

Laplace Transforms

DINAMICA E CONTROLLO DEI PROCESSI CHIMICI
 A.A. 2019-2020
 Corso di Laurea Magistrale in Ingegneria di Processo e dei Materiali



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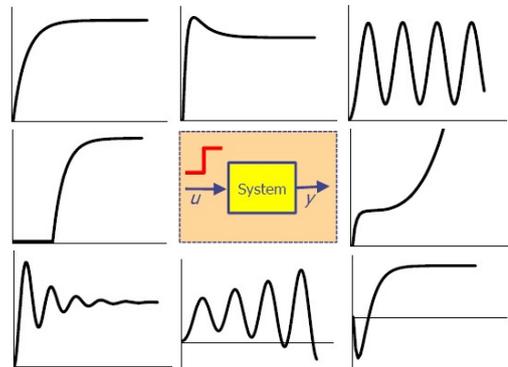
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CHARACTERIZING THE DYNAMIC RESPONSE

- Starting from **single** unit operation, we will get the response of the **entire process** system
- The dynamic response of a SISO system is quantified by the variation of (**output, y**) of a system to a change in the input (**forcing function; u**):
 1. We model the system through material/energy balances and constitutive equations
 2. Then we change the input through a forcing function (funzione «forzante»)
 3. We can then integrate the differential equations representing our system.
- Is it practical?
- How the response will be?



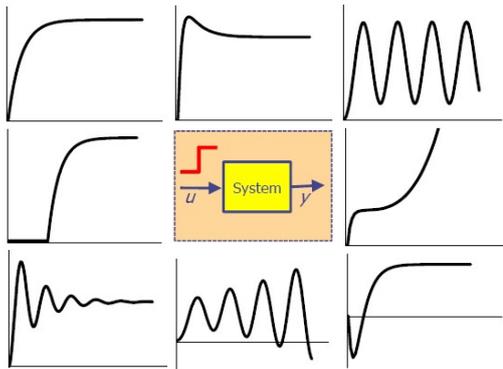
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CHARACTERIZING THE DYNAMIC RESPONSE

- Is the response monotonic or oscillatory?
- Is the response «stable»? Will it remain bounded when the system is excited by a bounded input?
- If stable, what will its final steady-state value be?
- If stable, how long will it take for the transient to die out?
- If stable and oscillatory, what is the period of oscillation? How long will it take for the oscillations to die out?



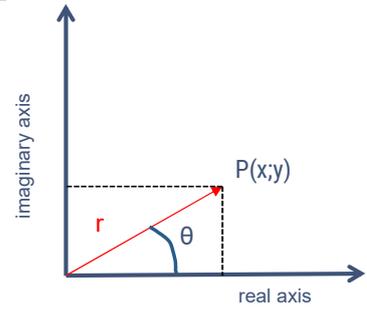
- The dynamic response is modeled using **differential equations**

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REFRESHING MATH

- Cartesian representation: $\boxed{z = x + jy}$ $j^2 \triangleq -1$
 - ▷ Modulus: $|z| = \sqrt{x^2 + y^2}$
 - ▷ Conjugate pairs: $z = x + jy; \quad \bar{z} = x - jy$
$$Re(z) = \frac{z + \bar{z}}{2}; \quad Im(z) = \frac{z - \bar{z}}{2j}; \quad z\bar{z} = |z|^2$$
- Polar representation: $\boxed{z = r\angle\theta}$ $x = r \cos \theta; \quad y = r \sin \theta; \quad z = r(\cos \theta + j \sin \theta)$
 - ▷ Modulus: $|z| = r\sqrt{\cos^2 \theta + \sin^2 \theta} = r$
- Polar exponential form: $\boxed{e^{j\theta} = \cos \theta + j \sin \theta}$ $z = re^{j\theta}$



Notice: the exponential of a complex number gives rise to oscillatory terms

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LAPLACE TRANSFORM

$$\frac{dy}{dt} = f(t); \quad s = x + jz \in \mathbb{C}; \quad j^2 \triangleq -1$$

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

- Laplace transform $F(s)$ is a **complex** function of the **complex** variable s
- The couple $F(s) \leftrightarrow f(t)$ is **unique**: no two distinct functions $f(t)$ and $g(t)$ can give rise to the same Laplace transform
- \mathcal{L} is the Laplace operator. It is a linear mapping from the **time domain** to the **Laplace domain**
- $F(s)$ does **not** contain any information on the behavior of $f(t)$ for $t < 0$
- The **inverse transform** of $F(s)$ is: $f(t) = \mathcal{L}^{-1}[F(s)]$
- s is called the **Laplace variable**

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LAPLACE TRANSFORM PROPERTIES

- \mathcal{L} is a **linear operator**. If a and b are **two real constants**, then:
 - ▷ $\mathcal{L}[af(t)] = a\mathcal{L}[f(t)] = aF(s)$
 - ▷ $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] = aF(s) + bG(s)$ ← **superposition principle**
 («principio di sovrapposizione degli effetti»)
 - ▷ $\mathcal{L}^{-1}[aF(s) + bG(s)] = af(t) + bg(t)$

- **Final value theorem**: if the limit for $t \rightarrow \infty$ exists

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} [sY(s)]$$

- **Initial value theorem**:

$$\lim_{t \rightarrow 0^+} y(t) = \lim_{s \rightarrow \infty} [sY(s)]$$

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WHAT IS THE LAPLACE TRANSFORM USED FOR?

- In the analysis of dynamic systems, the LT is used to **convert a linear differential equation into an algebraic equation**
 - Much easier to solve
 - After finding the solution in the Laplace domain, the solution must be inverse-transformed to recover it in the time domain

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REPRESENTATIVE FUNCTIONS

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

- «Constant» function: $f(t) = a, \quad a = \text{const}$

$$\mathcal{L}(a) = \int_0^{\infty} ae^{-st} dt = \frac{a}{-s} e^{-st} \Big|_0^{\infty} = \frac{a}{s}$$

$\mathcal{L}(a) = \frac{a}{s}$

*If $a=1$, $f(t)$ is called a **unit step function***

- «Unit Step» function: $S(t) = \begin{cases} 0; & t < 0 \\ 1; & t \geq 0 \end{cases}$

$\mathcal{L}[S(t)] = \frac{1}{s}$

 - This function incorporates the idea of initial time, time zero, zero time; this is the time at which the value of $S(t)$ switches from 0 to 1
 - In Italian it is called «scalino unitario» o «gradino unitario»

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REPRESENTATIVE FUNCTIONS
/cont'd

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

- «Time derivative» function: $\dot{f}(t) = \frac{df(t)}{dt}; \quad \ddot{f}(t) = \frac{d^2f(t)}{dt^2}; \quad f^{(n)}(t) = \frac{d^nf(t)}{dt^n}$

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2f(t)}{dt^2}\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

 - ▷ Information @ $t=0$ must be known
 - ▷ A multiplication by s appears

$$\mathcal{L}\left[\frac{d^3f(t)}{dt^3}\right] = s^3F(s) - s^2f(0) - sf'(0) - \ddot{f}(0)$$

 - ▷ A particular case: all initial conditions are zero

$y(0) = y^{(1)}(0) = y^{(2)}(0) = \dots = y^{(n-1)}(0) = 0$ this condition is frequently met in the study of dynamic systems

$$\mathcal{L}\left[\frac{d^nf(t)}{dt^n}\right] = s^nF(s)$$

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REPRESENTATIVE FUNCTIONS
/cont'd

$$F(s) = \mathcal{L}[f(t)] = \int_0^{\infty} f(t)e^{-st} dt$$

- «Time integral» function: $g(t) = \int_0^t f(t^*)dt^*$

$$\mathcal{L}\left[\int_0^t f(t^*)dt^*\right] = \frac{1}{s}F(s)$$

 - ▷ A division by s appears
- «Time delay» function (dead time): $f_d(t) = \begin{cases} 0, & t < t_0 \\ f(t - t_0), & t \geq t_0 \end{cases}$
 $= f(t - t_0)S(t - t_0)$

$$\mathcal{L}[f_d(t)] = e^{-st_0}F(s)$$

- See SEMD, Table 3.1 pp. 40-41 (pp 42-42, 3°) for a list of representative functions

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EXAMPLES OF APPLICATION OF THE LT

- Analytical solution of differential equations using the Laplace transform
- Concentration response in a blender (SEMD, 4° Exercise 3.18, p.53; 3°-3.17 p.57)

3.17 A stirred-tank blending system initially is full of water and is being fed pure water at a constant flow rate, q . At a particular time, an operator shuts off the pure water flow and adds caustic solution at the same volumetric flow rate q but with concentration \bar{c}_i . If the liquid volume V is constant, the dynamic model for this process is

$$V \frac{dc}{dt} + qc = q\bar{c}_i$$

with $c(0) = 0$.

What is the concentration response of the reactor effluent stream, $c(t)$? Sketch it as a function of time.

$$\text{Data: } V = 2 \text{ m}^3; \quad q = 0.4 \text{ m}^3/\text{min}; \quad \bar{c}_i = 50 \text{ kg/m}^3$$

