Corso di Laurea in Fisica - UNITS

ISTITUZIONI DI FISICA

PER IL SISTEMA TERRA

# FREE MODES OF THE EARTH

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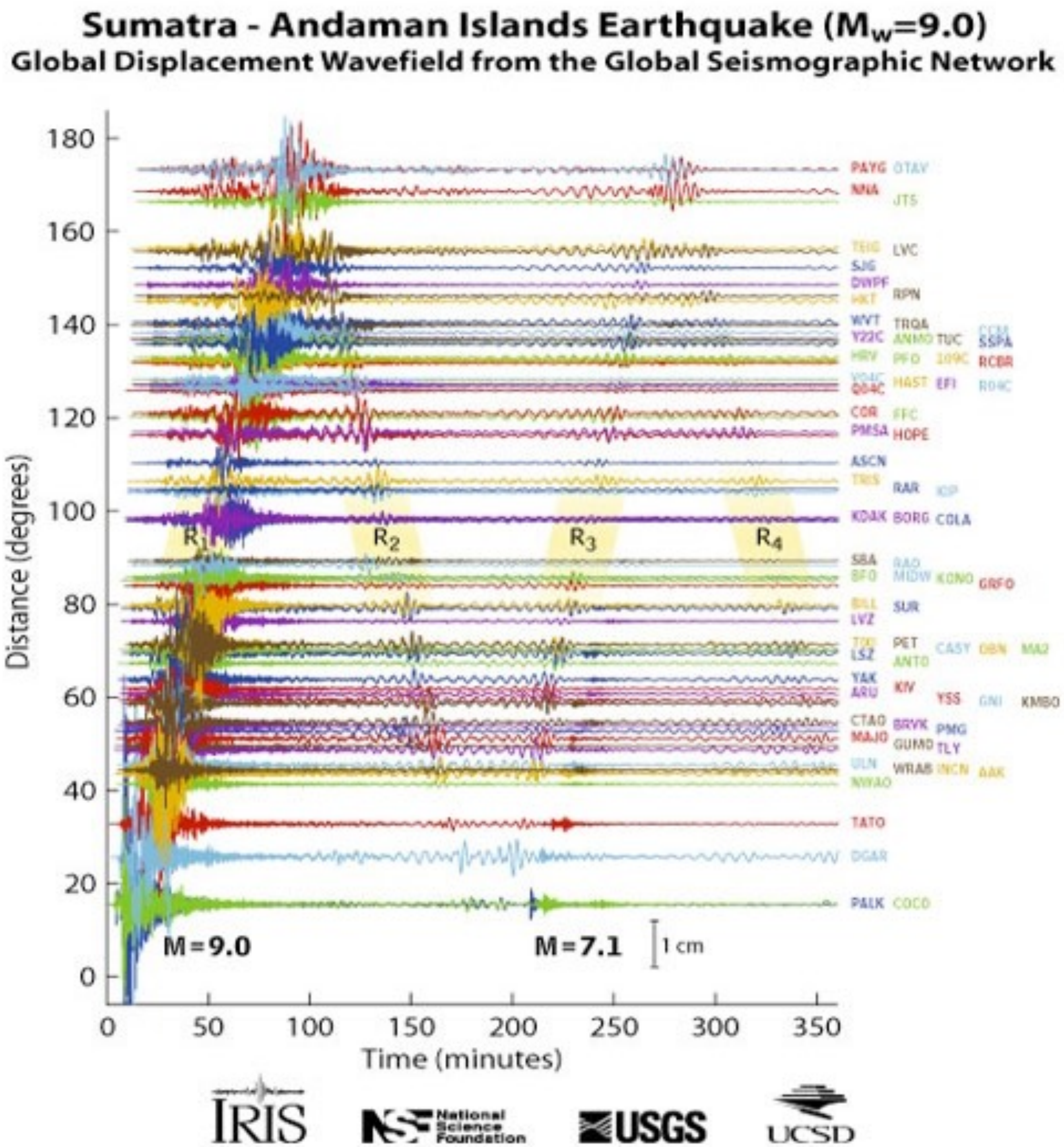
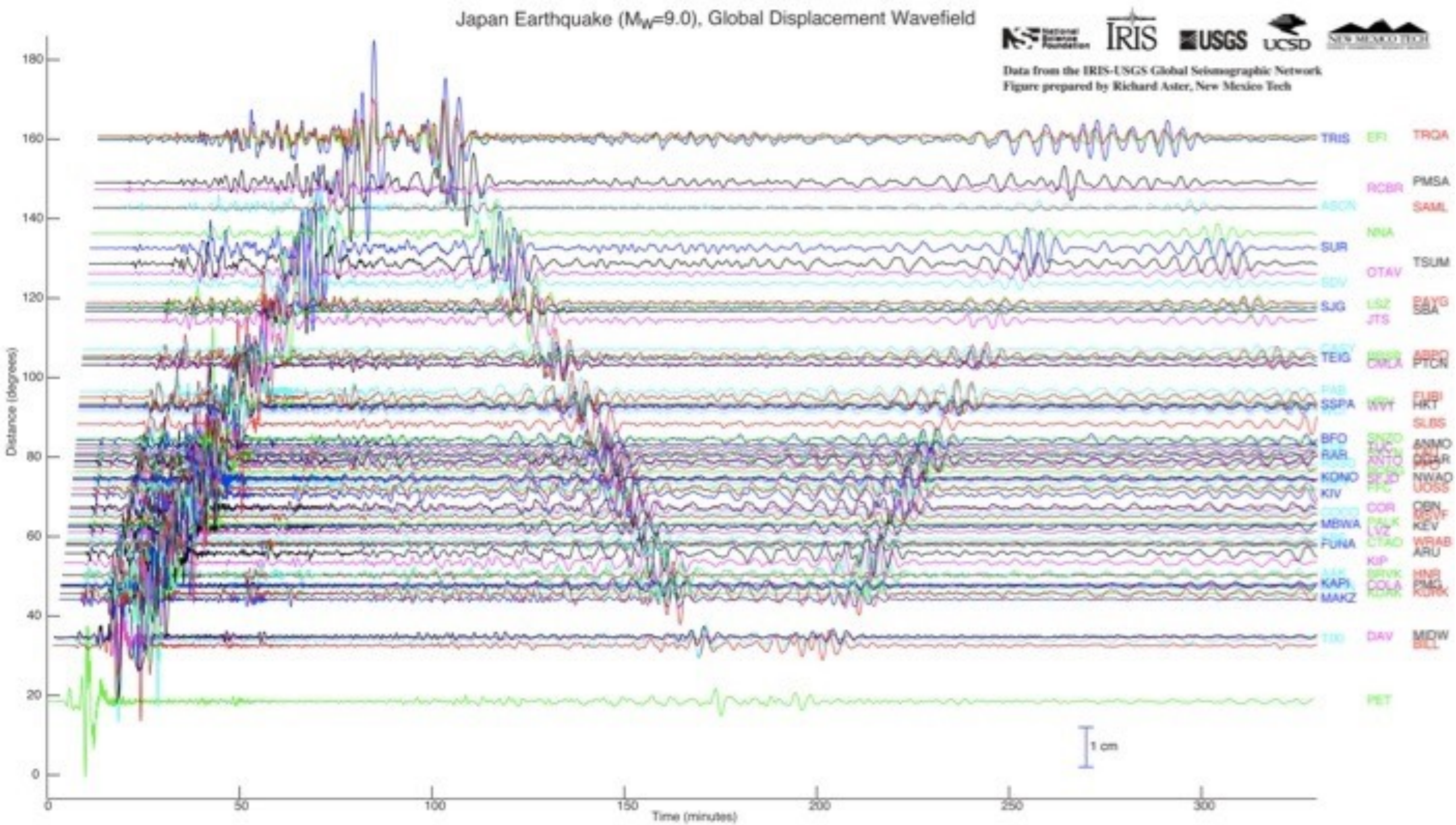
# Traveling surface waves




Vertical displacements of the Earth's surface recorded by seismometers.
The traces are arranged by distance from the epicenter in degrees. The earliest, lower amplitude, signal is that of the compressional (P) wave, which takes about 22 minutes to reach the other side of the planet (the antipode).
The largest amplitude signals are seismic surface waves that reach the antipode after about 100 minutes. The surface waves can be clearly seen to reinforce near the antipode (with the closest seismic stations in Ecuador), and to subsequently circle the planet to return to the epicentral region after about 200 minutes.
A major aftershock (magnitude 7.1) can be seen at the closest stations starting just after the 200 minute mark (note the relative size of this aftershock, which would be considered a major earthquake under ordinary circumstances, compared to the mainshock).

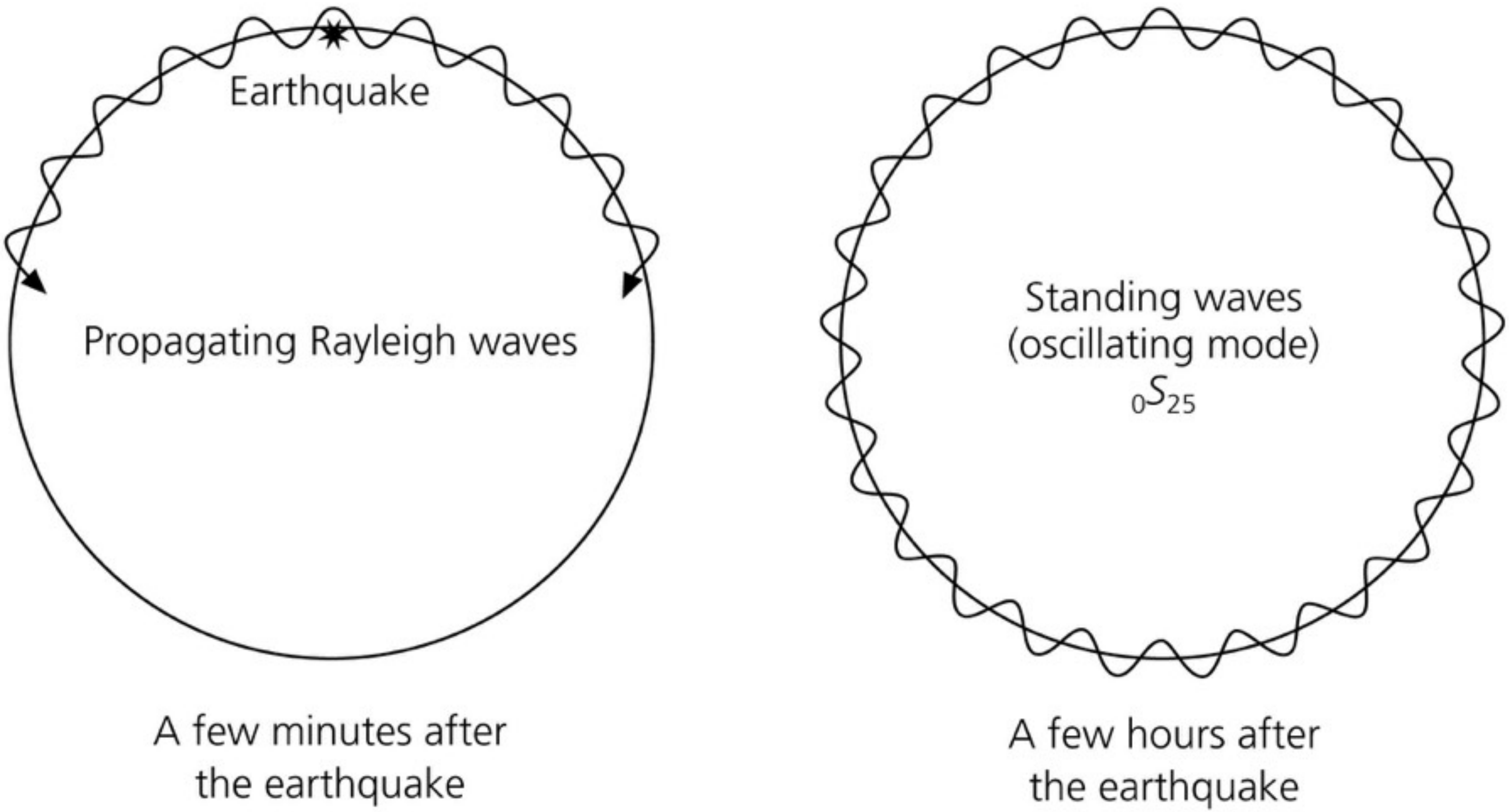
# Traveling surface waves

# Traveling surface waves



# Surface waves and free modes

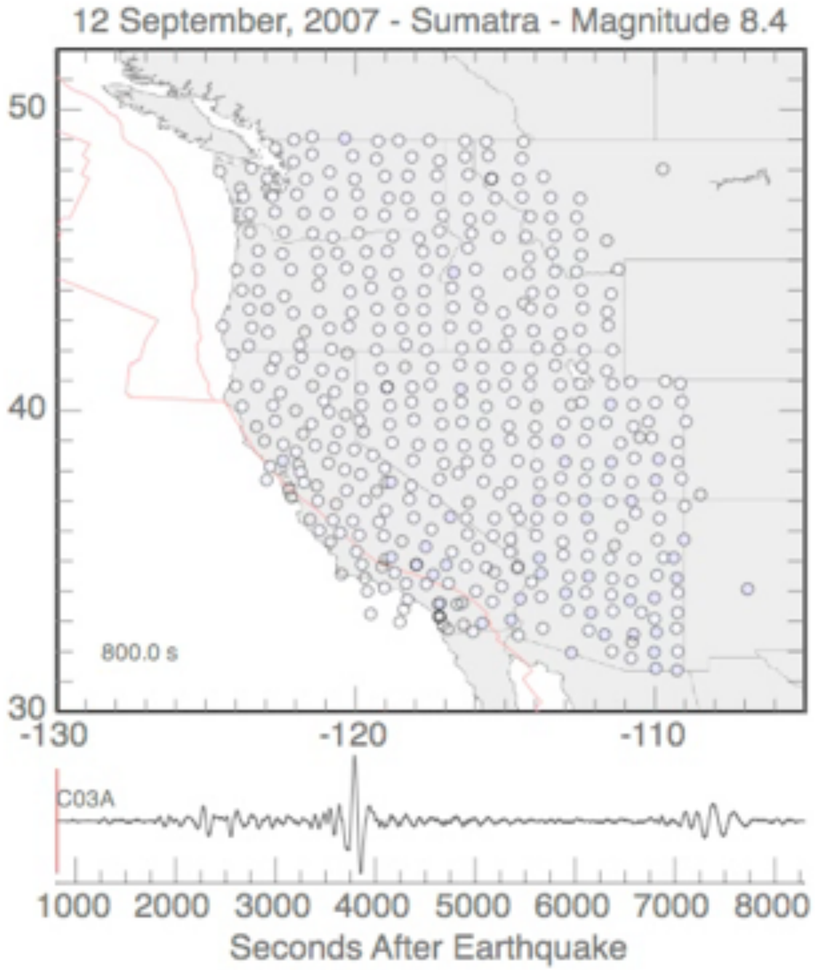
Figure 2.9-8: Cartoon of the equivalence of surface waves and normal modes.




https://jbrussell.github.io/other/normal_modes/

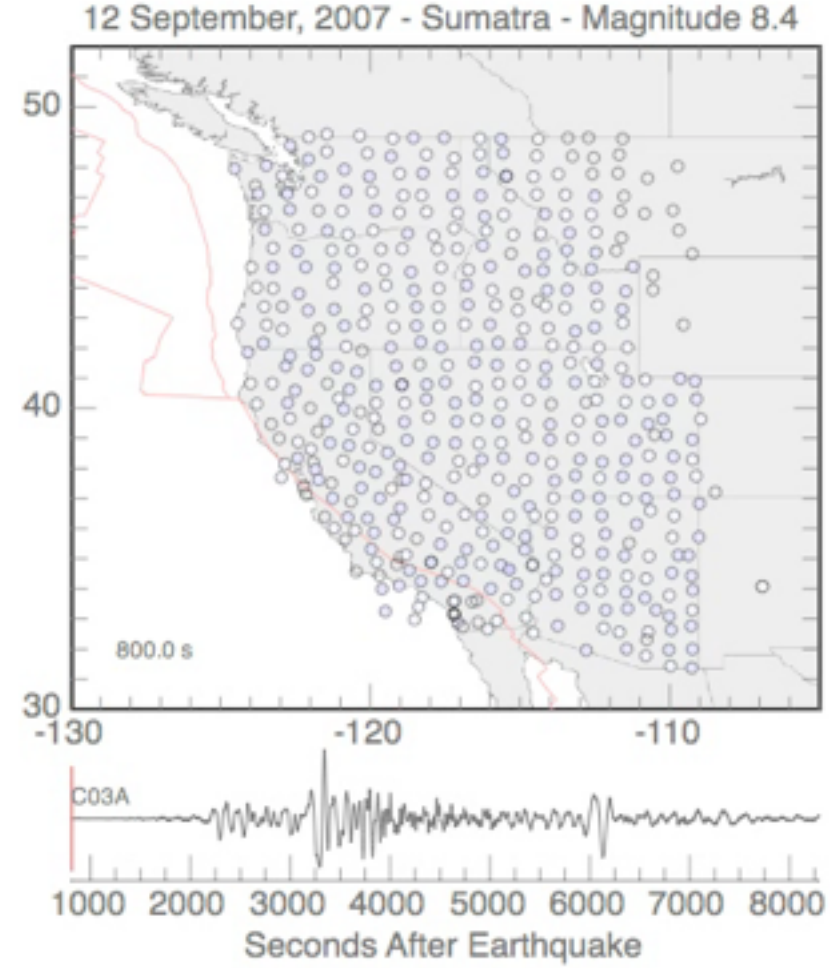
# Traveling and standing waves

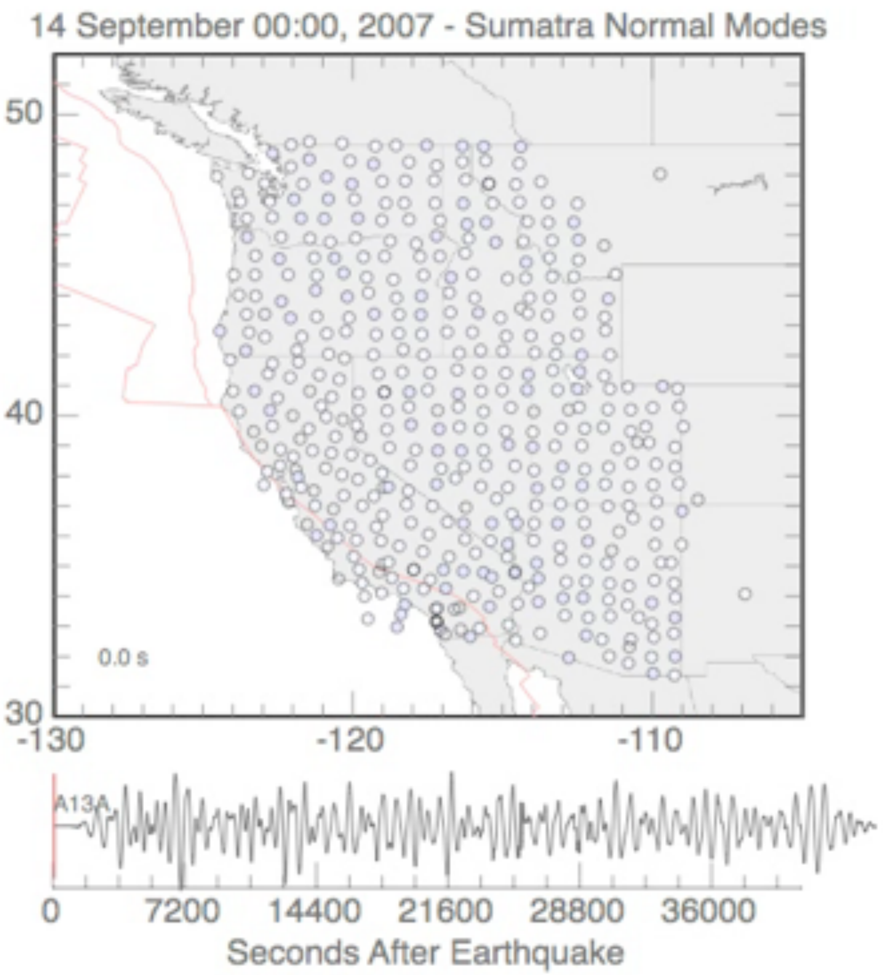
Periods from 200s-50s-R1-to-R2




Vertical ground velocity in the 750-50s period range observed across the western United States by the EarthScope transportable array and generated by the great 12 September, 2007 earthquake offshore southern Sumatra. Each circle represents a seismometer and the colors change to reflect variations in the signal amplitude crossing the array. Near the end of this animation you can see the waves that traveled the long way around Earth to reach the western United States (they propagate from NW to SE). Station 319A is located at the Douglas, AZ.

Tcomp: Periods from 200s-50s-G1-to-G2




If you watch closely, you'll see the waves from the 06:01:34 Magnitude 6.4 aftershock sweep through. This movie starts several just over one day after the Mw 8.4 earthquake (the horizontal axis label is incorrect). The time step for this longer animation is 20 seconds per frame. Each second of this animation represents almost 7 minutes. I didn't screen the data so some seismic stations with glitches more or less have large amplitudes throughout the animation. The amplitude scale for this animation is about 1000 times smaller than the main-shock animations.

http://eqseis.geosc.psu.edu/~cammon/QA/sumatra12Sep2007.html

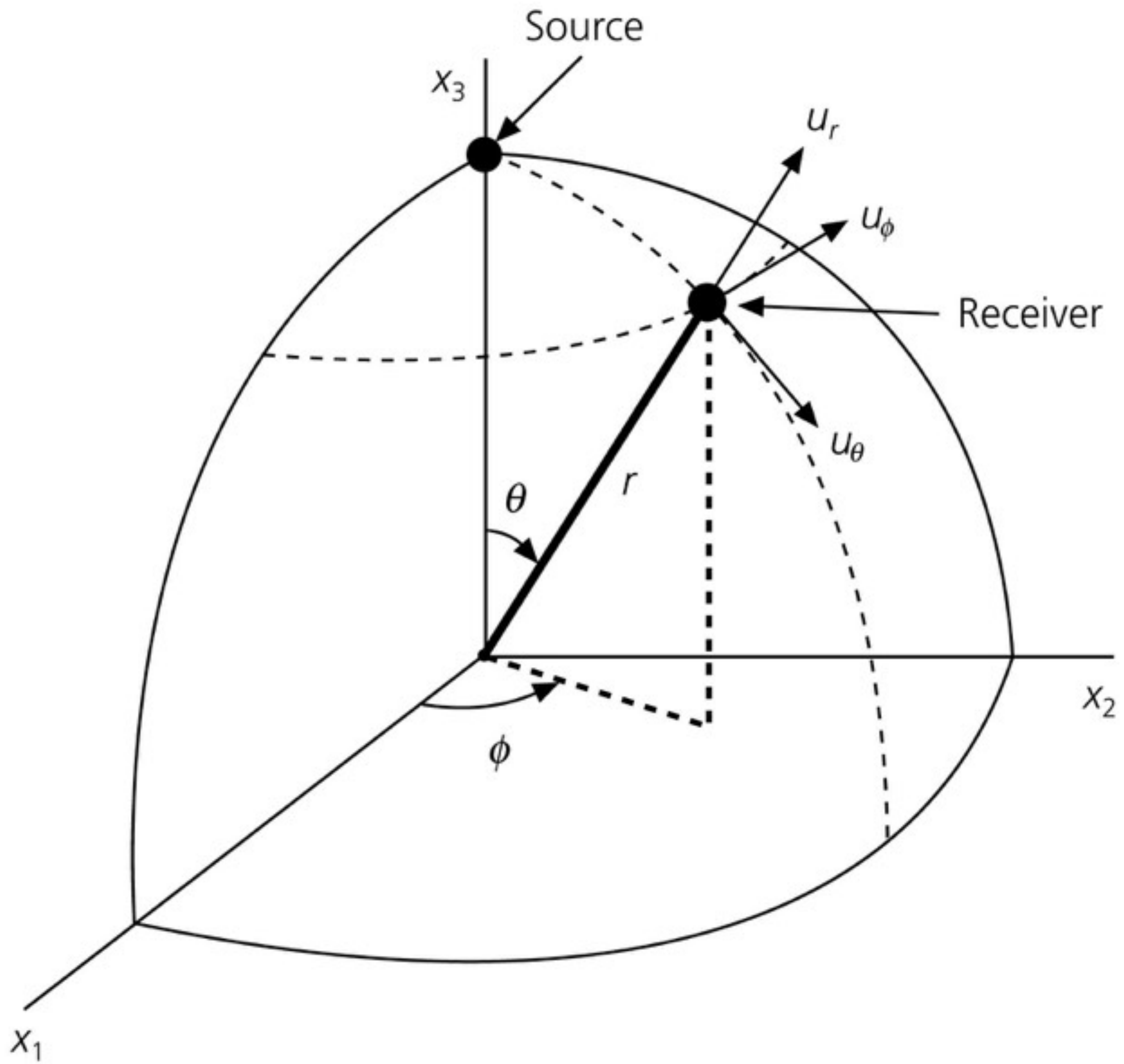
# Seismic waves

| Seismic domain | Wavefield type | | Data | Application | Boundary conditions |
|---|---|---|---|---|---|
| Body waves | P-SV | SH | Travel times; Waveforms. (>1 Hz) | Local, regional tomography | Unbounded. Free surface. |
| Surface waves | Rayleigh | Love | Dispersion; Waveforms. (0.05-1 Hz) | Local, regional tomography. Crustal, lithospheric | Free surface; interfaces. Flat geometry |
| Normal modes | Spheroidal | Torsional | Power spectra (mHz) | Global seismology | Free surface; interfaces; spherical geometry |

https://jbrussell.github.io/other/normal_modes/

# Spherical geometry

**Figure 2.9-1: Spherical coordinate geometry for normal modes.**



# Wave equation & Laplacian

- Wave equation

$$v^2\nabla^2 u = v^2\Delta u = u_{tt}$$

- Laplacian in Spherical system

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

# Separation of variables

$$u(r,\theta,\phi,t) = R(r)\Theta(\theta)\Phi(\phi)T(t)$$

$$\Phi''(\phi) + m^2\Phi(\phi) = 0$$

$$\Phi(\phi) = C\cos(m\phi) + D\sin(m\phi)$$

m is a positive integer

$$T''(t) + c^2k^2T(t) = 0$$

$$T(t) = A\cos(\omega t) + B\sin(\omega t)$$

ω=ck

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0$$

$$\frac{1}{r^2}\frac{\partial}{dr}\left(r^2\frac{dR}{dr}\right) + \left[k^2 - \frac{l(l+1)}{r^2}\right]R = 0$$

In mathematics, the **spherical harmonics** are the angular portion of an orthogonal set of solutions to Laplace's equation represented in a system of spherical coordinates.

Spherical harmonics are orthogonal:

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta\, Y_{l'}^{m'*}(\theta,\phi)\, Y_l^m(\theta,\phi)\, d\theta d\phi = \delta_{l'l}\delta_{m'm}$$

The spherical harmonics are easily visualized by counting the number of zero crossings they possess in both the latitudinal and longitudinal directions. For the latitudinal direction, the associated Legendre functions possess l − |m| zeros, whereas for the longitudinal direction, the trigonomentric sin and cos functions possess 2 |m| zeros.

When the spherical harmonic order m is zero, the spherical harmonic functions do not depend upon longitude, and are referred to as **zonal**. When l = |m| , there are no zero crossings in latitude, and the functions are referred to as **sectoral**. For the other cases, the functions checker the sphere, and they are referred to as **tesseral**.

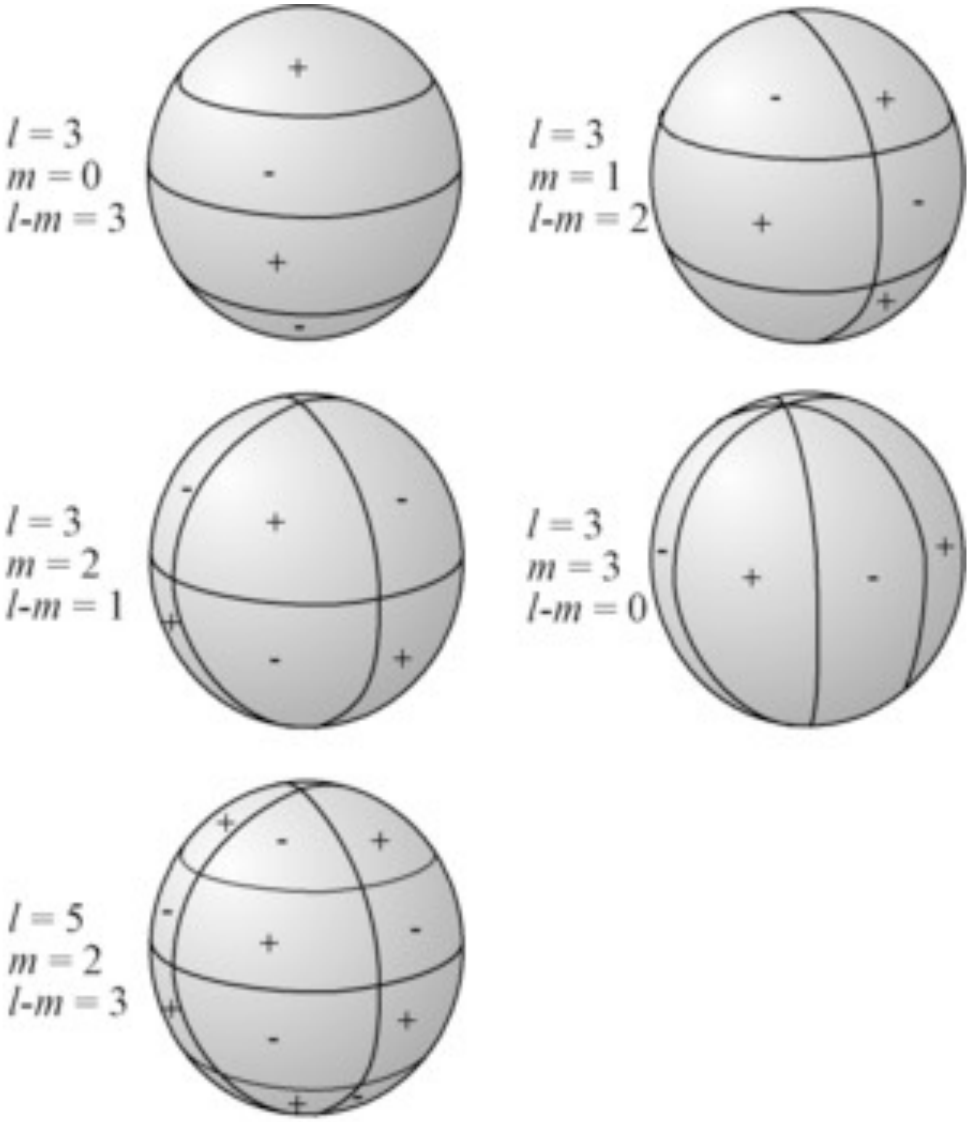
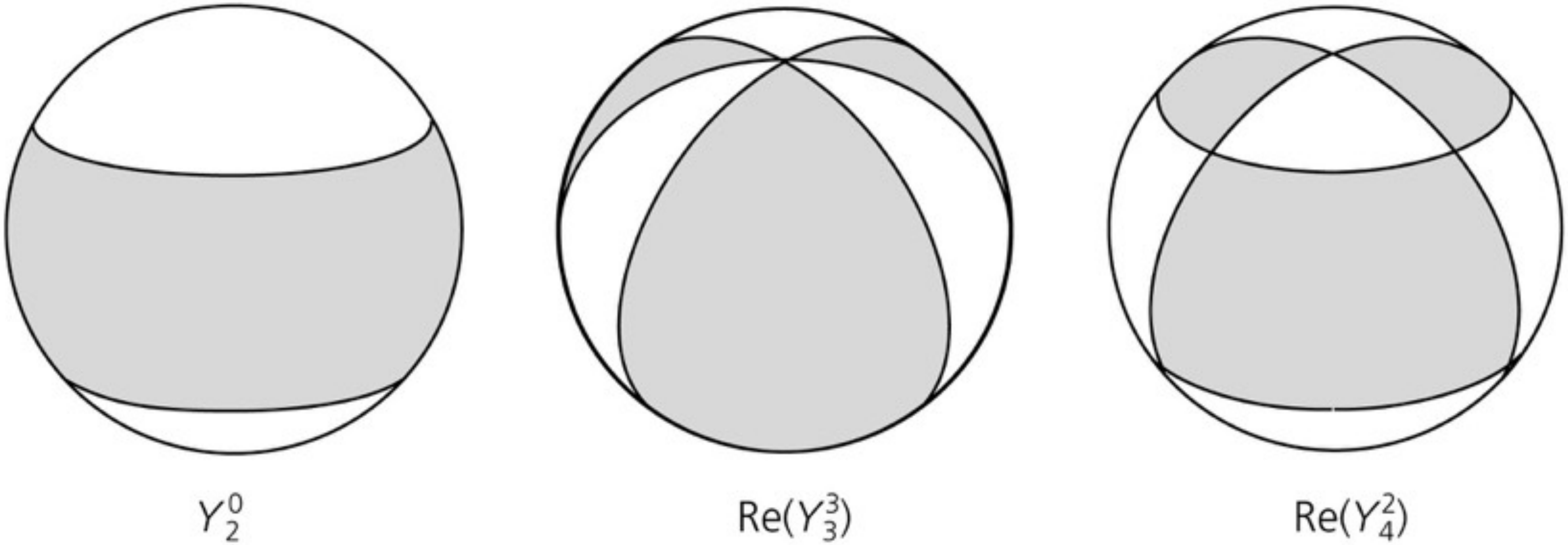


**Figure 2.9-4: Examples of spherical harmonics.**




$$Y_l^m(\theta, \phi) = (-1)^m \left[ \left( \frac{2l+1}{4\pi} \right) \frac{(l-m)!}{(l+m)!} \right]^{1/2} P_l^m(cos\theta) e^{im\phi}$$

The angular order, $l$, gives the number of nodal lines on the surface.

If the azimuthal order $m$ is zero, the nodal lines are small circles about the pole. These are called *zonal* harmonics, and do not depend on $\phi$

If $m = l$, then all of the surface nodal lines are great circles through the pole. These are called *sectoral* harmonics.

When $0 < |m| < l$, there are combined angular and azimuthal (colatitudinal and longitudinal) nodal patterns called *tesseral* harmonics.

# Torsional & Spheroidal modes

## Torsional modes ${}_nT^m_l$ :

- No radial component: tangential only, normal to the radius: motion confined to the surface of n concentric spheres inside the Earth (**SH, Love waves**).
- Changes in the shape, not of volume
- Do not exist in a fluid: so only in the mantle (and the inner core?)

n - radial : nodal planes with depth

l - polar : # nodal planes in latitude
! Max nodal planes = l - 1

m - azimuthal : # nodal planes in longitude

## Spheroidal modes ${}_nS^m_l$ :

- Horizontal components (tangential) et vertical (radial) (**P-SV, Rayleigh waves**)
- No simple relationship between n and nodal spheres
- ${}_0S_2$ is the longest "fundamental"
- Affect the whole Earth (even into the fluid outer core !)

n : no direct relationship with nodes with depth

l : # nodal planes in latitude
! Max nodal planes = l

m : # nodal planes in longitude

# Torsional modes

Torsional (toroidal) modes:
(analogous to *SH* waves)

Surface eignefunctions given by vector spherical harmonics:

$$\mathbf{T}_l^m(r,\theta,\phi) = \left(0, \frac{1}{\sin\theta}\frac{\partial Y_l^m(\theta,\phi)}{\partial\phi}, -\frac{\partial Y_l^m(\theta,\phi)}{\partial\theta}\right)$$

The displacements are given by:

$$\mathbf{u}^T(r,\theta,\phi) = \sum_n \sum_l \sum_{m=-l}^{l} {}_nA_l^m \; {}_nW_l(r) \;\; \mathbf{T}_l^m(\theta,\phi) \; e^{i\,{}_n\omega_l^m t}$$

${}_nW_l(r)$ - The radial eigenfunction (varies with depth)

# Torsional modes

For ${}_nT_l^m$:

$n$ = radial order, $l$ = angular order, $m$ = azimuthal order.

The $2l + 1$ modes of different azimuthal orders $-l \le m \le l$ are called *singlets,* and the group of singlets is called a *multiplet.*

If earth were perfectly spherically symmetric and non-rotating, all singlets in a multiplet would have the same eigenfrequency (called *degeneracy*).

For example, the period of ${}_nT_l^0$ would be the same for ${}_nT_l^{\pm 1}$, ${}_nT_l^{\pm 2}$, ${}_nT_l^{\pm 3}$, etc. In the real earth, singlet frequencies vary (called *splitting*).

The splitting is usually small enough to ignore, so we drop the $m$ superscript and refer to the entire ${}_nT_l^m$ multiplet as ${}_nT_l$, with eigenfrequency ${}_n\omega_l$.

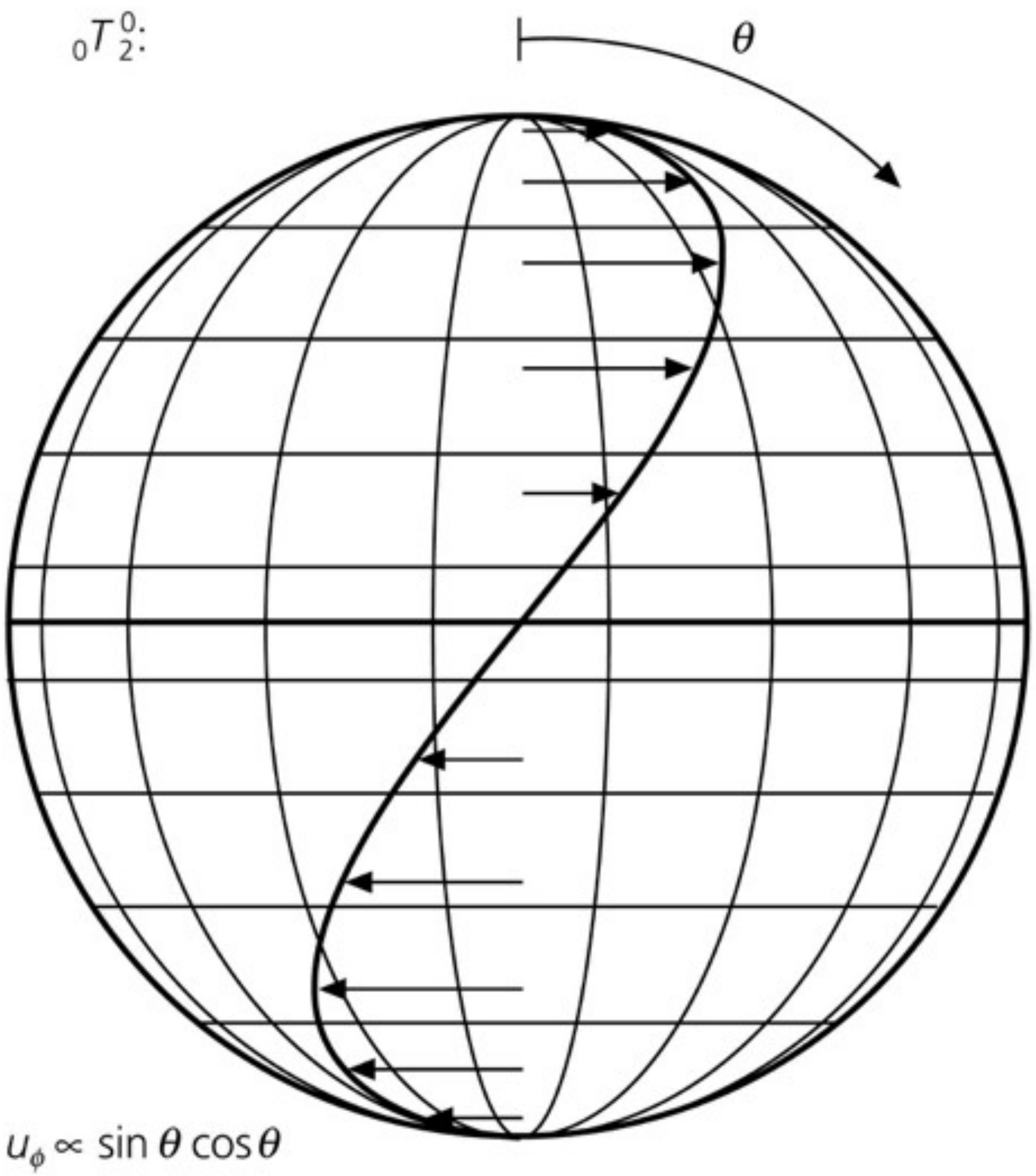
Example:

$$Y_l^m(\theta,\phi) = (-1)^m\left[\left(\frac{2l+1}{4\pi}\right)\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

$$\mathbf{T}_l^m = \left(0,\ \frac{1}{\sin\theta}\frac{\partial Y_l^m(\theta,\phi)}{\partial\phi},\ -\frac{\partial Y_l^m(\theta,\phi)}{\partial\theta}\right)$$

$$e^{im\phi}\frac{\partial}{\partial\theta}P_2^0(\cos\theta) = 3\sin\theta\cos\theta$$

Figure 2.9-5: Displacement associated with torsional mode ${}_0T_2$.



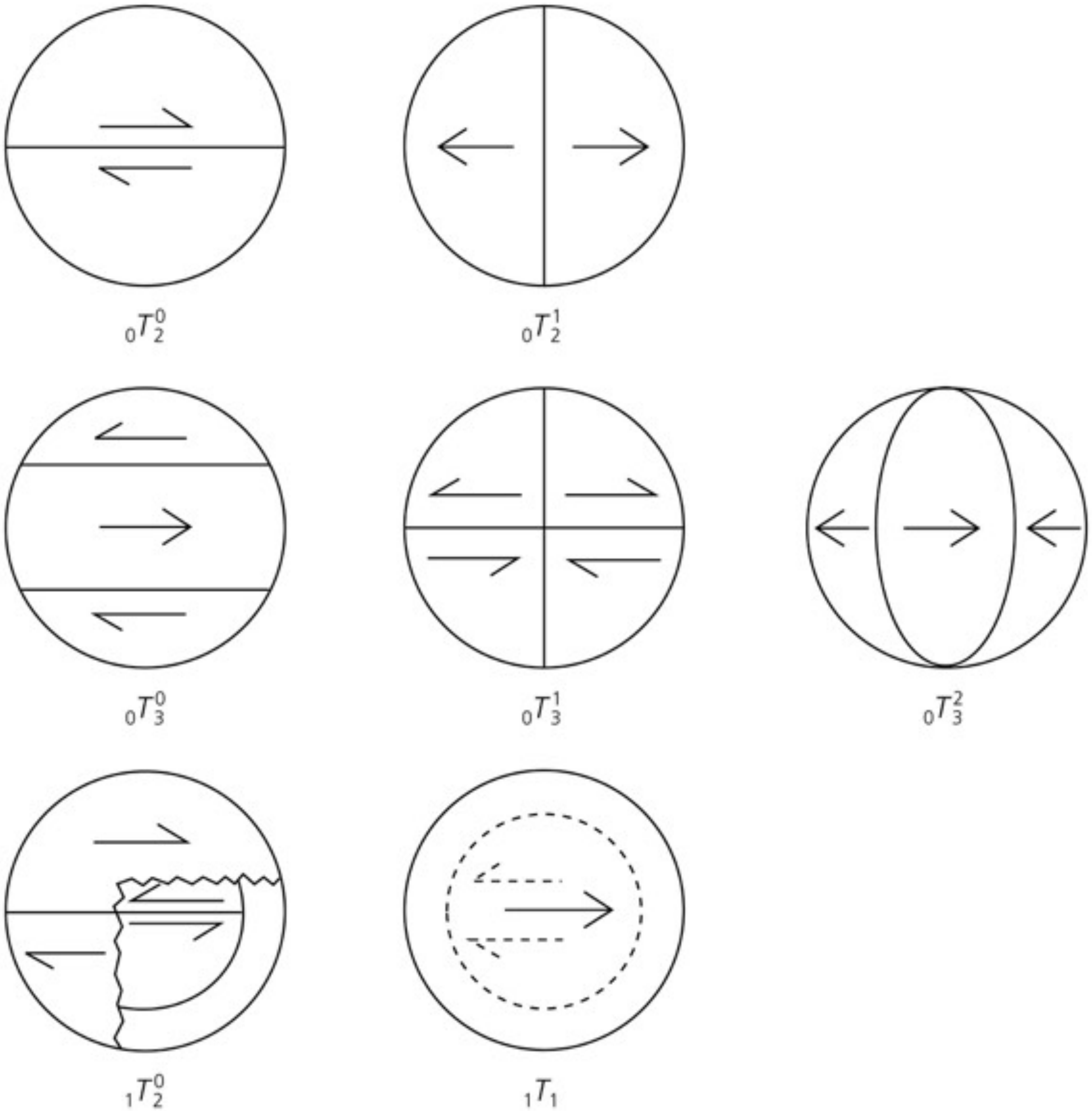
# Torsional modes

Torsional modes with $n = 0$ (${}_0T_l^m$) are called *fundamental modes*. (motions at depth in the same direction as at the surface).

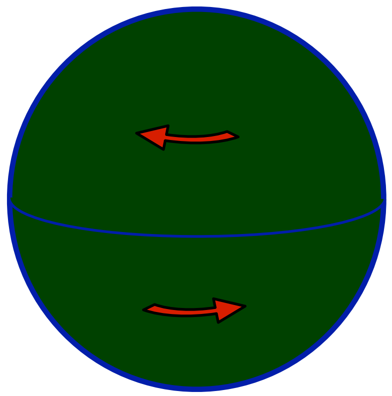
Modes with $n > 0$ are called *overtones*. (motions reverse directions at different depths)

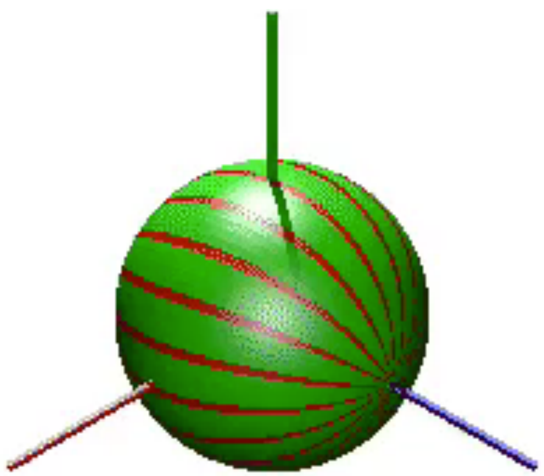
What happened to ${}_0T_1$ and ${}_0T_0$?

**Figure 2.9-6: Examples of the displacements for several torsional modes.**



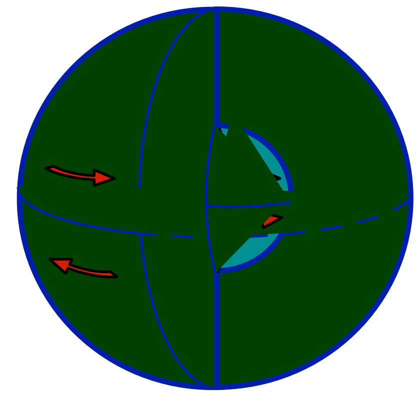
# Toroidal normal modes: examples





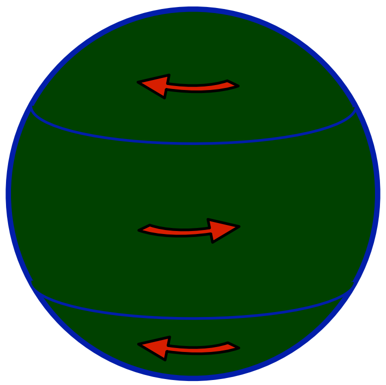
$_0T_2$ : «twisting» mode

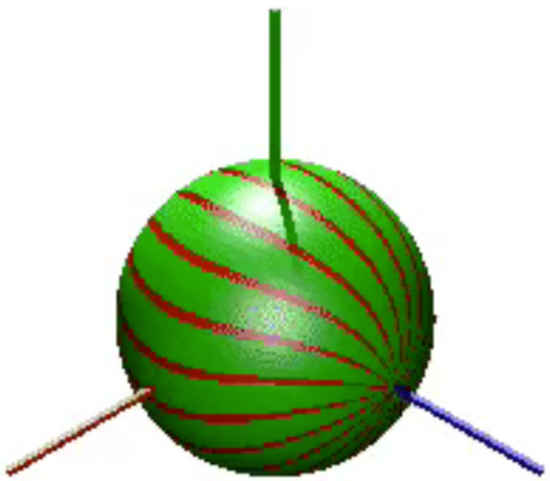
(44.2 minutes, observed in 1989 with an extensometer)



$_1T_2$

(12.6 minutes)





$_0T_3$

(28.4 minutes)

Animations from Lucien Saviot
https://saviot.cnrs.fr/terre/index.en.html

# Spheroidal modes

Spheroidal (poloidal) modes (involving *P-SV* motions):

The surface eigenfunctions are given by two other *vector spherical harmonics* with $(r, \theta, \phi)$ components

$$\mathbf{R}_l^m = (Y_l^m, 0, 0)$$

$$\mathbf{S}_l^m = \left(0, \frac{\partial Y_l^m(\theta, \phi)}{\partial \theta}, \frac{1}{\sin\theta}\frac{\partial Y_l^m(\theta, \phi)}{\partial \phi}\right)$$

Each corresponds to a different radial eigenfunction, ${}_nU_l(r)$ and ${}_nV_l(r)$, so the displacement for spheroidal modes is

$$\mathbf{u}^S(r, \theta, \phi) = \sum_n \sum_l \sum_{m=-l}^{l} {}_nA_l^m \left[ {}_nU_l(r)\, \mathbf{R}_l^m(\theta, \phi) + {}_nV_l(r)\, \mathbf{S}_l^m(\theta, \phi) \right] e^{i\,{}_n\omega_l^m t}$$

The radial eigenfunction ${}_nU_l(r)$ corresponds to radial motion and ${}_nV_l(r)$ corresponds to horizontal motion.

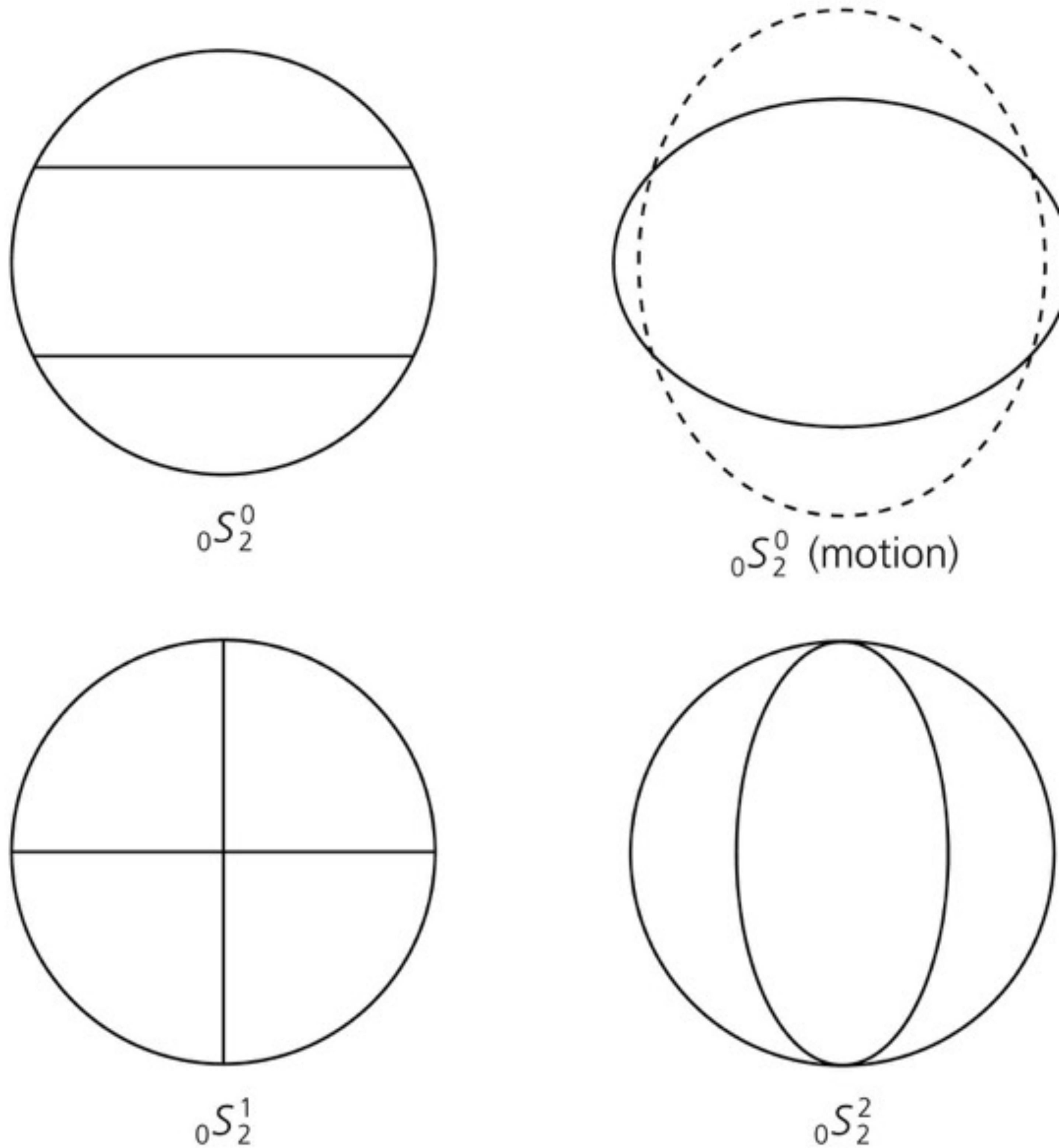
# Spheroidal modes

$_0S_2$ (football mode) is the gravest (lowest frequency or longest period) of earth's modes, with a period of 3233 s, or 54 minutes.

There is no $_0S_1$ mode, which would correspond to a lateral translation of the planet.

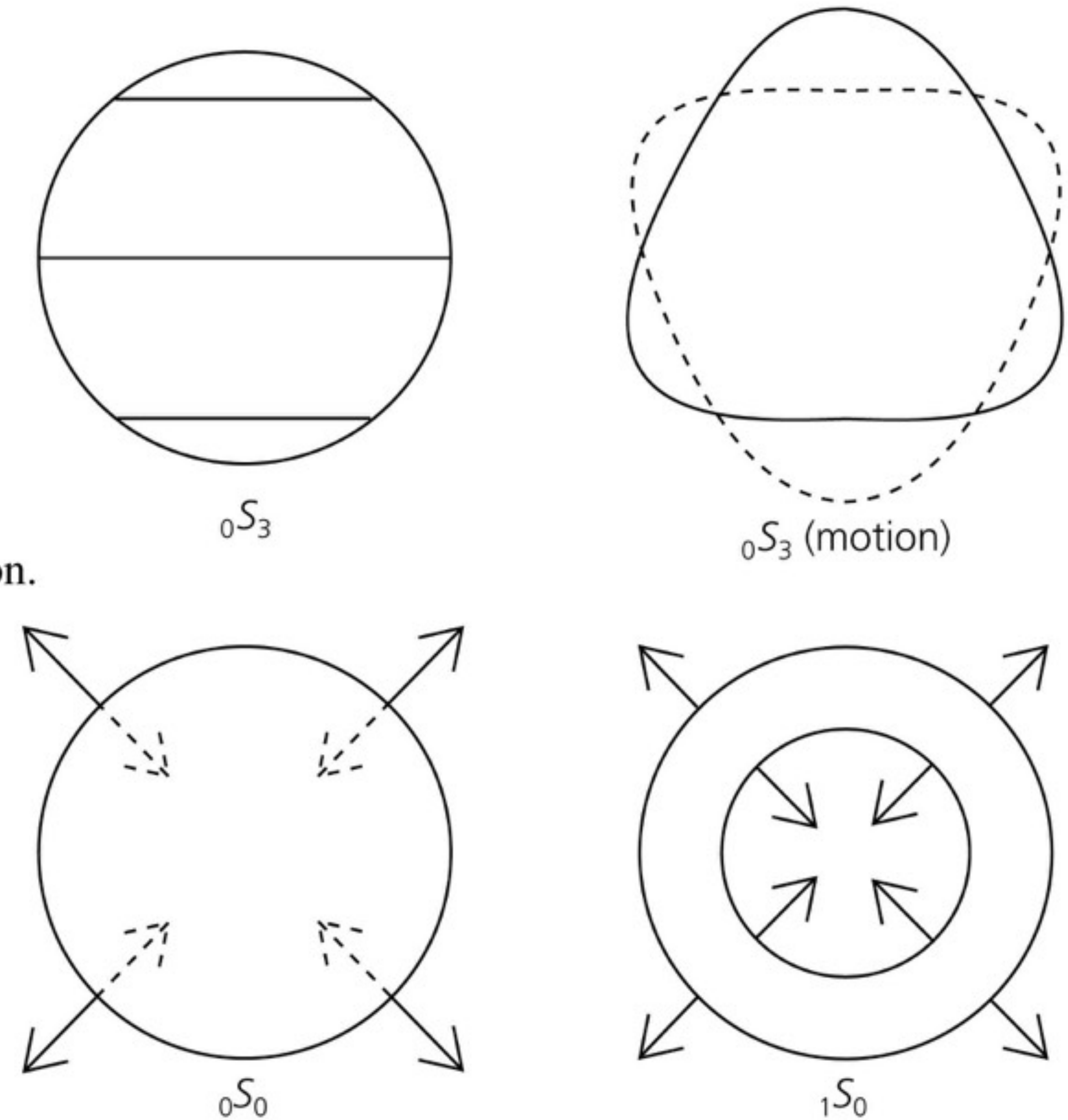
The $_1S_1$ Slichter mode due to lateral sloshing of the inner core through the liquid iron outer core, which has yet to be observed, should in theory have a period of about 5 1/2 hours.

**Figure 2.9-7: Examples of the displacements for several spheroidal modes.**

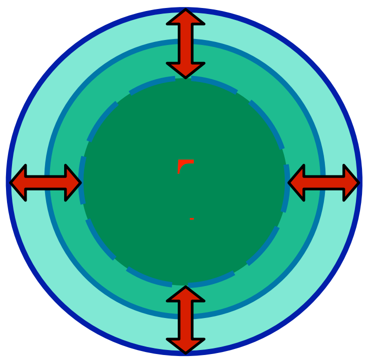


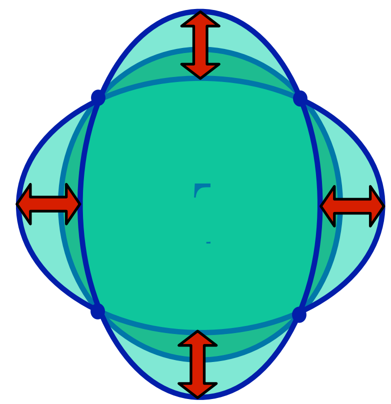
# Spheroidal modes

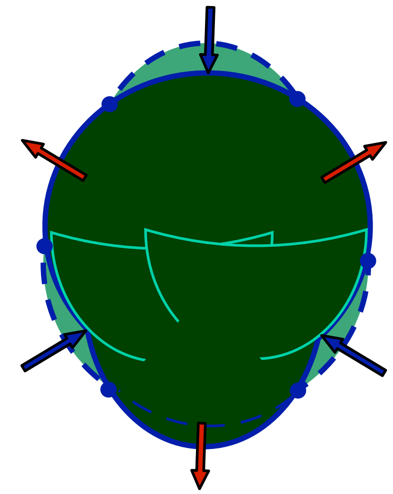
The "breathing" mode ${}_0S_0$ involves radial motions of the entire earth that alternate between expansion and contraction.



# Spheroidal normal modes: examples







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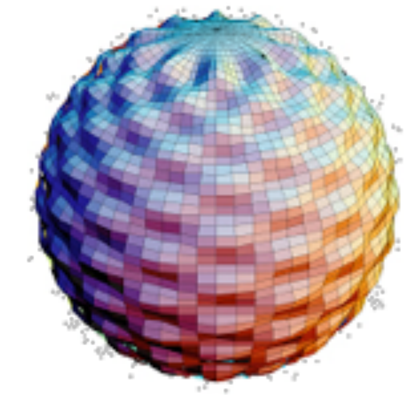

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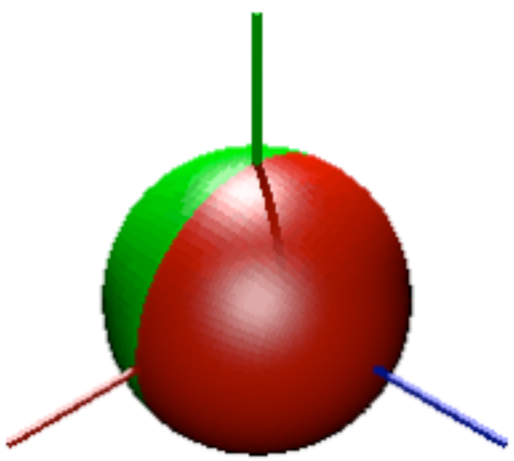
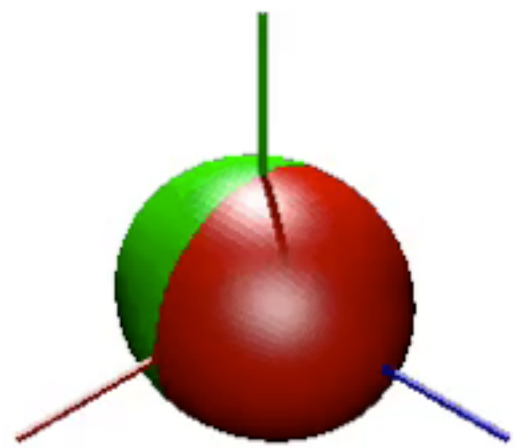
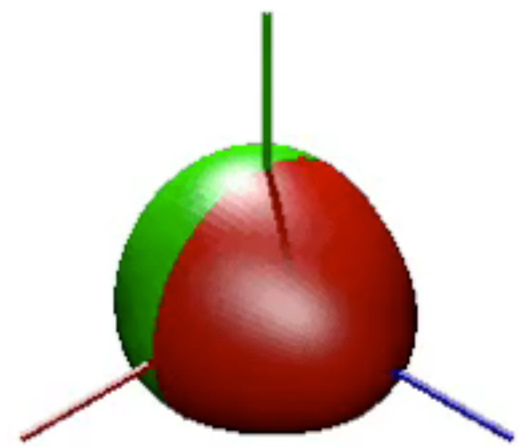






${}_0S_0$ : « balloon » or « breathing » : radial only (20.5 minutes)

${}_0S_2$ : « football » mode (Fundamental, 53.9 minutes)

${}_0S_3$ : (25.7 minutes)

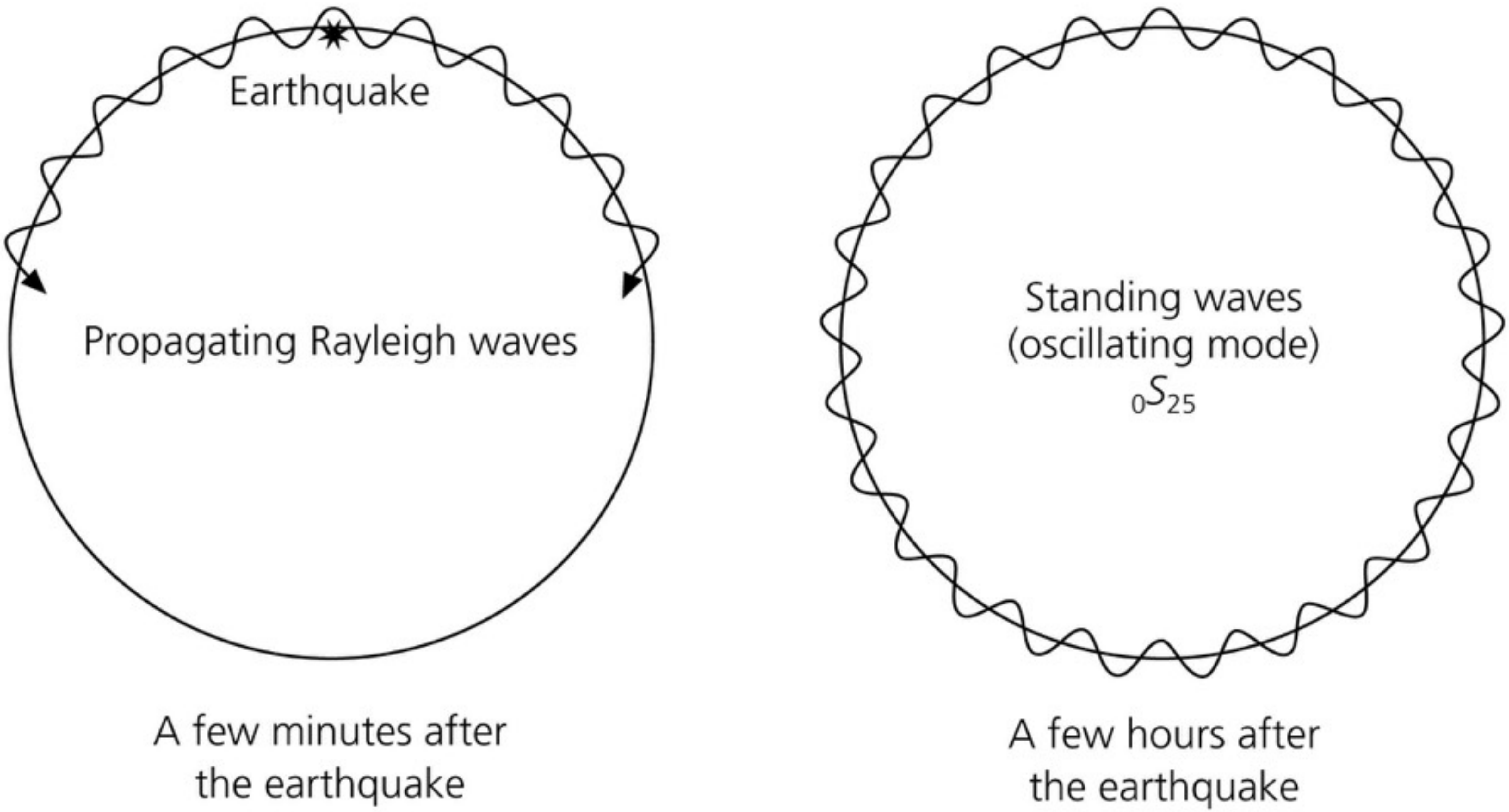
…

${}_0S_{29}$ : (4.5 minutes)

*Animations from Lucien Saviot*
https://saviot.cnrs.fr/terre/index.en.html

*See also Animations from Hein Haak*
http://www.knmi.nl/cms/content/64722/eigentrillingen_van_de_sumatra_aardbeving

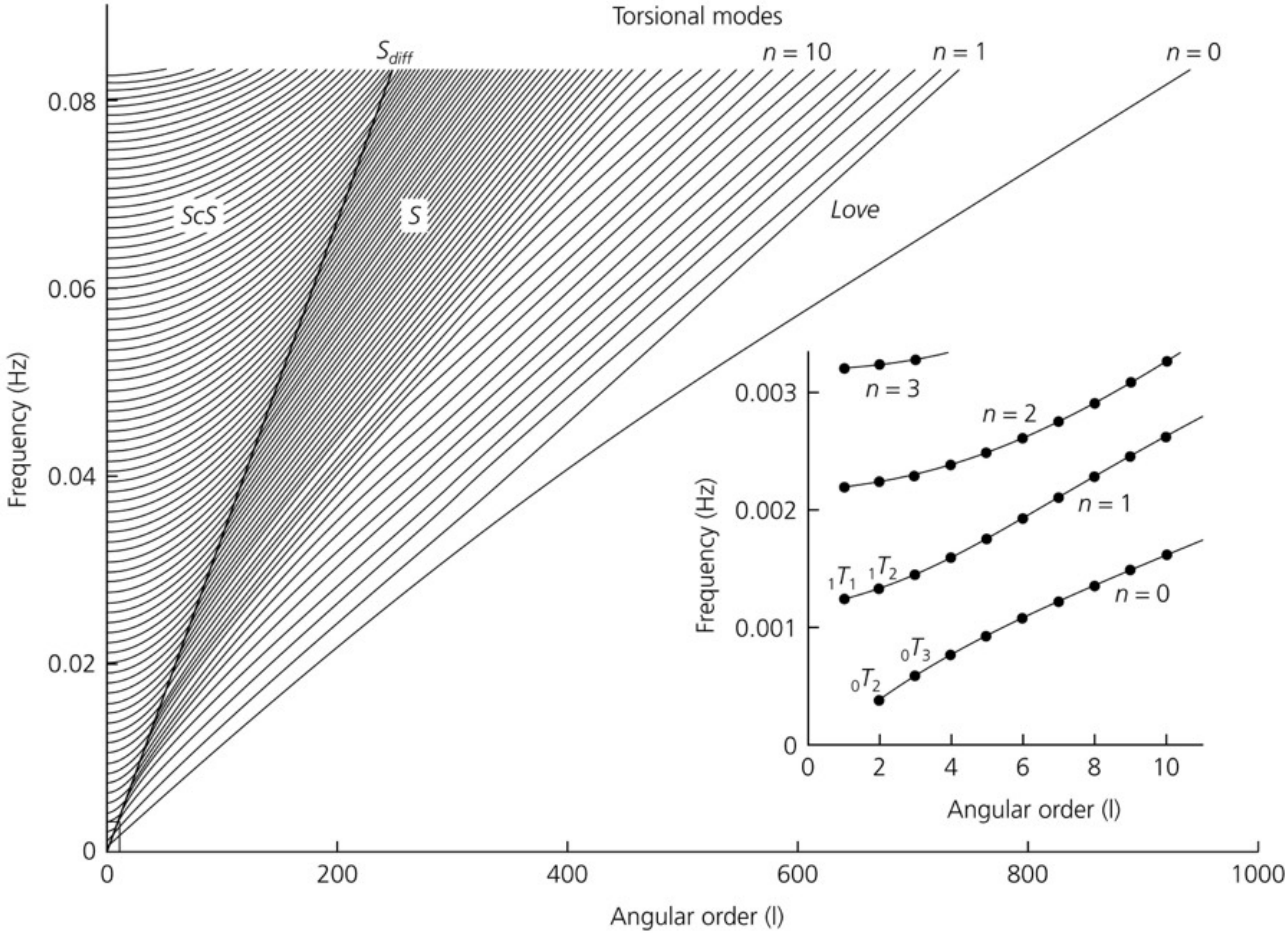
**Figure 2.9-8: Cartoon of the equivalence of surface waves and normal modes.**




The mode with angular order $l$ and frequency $_n\omega_l$ corresponds to a traveling wave with horizontal wavelength $\lambda_x = 2\pi/|\mathbf{k}_x| = 2\pi a/(l + 1/2)$ that has $l + 1/2$ wavelengths around the earth.

These waves travel at a horizontal phase velocity $c_x = {_n\omega_l}/|\mathbf{k}_x| = {_n\omega_l}\, a/(l + 1/2)$

# Torsional modes dispersion



# Splitting

If SNREI (Solid Not Rotating Earth Isotropic) Earth :
Degeneracy:
for n and l, same frequency for -l < m < l

For each m = one singlet.
The 2m+1 group of singlets = multiplet

**No more degeneracy if no more spherical symmetry :**

- **Coriolis**
- **Ellipticity**
- **3D**

**Different frequencies and eigenfunctions for each l, m**

**Figure 2.9-12: Synthesis of a body wave from normal mode summation.**

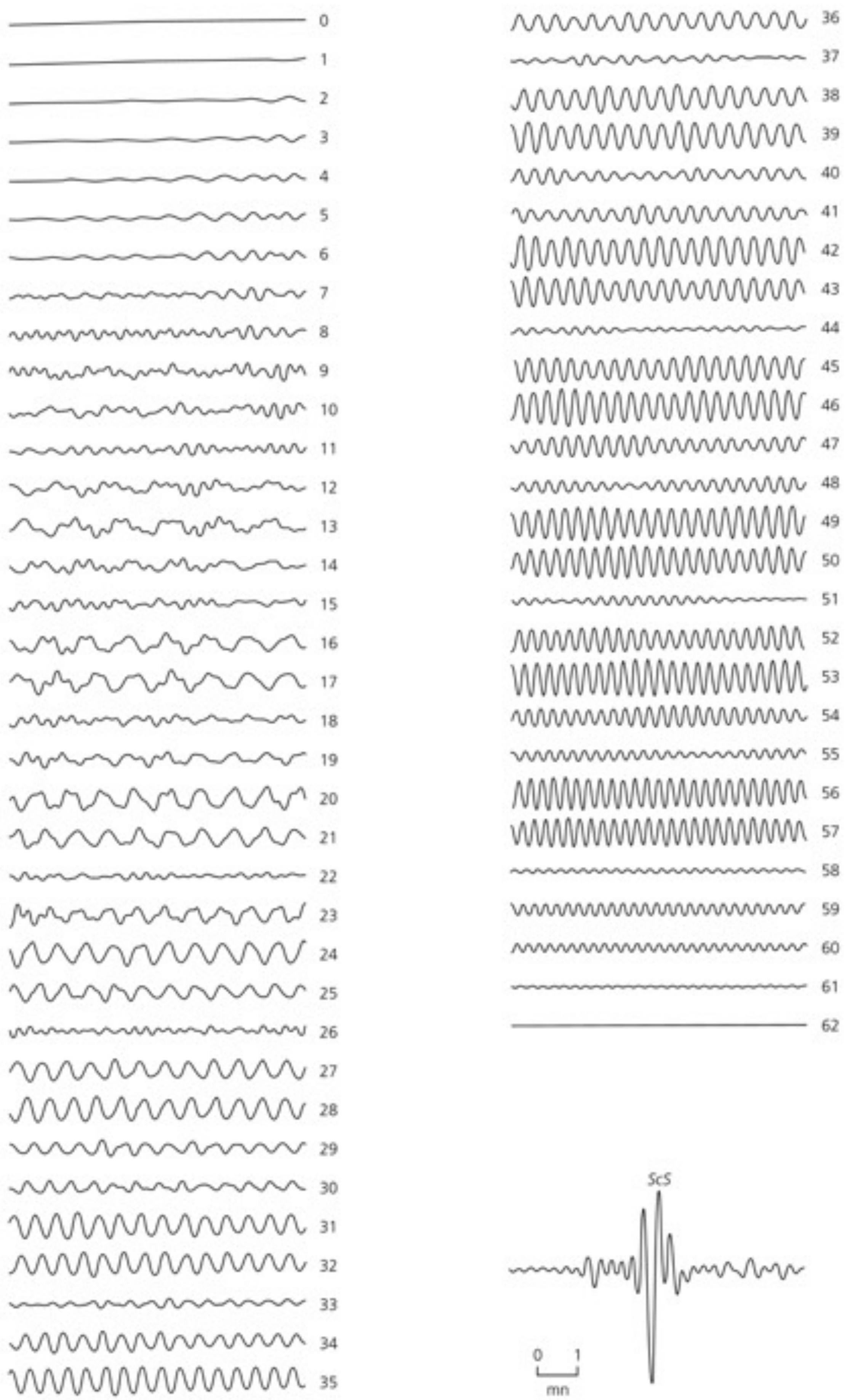


**Figure 2.9-13: Example of modeling data with normal mode synthetic seismograms.**