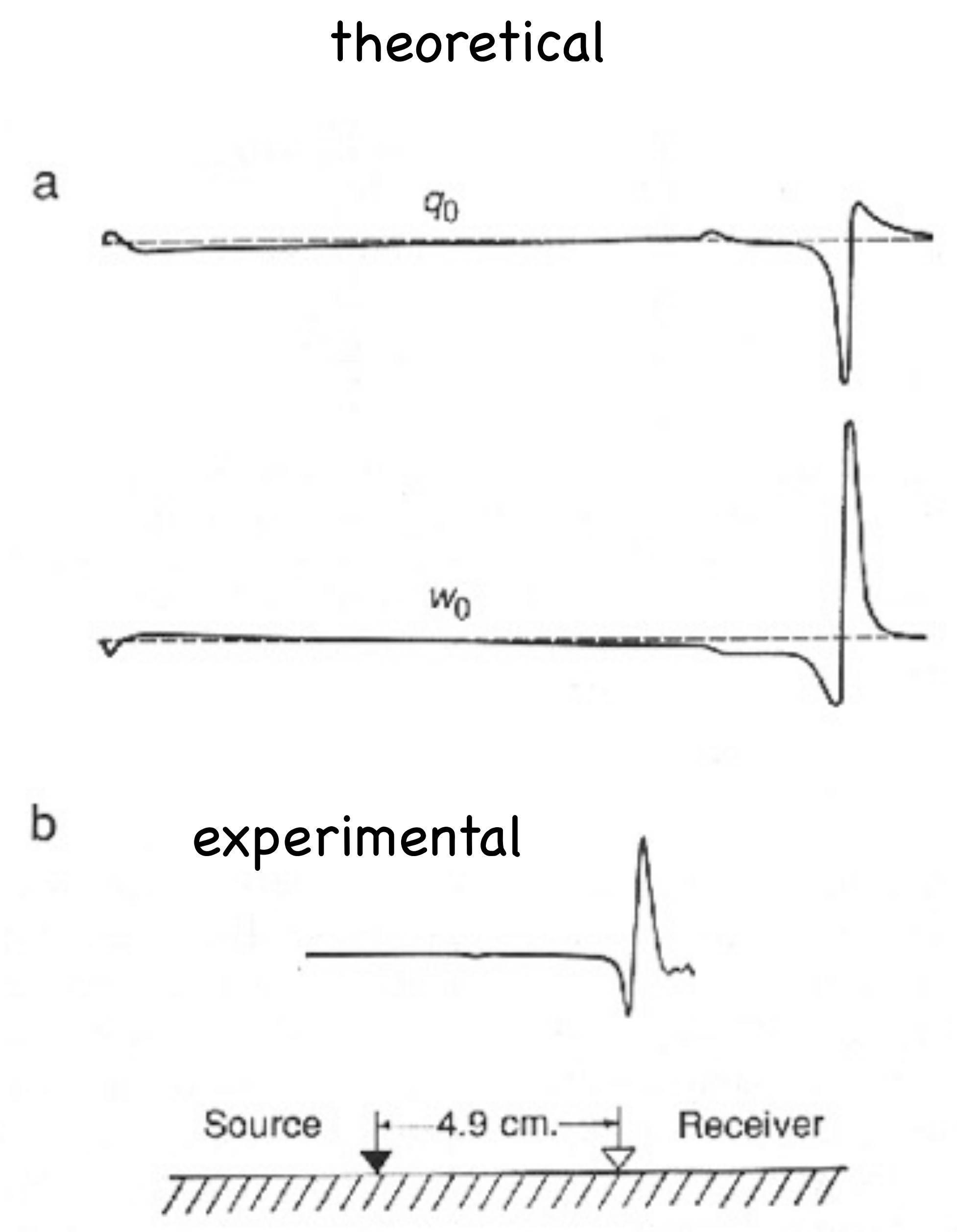
– for small values of z a particle describes an ellipse and the motion is retrograde

- at some depth z the motion is linear in z

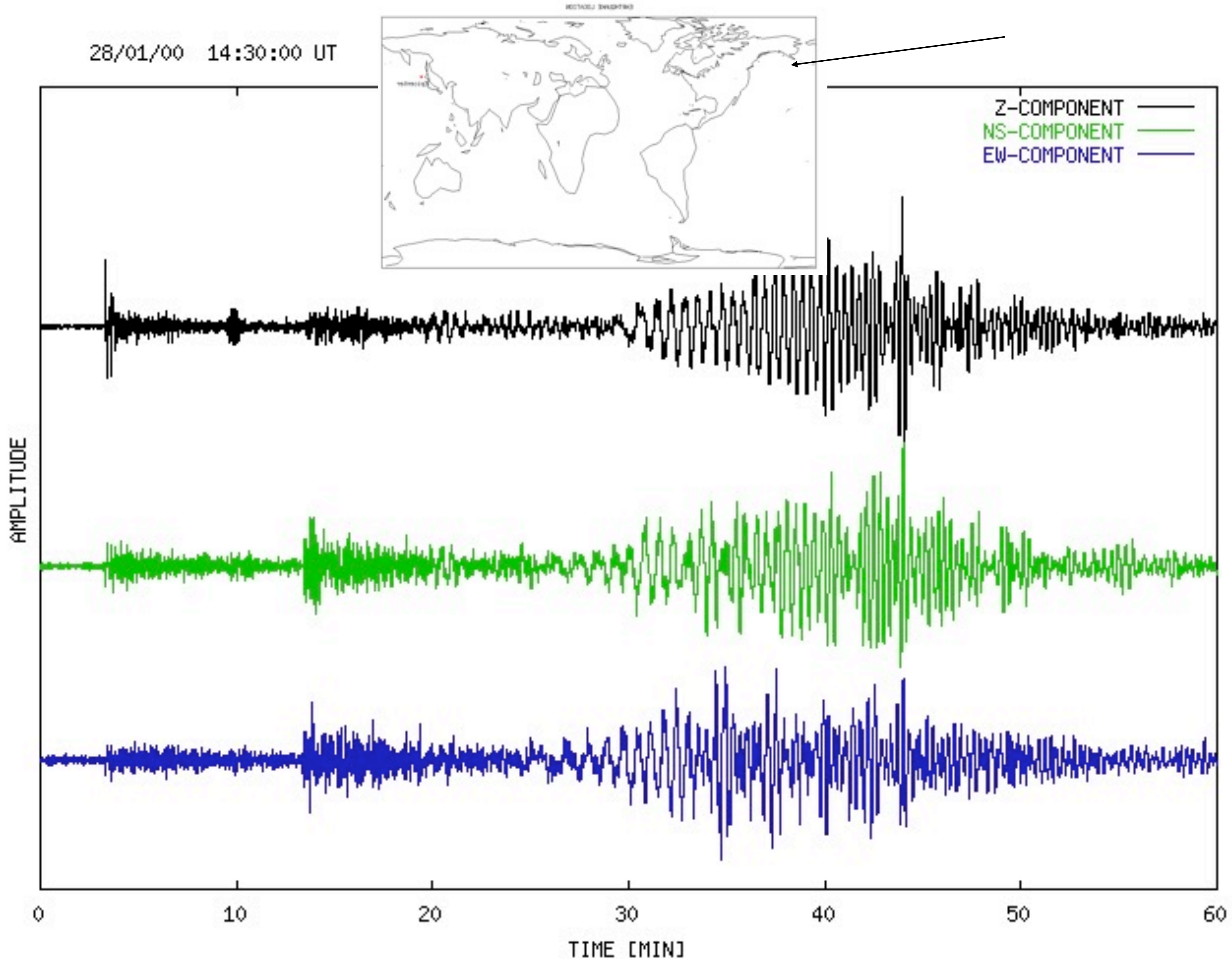
- below that depth the motion is again elliptical but prograde

- the phase velocity is independent of k: **there is no dispersion** for a homogeneous half space

- Right Figure: radial and vertical motion for a source at the surface



# Data Example



# Dispersion relation

- In physics, the dispersion relation is the relation between the energy of a system and its corresponding momentum. For example, for massive particles in free space, the dispersion relation can easily be calculated from the definition of kinetic energy:

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

- For electromagnetic waves, the energy is proportional to the frequency of the wave and the momentum to the wavenumber. In this case, Maxwell's equations tell us that the dispersion relation for vacuum is linear: $\omega = ck$.

- The name "**dispersion relation**" originally comes from optics. It is possible to make the effective speed of light dependent on wavelength by making light pass through a material which has a non-constant index of refraction, or by using light in a non-uniform medium such as a waveguide. In this case, the waveform will spread over time, such that a narrow pulse will become an extended pulse, i.e. be dispersed.

# Dispersion relation

- In classical mechanics, the Hamilton's principle the perturbation scheme applied to an averaged Lagrangian for an harmonic wave field gives a characteristic equation: $\Delta(\omega, k_i)=0$

*Transverse wave in a string*

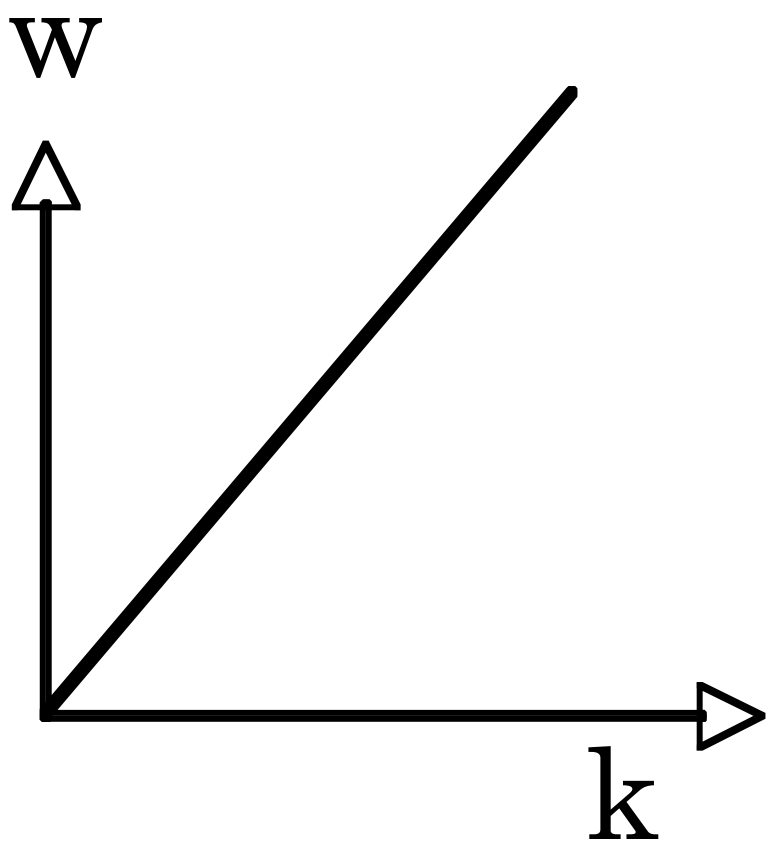
$$\left(\frac{\partial^2}{\partial x^2} - \frac{\mu}{F}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \Rightarrow \quad \omega = \pm kc$$

*Acoustic wave*

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\rho}{B}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \Rightarrow \quad \omega = \pm kc$$

*Longitudinal wave in a rod*

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\rho}{E}\frac{\partial^2}{\partial t^2}\right)\phi = 0 \Rightarrow \quad \omega = \pm kc$$



# Effect of dispersion...

Demonstration: sum two harmonic waves with slightly different angular frequencies and wavenumbers:

$$u(x, t) = \cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x)$$

$$\omega_1 = \omega + \delta\omega \qquad \omega_2 = \omega - \delta\omega \qquad \omega \gg \delta\omega$$

$$k_1 = k + \delta k \qquad k_2 = k - \delta k \qquad k \gg \delta k$$

Add the two cosines:

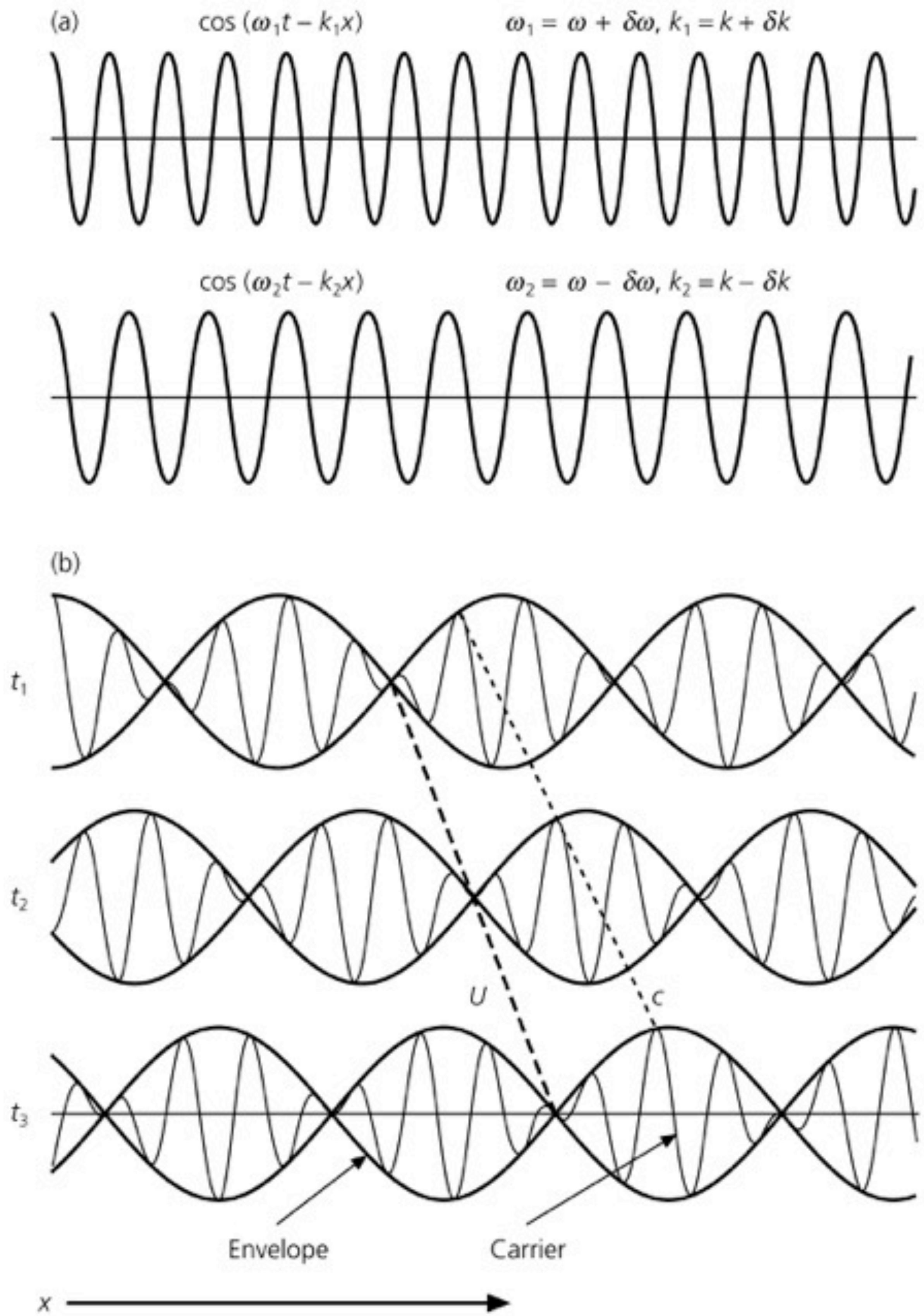
$$u(x, t) = \cos(\omega t + \delta\omega t - kx - \delta kx)$$
$$+ \cos(\omega t - \delta\omega t - kx + \delta kx)$$
$$= 2\cos(\omega t - kx)\cos(\delta\omega t - \delta kx)$$

The envelope (beat) has a *group velocity*:

$$U = \delta\omega / \delta k$$

The individual peaks move with a *phase velocity*:

$$c = \omega / k$$



# Dispersion examples

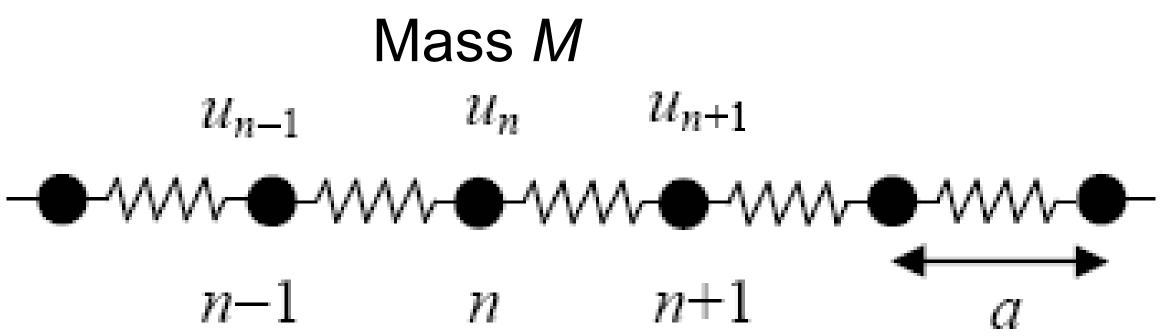
- Discrete systems: lattices
- Stiff systems: rods and thin plates
- Boundary waves: plates and rods

  Discontinuity interfaces are intrinsic in their propagation since they allow to store energy (not like body waves)!

# Monatomic 1D lattice

Let us examine the simplest periodic system within the context of harmonic approximation ($F = dU/du = Cu$) - a one-dimensional crystal lattice, which is a sequence of masses m connected with springs of force constant $C$ and separation $a$.

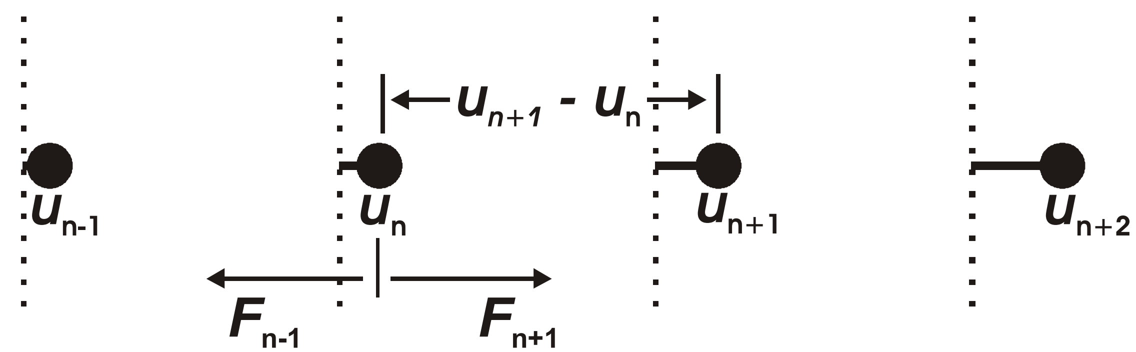
The collective motion of these springs will correspond to solutions of a wave equation. Note: by construction we can see that 3 types of wave motion are possible,
2 transverse, 1 longitudinal (or compressional)




How does the system appear with a longitudinal wave?:

The force exerted on the $n$-th atom in the lattice is given by

$F_n = F_{n+1,n} - F_{n-1,n} = C[(u_{n+1} - u_n) - (u_n - u_{n-1})]$.




Applying Newton's second law to the motion of the $n$-th atom we obtain

$$M\frac{d^2u_n}{dt^2} = F_n = -C(2u_n - u_{n+1} - u_{n-1})$$

Note that we neglected hereby the interaction of the $n$-th atom with all but its nearest neighbors. A similar equation should be written for each atom in the lattice, resulting in $N$ coupled differential equations, which should be solved simultaneously ($N$ - total number of atoms in the lattice). In addition the boundary conditions applied to end atoms in the lattice should be taken into account.

# Dispersion in lattices

## Monatomic 1D lattice - continued

Now let us attempt a solution of the form: $u_n = Ae^{i(kx_n - \omega t)}$ ,

where $x_n$ is the equilibrium position of the $n$-th atom so that $x_n = na$. This equation represents a traveling wave, in which all atoms oscillate with the same frequency $\omega$ and the same amplitude $A$ and have a wavevector $k$. Now substituting the guess solution into the equation and canceling the common quantities (the amplitude and the time-dependent factor) we obtain

$$M(-\omega^2)e^{ikna} = -C[2e^{ikna} - e^{ik(n+1)a} - e^{ik(n-1)a}].$$

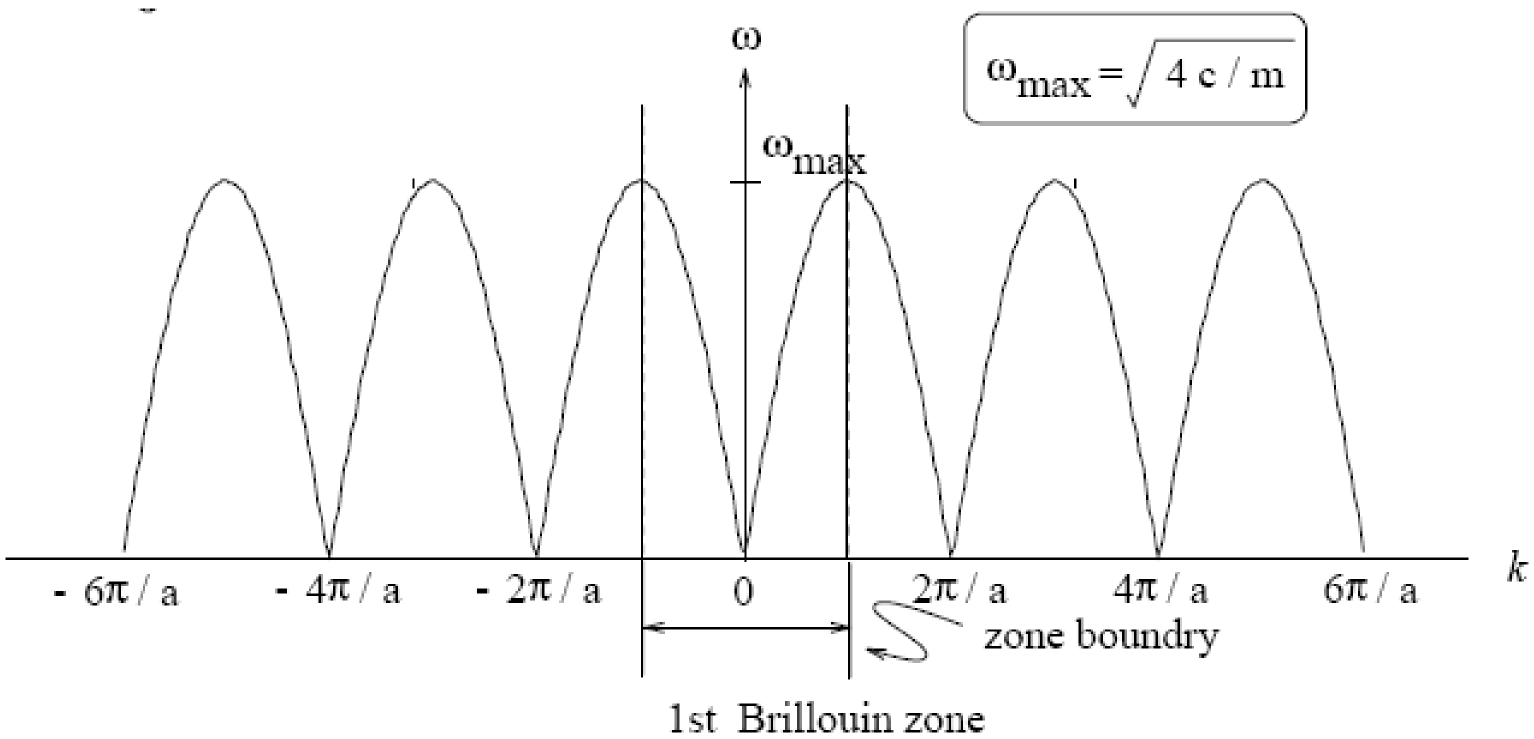
This equation can be further simplified by canceling the common factor $e^{ikna}$ , which leads to

$$M\omega^2 = C\left(2 - e^{ika} - e^{-ika}\right) = 2C(1 - \cos ka) = 4C\sin^2\frac{ka}{2}.$$

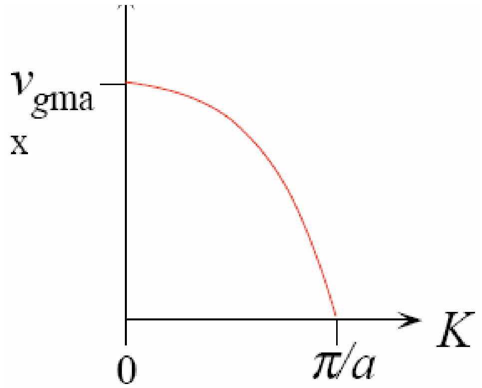
We find thus the dispersion relation for the frequency:

$$\omega = \sqrt{\frac{4C}{M}}\left|\sin\frac{ka}{2}\right|$$

which is the relationship between the frequency of vibrations and the wavevector $k$. The dispersion relation has a number of important properties.



# Monatomic 1D lattice – continued




*Phase and group velocity*. The phase velocity is defined by

$v_p = \frac{\omega}{k}$ and the group velocity by $v_g = \frac{d\omega}{dk}$

The physical distinction between the two velocities is that $v_p$ is the velocity of propagation of the plane wave, whereas the $v_g$ is the velocity of the propagation of the wave packet. The latter is the velocity for the propagation of energy in the medium. For the particular dispersion relation $\omega = \sqrt{\frac{4C}{M}}\left|\sin\frac{ka}{2}\right|$ the group velocity is given by $v_g = \sqrt{\frac{Ca^2}{M}}\cos\frac{ka}{2}$.

Apparently, the group velocity is zero at the edge of the zone where $k = \pm\,\pi/a$. Here the wave is standing and therefore the transmission velocity for the energy is zero.

*Long wavelength limit*. The long wavelength limit implies that $\lambda \gg a$. In this limit $ka \ll 1$.

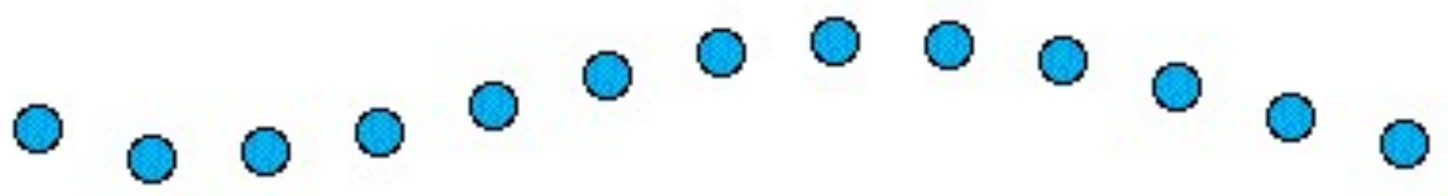
We can then expand the sine in '$\omega$' and obtain for the positive frequencies: $\omega = \sqrt{\frac{C}{M}}ka$.

We see that the frequency of vibration is proportional to the wavevector. This is equivalent to the statement that velocity is independent of frequency. In this case:

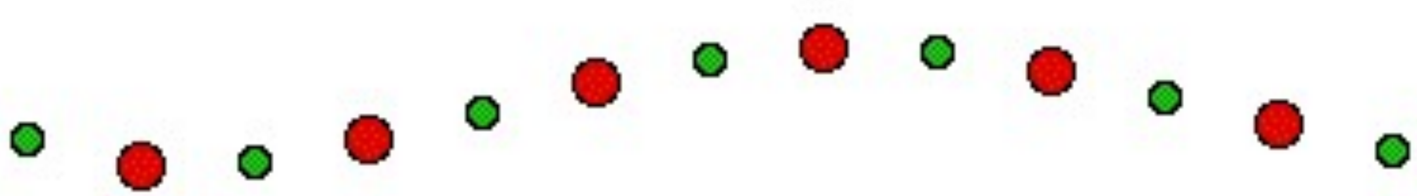
$v_p = \frac{\omega}{k} = \sqrt{\frac{C}{M}}a$. This is the velocity of sound for the one dimensional lattice which is consistent with the expression we obtained earlier for elastic waves.

# Acoustic and optical modes

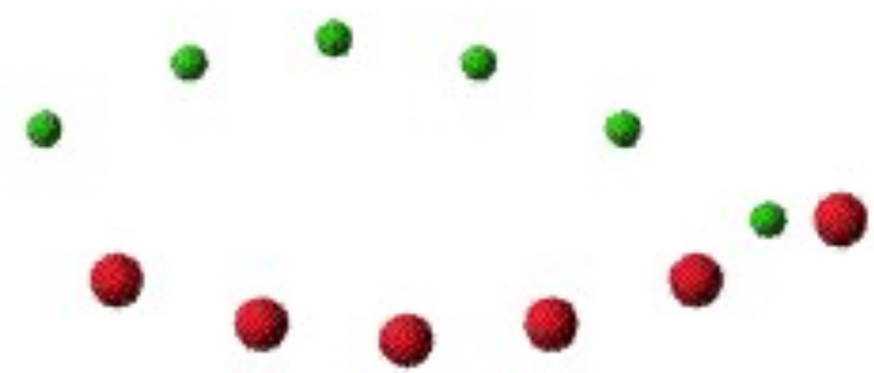
Monoatomic chain
**acoustic longitudinal** mode



Monoatomic chain
**acoustic transverse** mode



Diatomic chain
**acoustic transverse** mode



Diatomic chain
**optical transverse** mode

# Dispersion examples

- Discrete systems: lattices

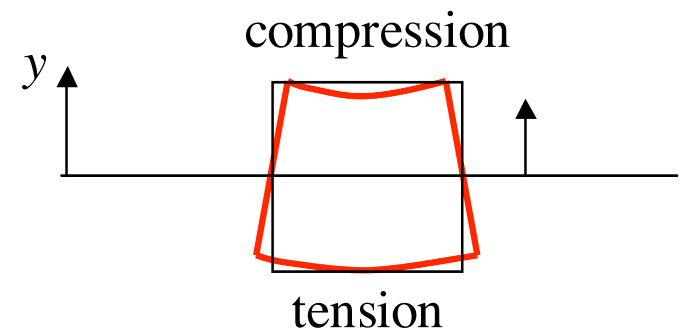
- Stiff systems: rods and thin plates

- Boundary waves: plates and rods

  Discontinuity interfaces are intrinsic in their propagation since they allow to store energy (not like body waves)!

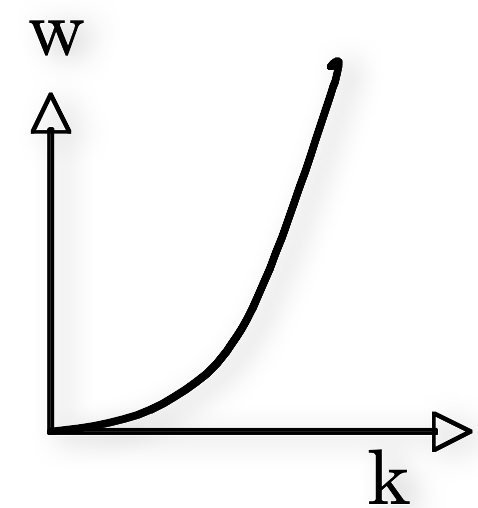
# Stiffness...

- How "**stiff**" or "flexible" is a material? It depends on whether we pull on it, twist it, bend it, or simply compress it. In the simplest case the material is characterized by two independent "stiffness constants" and that different combinations of these constants determine the response to a pull, twist, bend, or pressure.




Euler-Bernoulli equation

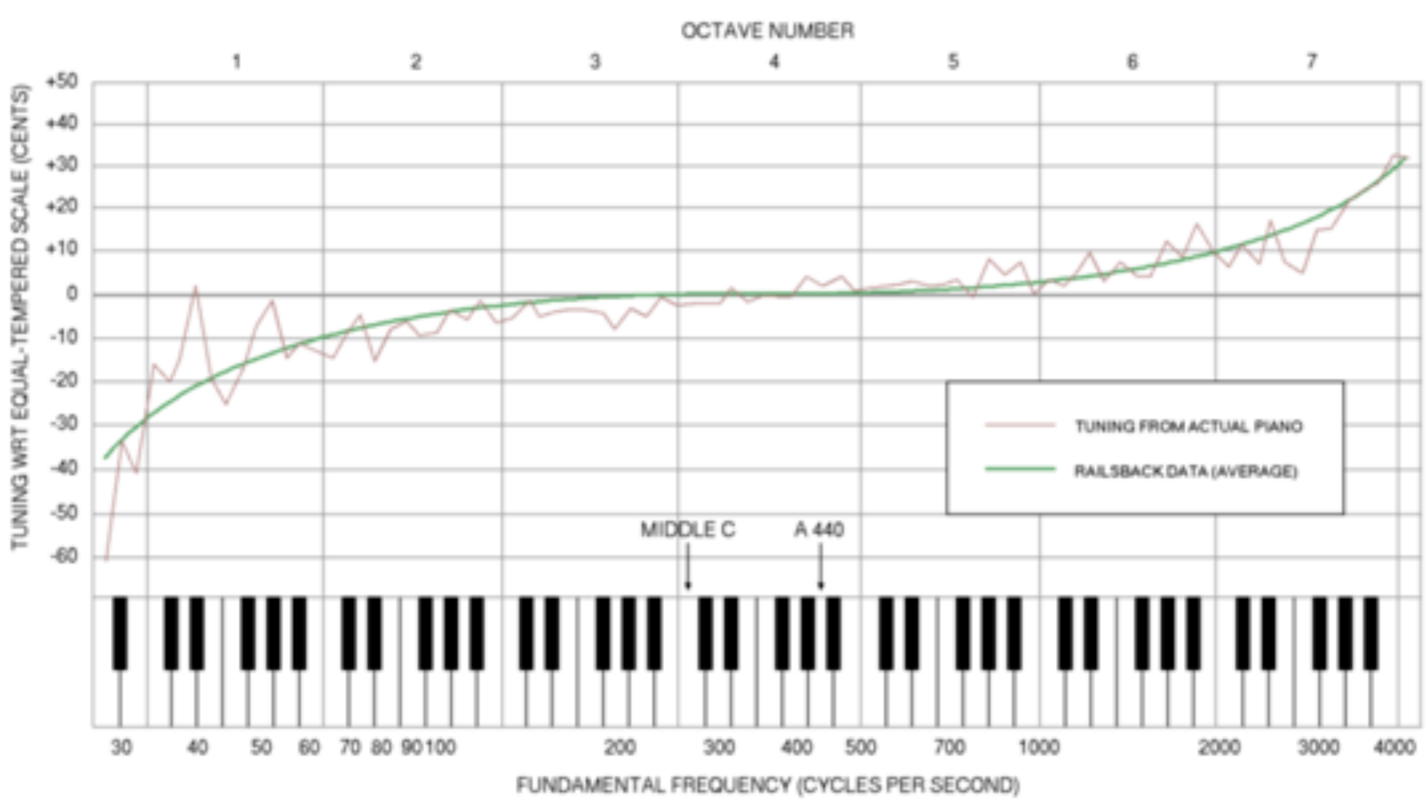
$$\left(\frac{\partial^4}{\partial x^4} - \frac{\rho A}{EI}\frac{\partial^2}{\partial t^2}\right)w = 0 \Rightarrow \quad \omega = \pm k^2\sqrt{\frac{EI}{\rho A}}$$



# Stiffness...

- Stiffness in a vibrating string introduces a restoring force proportional to the bending angle of the string and the usual stiffness term added to the wave equation for the ideal string. Stiff-string models are commonly used in piano synthesis and they have to be included in tuning of piano strings due to inharmonic effects.

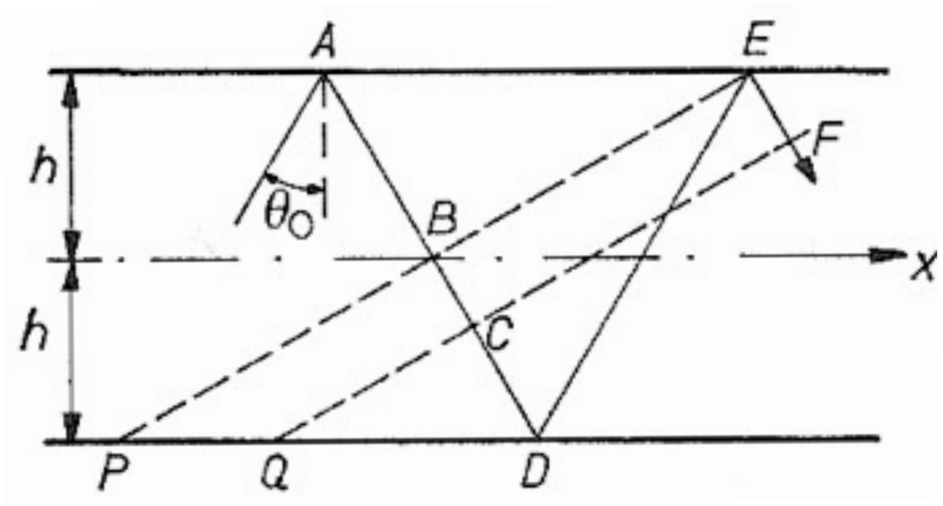
$$\left( \frac{\partial^4}{\partial x^4} + \frac{E}{\rho}\frac{\partial^2}{\partial x^2} - \frac{\rho A}{EI}\frac{\partial^2}{\partial t^2} \right) w = 0 \Rightarrow \quad \omega = \pm k\sqrt{\frac{E}{\rho}}\left(1 + k^2\sqrt{\frac{I}{A}}\right)^{1/2}$$




$$\Rightarrow \quad \omega \approx \pm k\sqrt{\frac{E}{\rho}}\left(1 + \frac{1}{2}k^2\sqrt{\frac{I}{A}}\right)$$

# SH Waves in plates: Geometry

Repeated reflection in the layer allow interference between incident and reflected SH waves: SH reverberations can be totally trapped.




The condition of interference of multiply reflected waves at the rigid boundaries is:

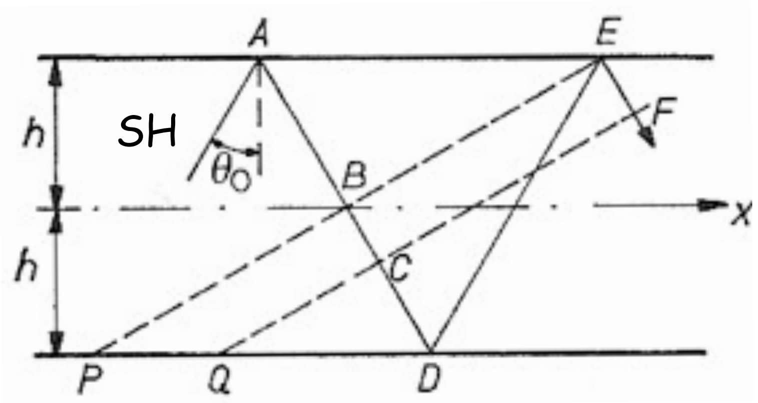
$$\frac{BDE}{\lambda} = \frac{CDEF}{\lambda} = n \qquad BDE = 2(2h)\cos\theta_0 \qquad \cos\theta_0 = n\frac{\lambda}{2(2h)} = n\frac{\pi}{(2h)k}$$

$$k\cos\theta_0(2h) = k_z(2h) = k_x r_\beta(2h) = n\pi$$

Examples: Sound waves in a duct; SH (P-SV) waves in a plate; TEM modes

# SH waves: trapping

$$u_y = A\exp[i(\omega t + \omega\eta_\beta z - kx)] + B\exp[i(\omega t - \omega\eta_\beta z - kx)]$$




$$k = k_x = \frac{\omega}{c}; \quad \omega\eta_\beta = k_z = \frac{\omega}{c}\sqrt{\frac{c^2}{\beta^2} - 1} = kr_\beta$$

$$u_y = A\exp[i(\omega t + kr_\beta z - kx)] + B\exp[i(\omega t - kr_\beta z - kx)]$$

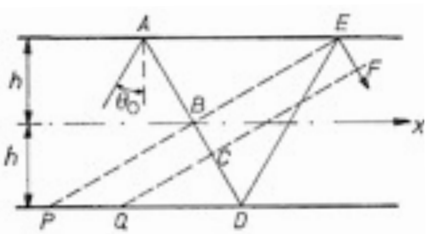
The formal derivation is very similar to the derivation of the Rayleigh waves. The conditions to be fulfilled are: free surface conditions

$$\sigma_{zy}(0) = \mu \left.\frac{\partial u_y}{\partial z}\right|_0 = ikr_\beta\mu\{A\exp[i(\omega t - kx)] - B\exp[i(\omega t - kx)]\} = 0$$

$$\sigma_{zy}(2h) = \mu \left.\frac{\partial u_y}{\partial z}\right|_{2h} = ikr_\beta\mu\{A\exp[i(\omega t + kr_\beta 2h - kx)] - B\exp[i(\omega t - kr_\beta 2h - kx)]\} = 0$$

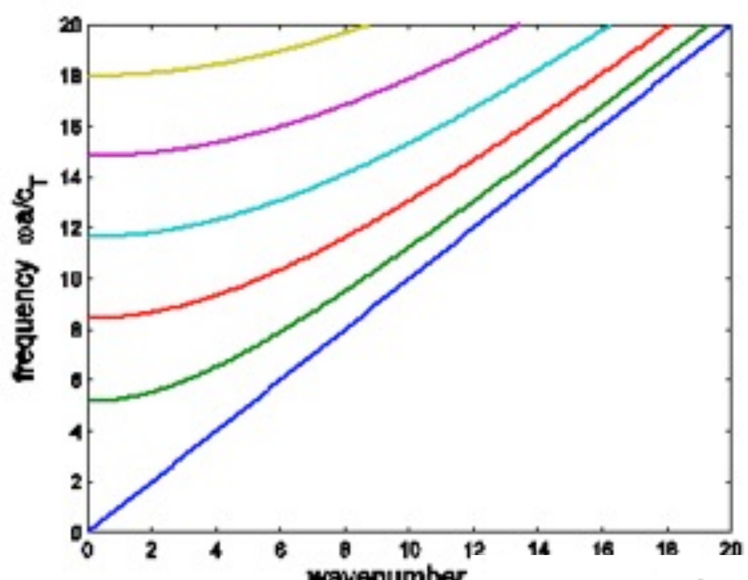
# SH waves: eigenvalues...

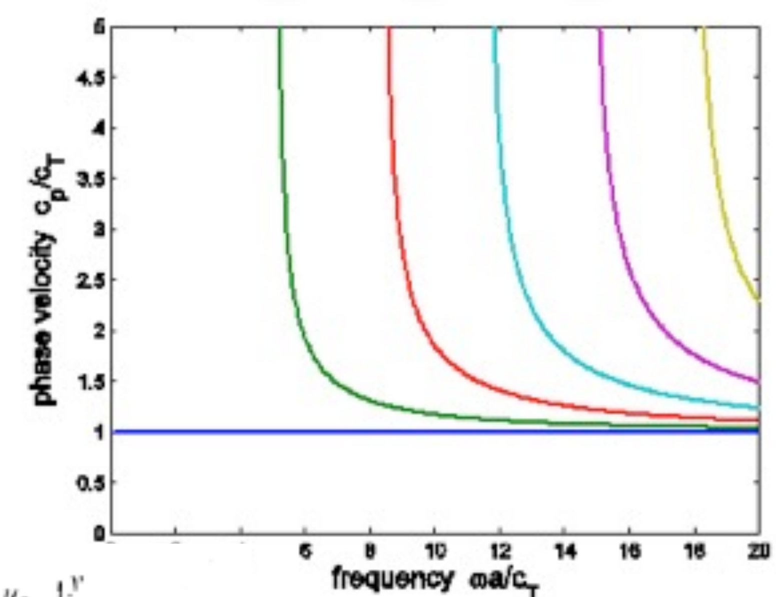
that leads to: $kr_\beta 2h = n\pi$ with n=0,1,2,... NB: REMEMBER THE "STRING PROBLEM":$kL=n\pi$

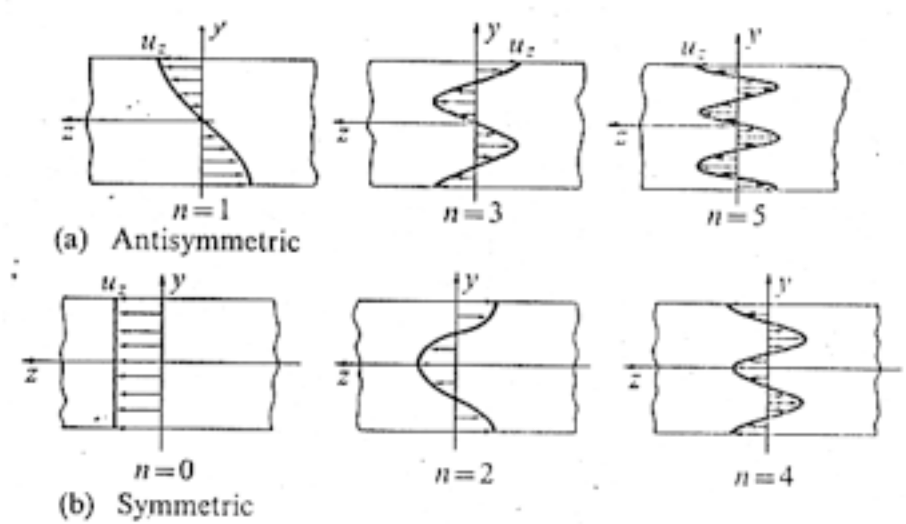



$$\omega^2 = k^2\beta^2 + \left(\frac{n\pi\beta}{2h}\right)^2$$

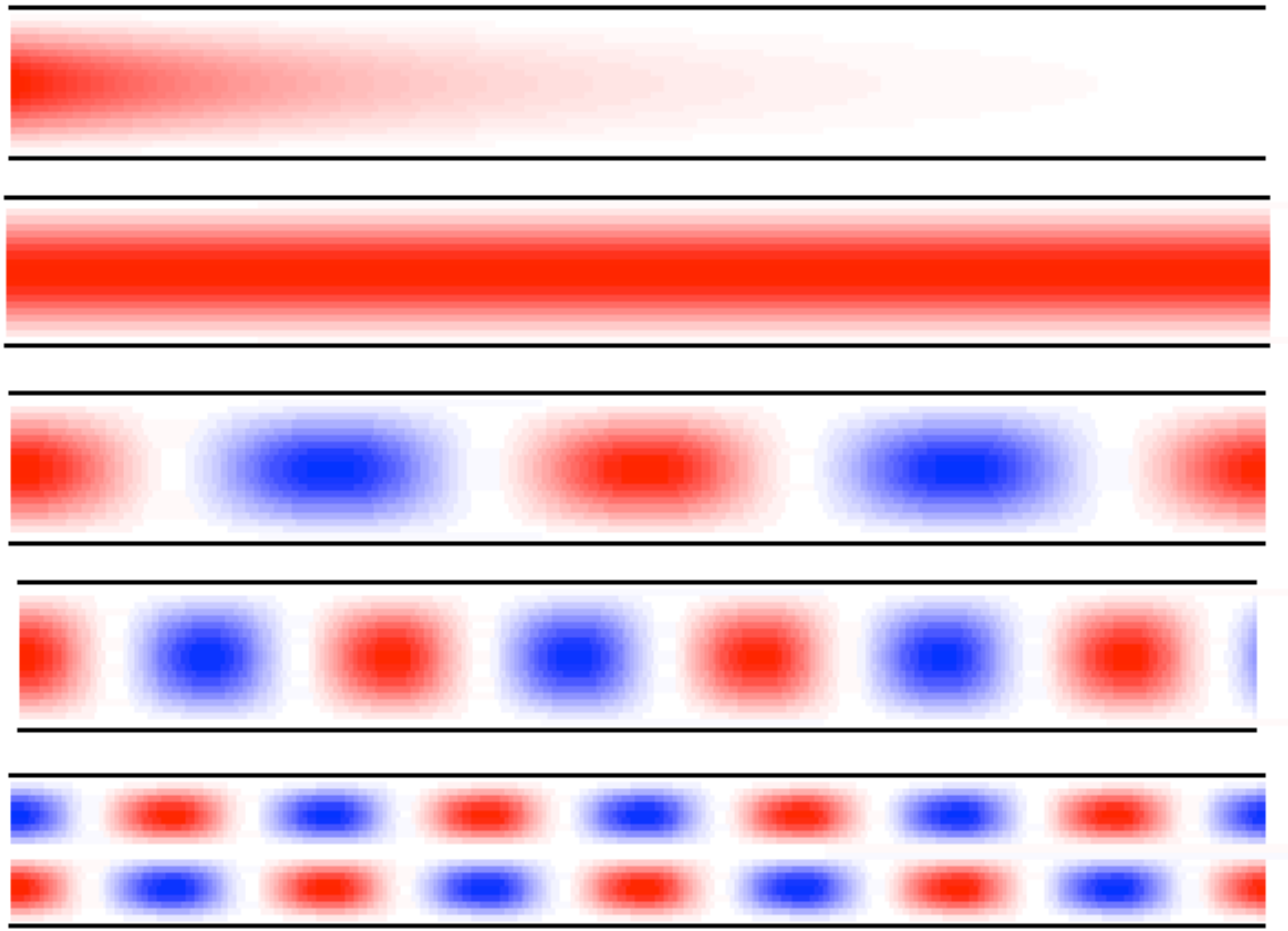
$$c = \frac{\beta}{\sqrt{1-\left(\frac{n\pi\beta}{2h\omega}\right)^2}}$$



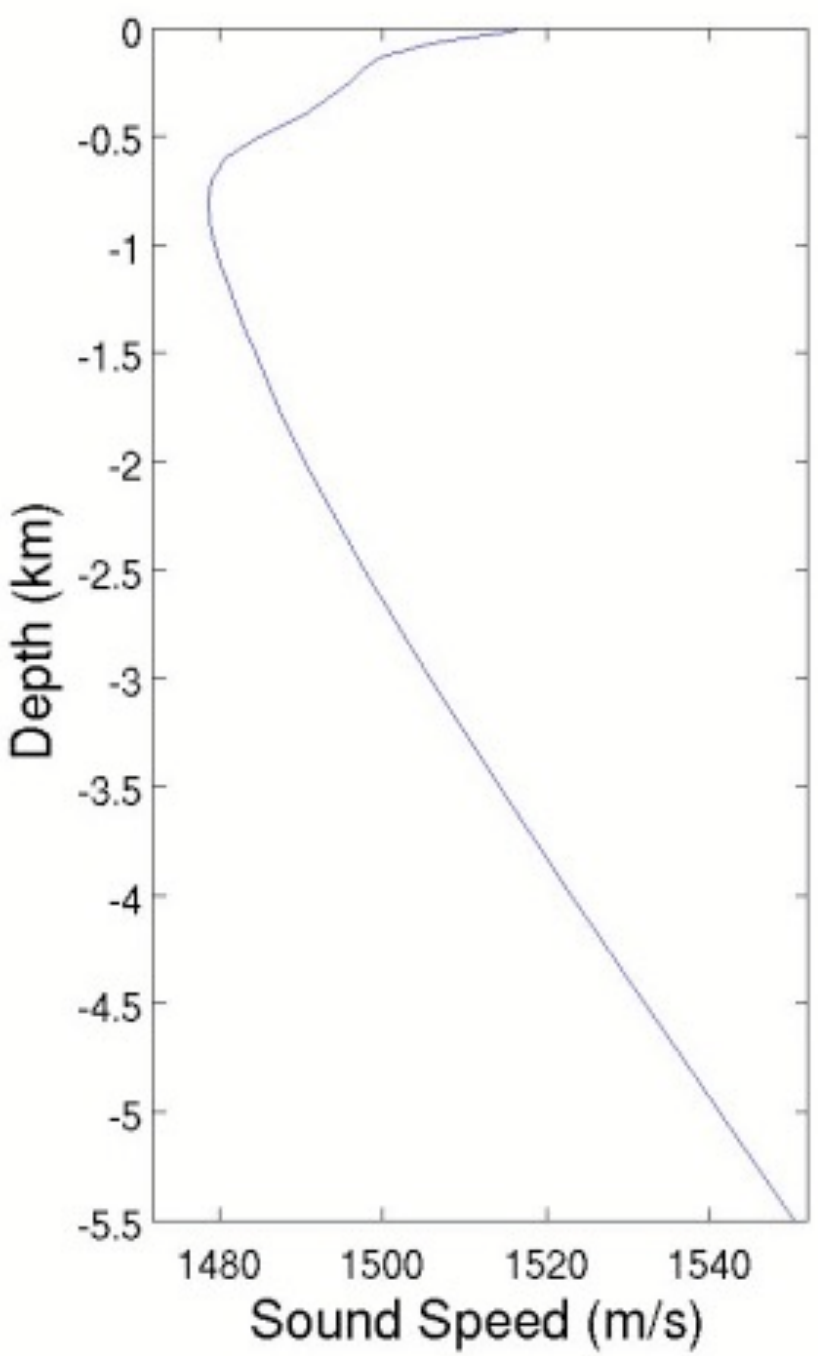
# EM waveguide animations

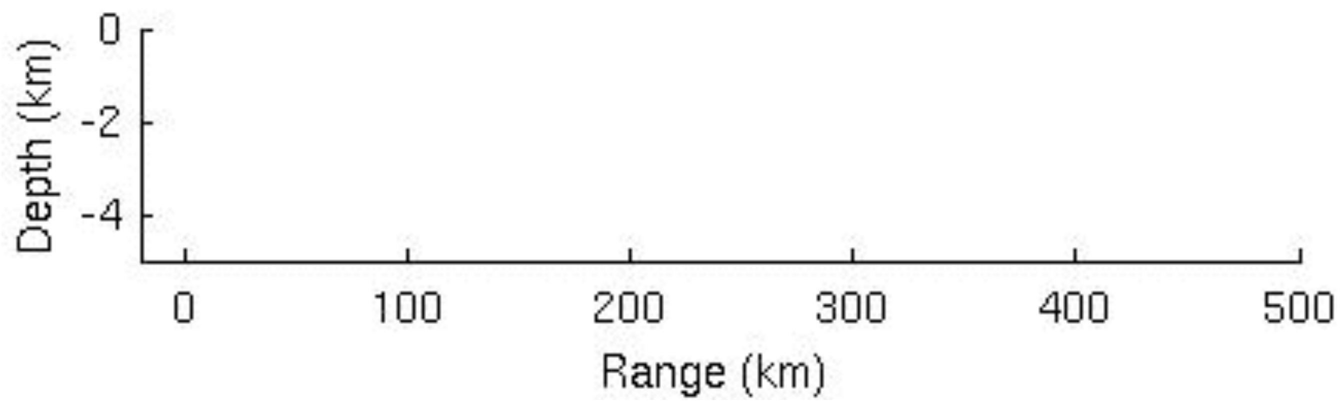



http://people.seas.harvard.edu/~jones/ap216/lectures/ls_1/ls1_u8/ls1_unit_8.html

# Acoustic waveguides...

**SOFAR channel** (Sound Fixing And Ranging channel)



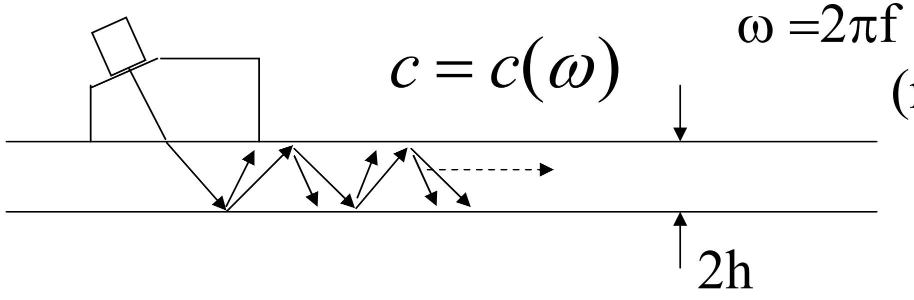

Sound speed as a function of depth at a position north of Hawaii in the Pacific Ocean derived from the 2005 World Ocean Atlas. The SOFAR channel axis is at ca. 750-m depth.

# Waves in plates

In low frequency plate waves, there are two distinct type of harmonic motion. These are called symmetric or **extensional** waves and antisymmetric or **flexural** waves.

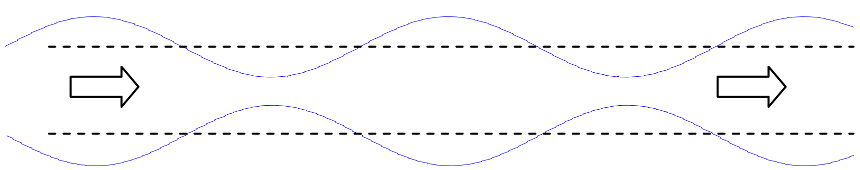
$c = c(\omega)$

$\omega = 2\pi f$ … frequency (rad/sec)




$$\phi = f(y)\exp[ik(x - ct)]$$

$$\psi = g(y)\exp[ik(x - ct)]$$

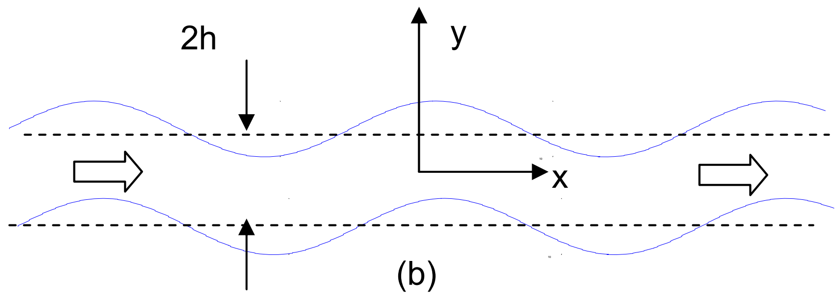


(a)

extensional waves

$$f = A\cosh(\alpha y)$$

$$g = B\sinh(\beta y)$$




(b)

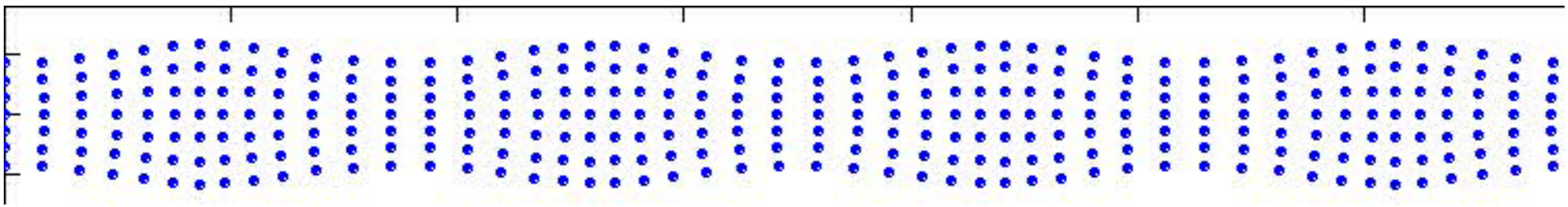
flexural waves

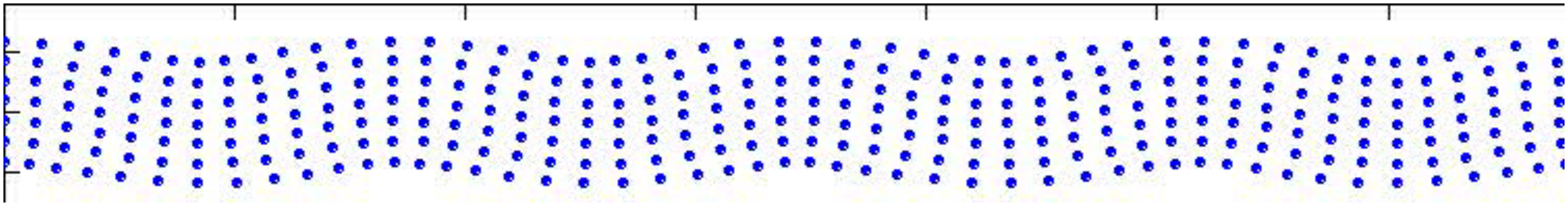
$$f = A'\sinh(\alpha y)$$

$$g = B'\cosh(\beta y)$$

# Lamb waves

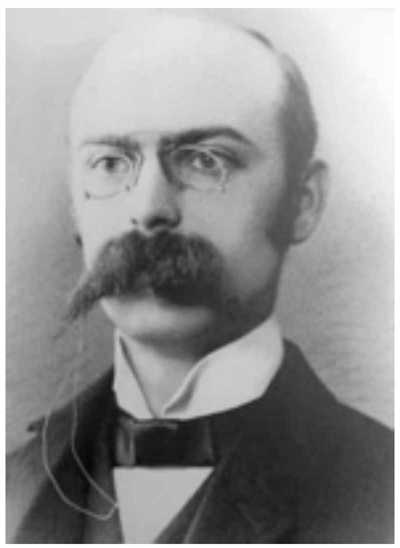
Lamb waves are waves of plane strain that occur in a free plate, and the traction force must vanish on the upper and lower surface of the plate. In a free plate, a line source along y axis and all wave vectors must lie in the x-z plane. This requirement implies that response of the plate will be independent of the in-plane coordinate normal to the propagation direction.





# Love Waves: Geometry

In an elastic half-space no SH type surface waves exist. Why?
Because there is total reflection and no interaction between an evanescent P wave and a phase shifted SV wave as in the case of Rayleigh waves. What happens if we have a layer over a half space (Love, 1911) ?



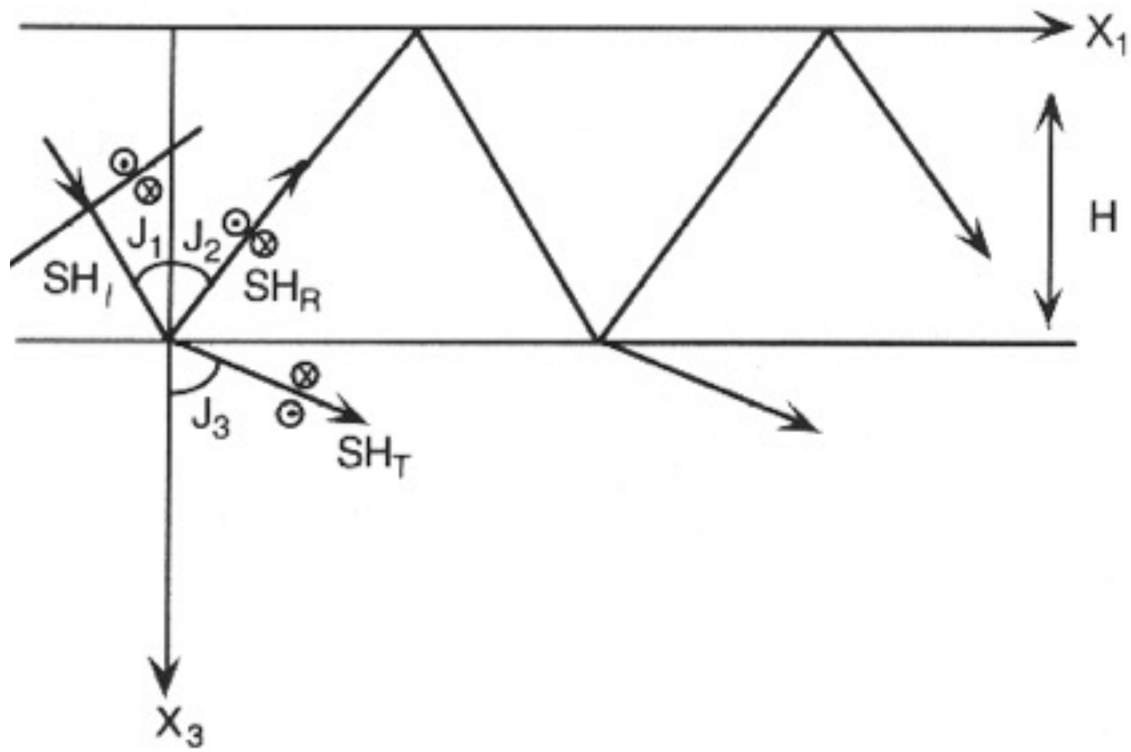
SOME PROBLEMS
OF
GEODYNAMICS

BEING AN ESSAY TO WHICH THE ADAMS PRIZE
IN THE UNIVERSITY OF CAMBRIDGE
WAS ADJUDGED IN 1911

BY

A. E. H. LOVE, M.A., D.Sc., F.R.S.
FORMERLY FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE
HONORARY FELLOW OF QUEEN'S COLLEGE, OXFORD

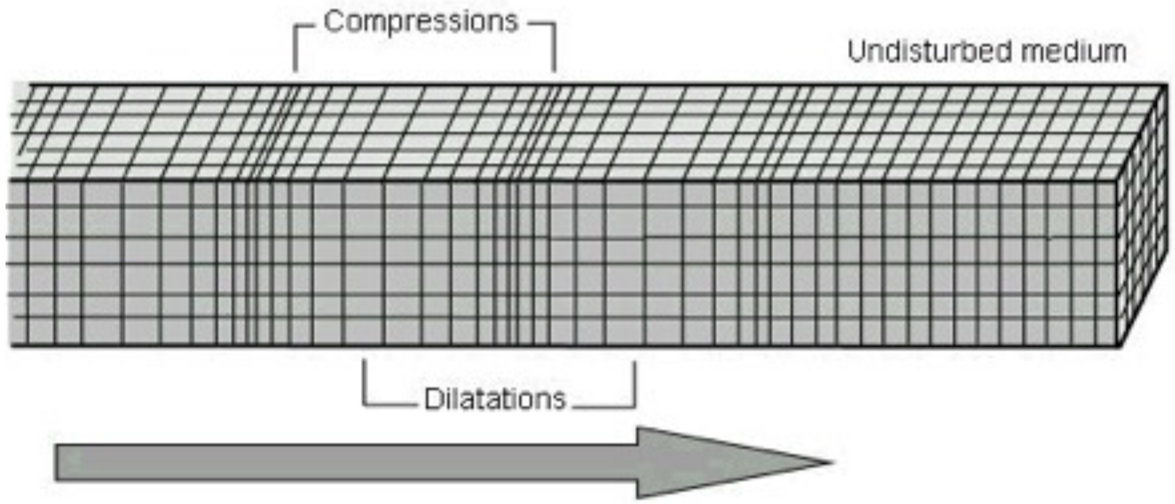
Cambridge:
at the University Press
1911



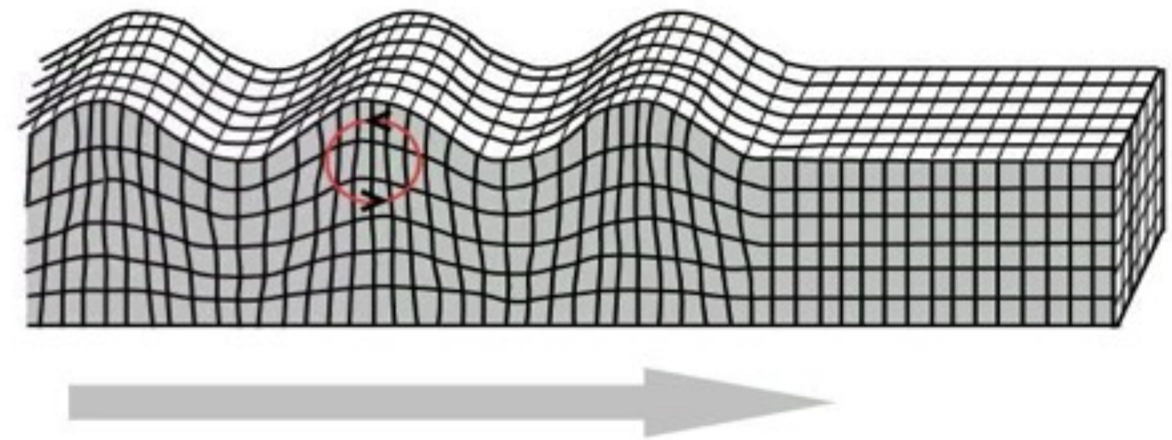

Repeated reflection in a layer over a half space.
Interference between incident, reflected and transmitted SH waves.
When the layer velocity is smaller than the halfspace velocity, then there is a critical angle beyond which SH reverberations will be totally trapped.

# Wavefields visualization

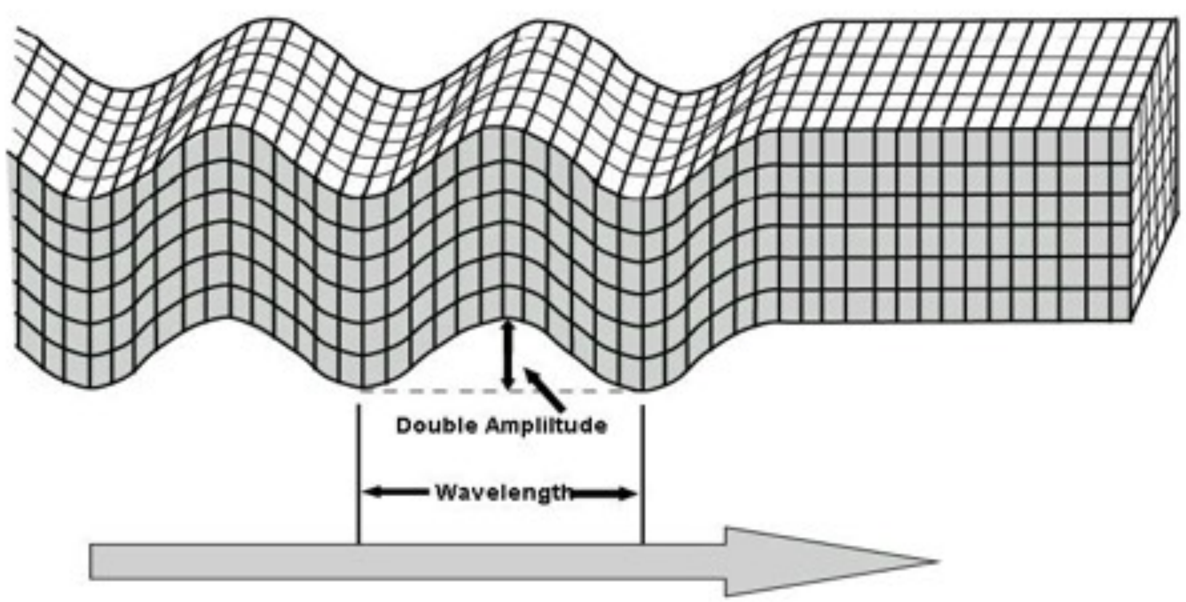
P Wave



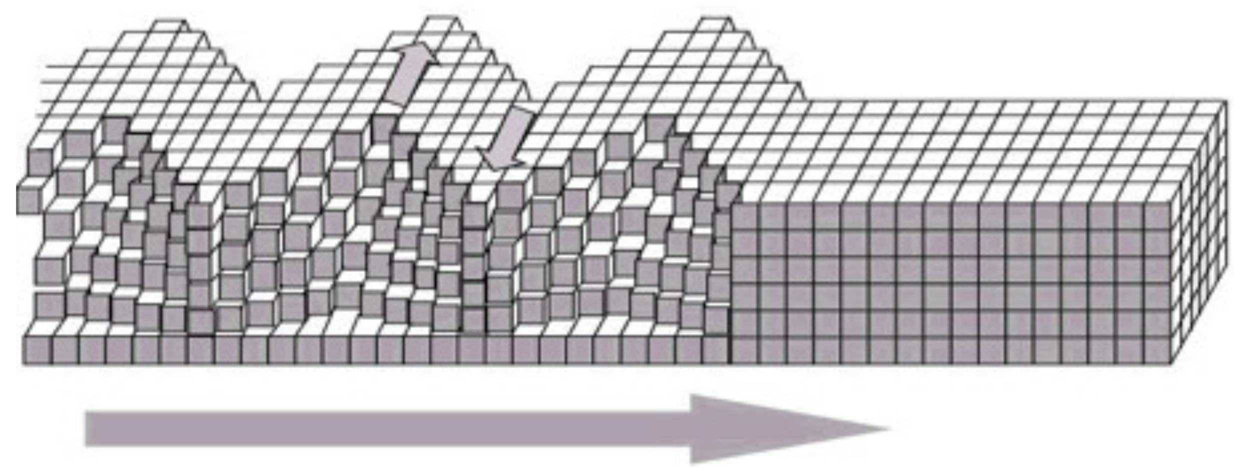

Rayleigh Wave



S Wave




Love Wave



# Surface Wave Tomography

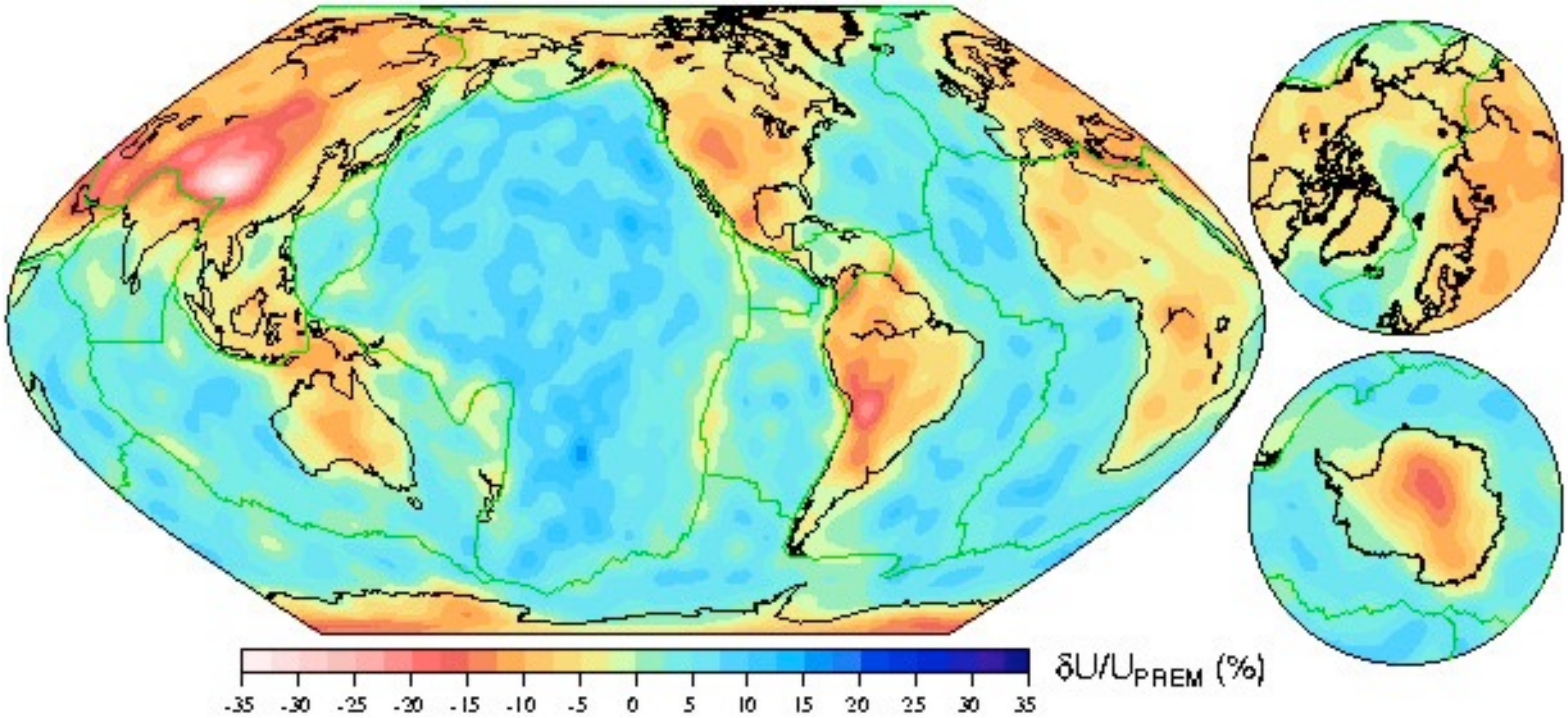
We describe a method to invert surface wave group or phase velocity measurements to estimate 2-D models of the distribution and strength of velocity variations.

Using ray theory, the forward problem for surface wave tomography consists of predicting a frequency dependent travel time $t_{R/L}(\omega)$. For both Rayleigh (R) and Love (L) waves from a set of 2-D phase or group velocity maps, $c(\mathbf{r}, \omega)$:

$$t_{R/L}(\omega) = \int_{ray} c_{R/L}^{-1}(r,\omega)\,ds$$

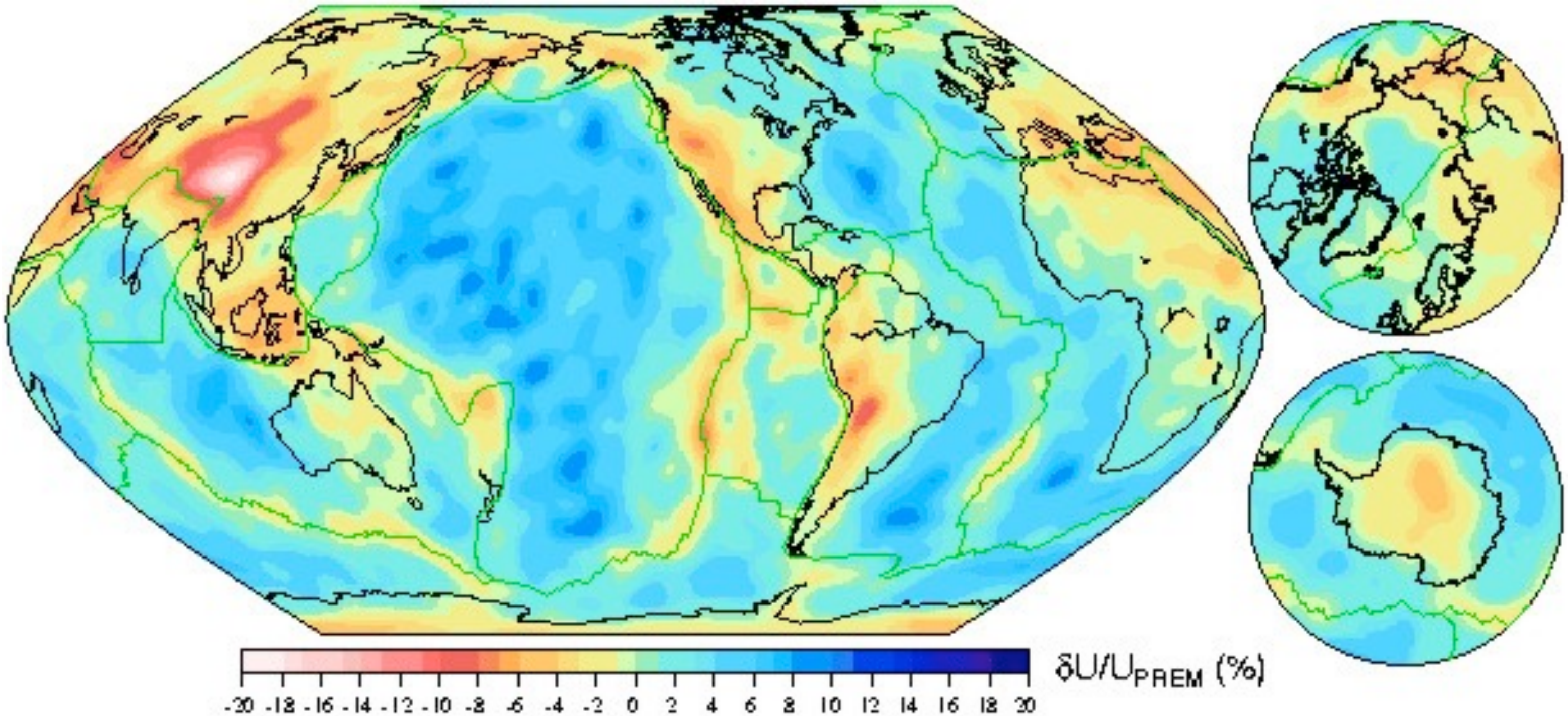
Where $\mathbf{r}=[\theta,\varphi]$ is the surface position vector, $\theta$ and $\varphi$ are colatitude and longitude, and ray specifies the path.

# Global scale



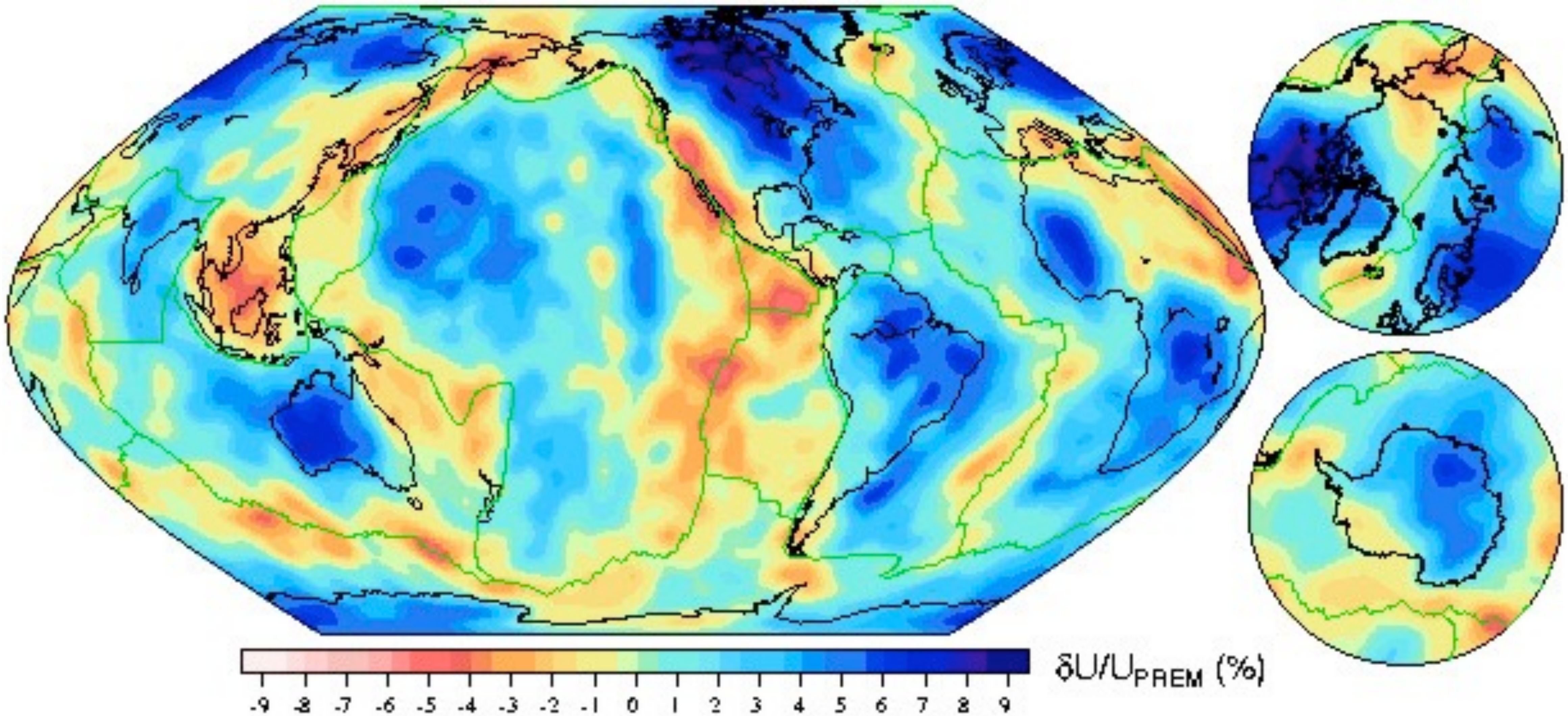

**RAYLEIGH WAVE 35s**

Larson, E.W.F. and G. Ekström, Global Models of Surface Wave Group Velocity, Pure Appl. Geophys. **158** (8), 1377-1400, 2001.




**RAYLEIGH WAVE 50s**

Larson, E.W.F. and G. Ekström, Global Models of Surface Wave Group Velocity, Pure Appl. Geophys. **158** (8), 1377-1400, 2001.




**RAYLEIGH WAVE 100s**

Larson, E.W.F. and G. Ekström, Global Models of Surface Wave Group Velocity, Pure Appl. Geophys. **158** (8), 1377-1400, 2001.

The reliability of the group velocity maps across large regions degrades sharply below 15 s and above 150-200 s for Rayleigh waves and 100-125 s for Love waves. Surface waves maps at and below 30 s period are particularly important because they provide significant constraints on crustal thickness by helping to resolve Moho depth from the average shear velocity of the crust. Although there have been numerous studies of surface wave dispersion that have produced measurements of group and/or phase velocities between 10 and 40 s period, these studies have typically been confined to areas of about 15° or less in lateral extent.

Phase and group velocity maps provide constraints on the shear velocity structure of the crust and uppermost mantle. Accurate high-resolution group velocity maps, in particular, are useful in monitoring clandestine nuclear tests.

Measurements of group velocities are much less sensitive to source effects than phase velocities because they derive from measurements of the wave packet envelopes rather than the constituent phases. This is particularly true at shorter periods and longer ranges. Group velocity sensitivity is compressed nearer to the surface than the related phase velocities, which should provide further help in resolving crustal from mantle structures.

# Surface waves

- Condition of existence:
  - Discontinuity (boundary waves, undispersed: Rayleigh, Stoneley)
  - Waveguide (interferential & dispersed: Love & Rayleigh)

| T (s) | f (Hz) | λ (km) | c (km/s) | d (km) | application |
|---|---|---|---|---|---|
| 0.02-0.1 | 10-50 | 0.002-0.05 | 0.1-0.5 | 0.02 | engineering, geophysics |
| 0.2-1 | 1-5 | 0.15-1.50 | 0.1-1.5 | 0.2 | upper sediments |
| 5-10 | 0.1-0.2 | 7-30 | 2-3 | 5 | sedimentary basins |
| 10-35 | 0.03-0.1 | 30-100 | 3.0-3.5 | 40 | crust |
| 35-350 | 0.005-0.03 | 200-1000 | 4-5 | 300 | upper mantle |