Transfer function models

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55

OBJECTIVE

- We would like to find a compact dynamic representation of the process, of the input-output type

$$Y(s) = G(s) \cdot U(s)$$
$$G(s) = \frac{Y(s)}{U(s)}$$

- We wish to represent the system in the Laplace domain, using a transfer function G(s)
  - It connects **one input** U(s) to **one output** Y(s)
  - Input ↔ output
  - Forcing function ↔ response
  - Independent variable ↔ dependent variable
  - Cause ↔ effect
- It is expected that we have to transform the differential equations describing the system dynamics

56

56

EFFECT OF THE INITIAL CONDITIONS

- As an example, assume that a dynamic model is:

$$a_2 \frac{d^2y(t)}{dt^2} + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b \cdot u(t)$$

  - y(t) and u(t) are the output and the input, respectively
  - a_i and b are the model parameters (constant and known)
- Upon Laplace trasformation:

$$a_2 \mathcal{L}\left[\frac{d^2y(t)}{dt^2}\right] + a_1 \mathcal{L}\left[\frac{dy(t)}{dt}\right] + a_0 \mathcal{L}[y(t)] = b\mathcal{L}[u(t)]$$

- After algebraic manipulation:

$$Y(s) = \frac{b}{a_2 s^2 + a_1 s + a_0} U(s) + \frac{(a_2 s + a_1) y(0) + a_2 \dot{y}(0)}{a_2 s^2 + a_1 s + a_0}$$

Initial condition on the output

Initial condition on the derivative of the output

The final result depends on the initial conditions!

57

57

DEVIATION VARIABLES

- We have seen that when a time derivative (i.e., a differential equation) appears, the Laplace Transform results depends on the initial conditions on the output and on its derivative(s)
- Hence, given a differential equation, whatever the system initial conditions change, the resulting algebraic equation (in the Laplace domain) changes as well
- To avoid this inconvenience:

$y' = y(t) - \bar{y}$

  1. We assume that we are indeed studying the system dynamics, but the system is «started» from an initial steady-state condition → the time derivatives of y(t) are null at t=0
  2. The differential equations are written using «auxiliary» variables that represent the original ones, but are such that auxiliary variable is null at time t=0
     - Since a steady-state condition exists at t=0, we define these auxiliary variables as deviation variables (they deviate with respect to their initial steady-state values)
     - The prime symbol (') will be used to indicate a deviation variable (also called perturbation variable)

58

58

DEVIATION VARIABLES /cont'd

$$u'(t) = u(t) - \bar{u}$$
$$y'(t) = y(t) - \bar{y}$$

where:

- u'(t) and y'(t) are the deviation variables of u(t) and y(t), respectively
- $\bar{u}$ and $\bar{y}$ are the nominal steady-state values of u(t) and y(t), respectively

- It follows that: $u'(0) = 0; \quad y'(0) = 0$
- Additionally (same for u):

$$\frac{dy'}{dt} = \frac{d(y - \bar{y})}{dt} = \frac{dy}{dt}$$

- Hence, for t=0:

$$\left.\frac{dy'}{dt}\right|_{t=0} = 0$$

Notice: in this context, the prime symbol (') does **not** denote a first derivative!

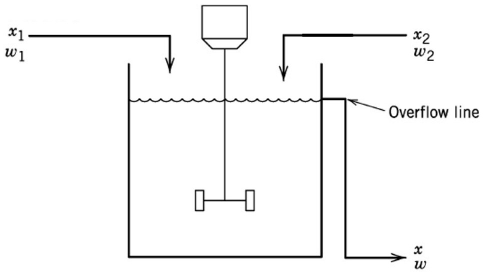
59

59

TRANSFER FUNCTION DETERMINATION

- The transfer function (TF) we are looking for provides a link between the input and the output, when both are expressed in deviation variable form
  - It describes the dynamic behavior of the system when it moves away from the nominal steady state
  - This is exactly what one is interested to know from a process control perspective

- Example:
  - Derive the transfer function for a blender



60

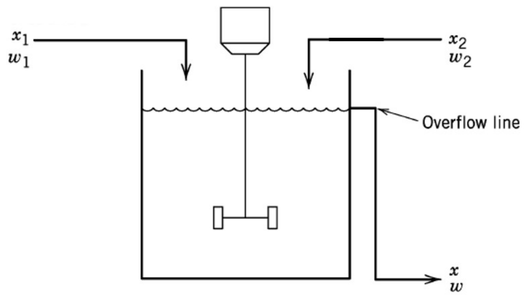
60

SUMMARIZING THE EXAMPLE

- Perfect mixing of two liquid streams
  - Same solute A
  - Volume V=const
  - Density ρ=const
  - Mass flow rates w1=const and w2=const

$$\rho V \frac{dx(t)}{dt} = w_1 x_1(t) + w_2 x_2(t) - w x(t)$$

- «General» case
  - Independent inputs are: $x_1(t)$ and $x_2(t)$
  - The output is: $x(t)$
- We would like to show how, in the Laplace domain, the output depends on each of the inputs



61

61

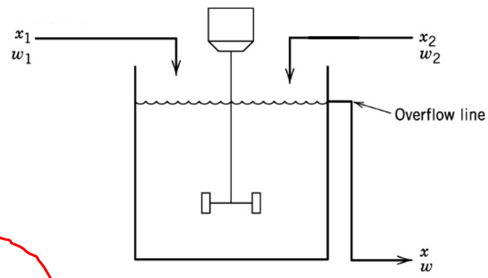
EXAMPLE: DETERMINATION OF A TF

$$\left.\frac{X'(s)}{X_1'(s)}\right|_{x_2=con} = G_1(s) = \frac{K_1}{\tau s + 1} \quad (\text{for } X_2'(s) = 0)$$

$$\left.\frac{X'(s)}{X_2'(s)}\right|_{x_1=const} = G_2(s) = \frac{K_2}{\tau s + 1} \quad (\text{for } X_1'(s) = 0)$$

$$\tau = \frac{\rho V}{w}; \quad K_1 = \frac{w_1}{w}; \quad K_2 = \frac{w_2}{w}$$

- Two transfer functions are obtained
  - A variation in one independent input is «transferred» to the output through the TF related to that specific input
  - Each TF is called a «first-order» TF, because the polynomial in s appearing at the TF denominator is a first-order polynomial
- The knowledge of the initial conditions is not needed anymore!
  - Each TF provides a quantitative indication of how the output changes, with respect to its initial steady-state value, in response to a change of a given input (with respect to its initial steady-state value)



Independent inputs: $x_1(t), x_2(t)$
Dependent output: $x(t)$

62

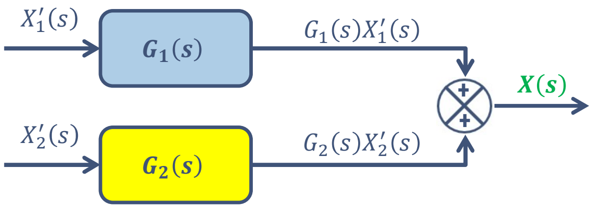
62

THE SUPERPOSITION PRINCIPLE

- What happens if both inputs change simultaneously?
  - The Laplace transform is a linear operator

$$X'(s) = G_1(s)X_1'(s) + G_2(s)X_2'(s)$$

Block diagram representation



- The effects of the two inputs can be evaluated separately, and then they can be re-combined by simply summing them
  - Additive property of transfer functions (superposition principle)

63

63

USEFULNESS OF THE TRANSFER FUNCTION

- It completely characterizes the input-output dynamics of the system
- It allows knowing in detail the system response to any change of the input
  - It is independent of the shape of the input, U'(s)
    - The TF is «normalized» with respect to the input
  - If more than one independent inputs exists, both the single effect and the combined effects can be described
- It is written in the terms of deviation variables
  - It does not depend on the initial conditions (provided that the system initial condition is a steady-state one)
  - Which initial steady state to start from does not influence the final result (provided that the system is a linear one)
- Notice
  - In the following, we will **always** assume that TFs are written in deviation variable form
  - We will usually not use the prime symbol (') to indicate a deviation variable

64

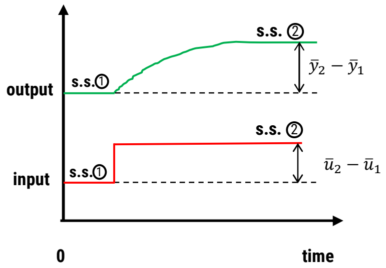
64

THE STEADY-STATE GAIN

- Steady-state gain K
  - It is the ratio between two changes, both calculated with respect to two steady-state conditions (steady- state ① and steady-state ②)

$$K = \frac{output\ change\ following\ an\ input\ change(1 \to 2)}{input\ change(1 \to 2)} = \frac{\bar{y}_2 - \bar{y}_1}{\bar{u}_2 - \bar{u}_1}$$

- In truly linear systems:
  - The gain value does not depend on the reference steady state
  - The gain value does not depend on the amplitude of the input perturbation
- The perturbation «shape» is not important
  - What matters is that the input eventually reaches a steady-state value
  - Usually, a step change is used
- A distinct gain exists for each input-output pair
  - But not all real systems have a steady-state gain!
- Each steady-state gain has its own measurement units (output u.m./input u.m.)
  - And its value can be positive or negative



THE CALCULATION OF THE STEADY-STATE GAIN

- If the TF of a system is known, determining its gain is simple:

$$K = \frac{\bar{y}_2 - \bar{y}_1}{\bar{u}_2 - \bar{u}_1}; \qquad G(s) = \frac{Y(s)}{U(s)}$$

To find the steady-state gain (if it exists) **with respect to a given input, one only needs to evaluate the relevant TF @ s=0: $K = G(0)$**

  - Assume a unit step change of input u
    - Hence, the denominator in the K expression is 1 (with own units)
  - It follows that $U(s) = \frac{1}{s} \to Y(s) = G(s) \cdot U(s) = \frac{1}{s} G(s)$
  - We are interested at the steady-state change in y; in terms of deviation variables (with respect to the starting steady-state 1), it follows that:

$$K = \frac{\bar{y}_2 - \bar{y}_1}{1} \quad \to \quad K = \bar{y}_2 - \bar{y}_1 = y'(t)\Big|_{s.s} = \lim_{t\to\infty} y'(t)$$

  - Final value theorem: $\lim_{t\to\infty} y'(t) = \lim_{s\to 0}[sY(s)] \quad \to \quad K = \lim_{s\to 0} G(s)$

ORDER OF DYNAMIC SYSTEMS

- Let's consider a generic dynamic system:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

  - Upon Laplace transformation it follows that:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

Notice the order of numerator and denominator

- The TF appears (in deviation variable form) as a ratio of two polynomials (of different orders) in s
- The order of the dynamic system is the degree (n) of the denominator polynomial
  - The relative order (or relative degree) of the system is the difference (n-m)
  - Roots of the denominator polynomial are called poles
  - Roots of the numerator polynomial are called zeroes
- The steady-state gain of G(s) is equal to b_0 / a_0
  - Simply set s=0 on the G(s) expression

67

67

CHARACTERISTIC POLYNOMIAL

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_0}$$

- By dividing both the numerator and the denominator by a_0, the system characteristic polynomial is obtained at the denominator
  - It can be factoed as the product of a sequence: $\prod_i (\tau_i s + 1)$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{KB(s)}{(\tau_1 s + 1)(\tau_2 s + 1) \ldots (\tau_n s + 1)}$$

  - The B(s) polynomial has order m
- This form of G(s) is called «Time constant form»
  - It highlights the individual time constants τ_i that determine the dynamic response of the system
    - Strictly speaking, τ_i is called «time constant» only if it is real and positive; on a general basis, τ_i might be negative or complex
  - Inspection of the individual τ_i provides information about the speed and the shape of the dynamic response, as we will see later during this course

68

68

A PROPERTY OF TFs

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

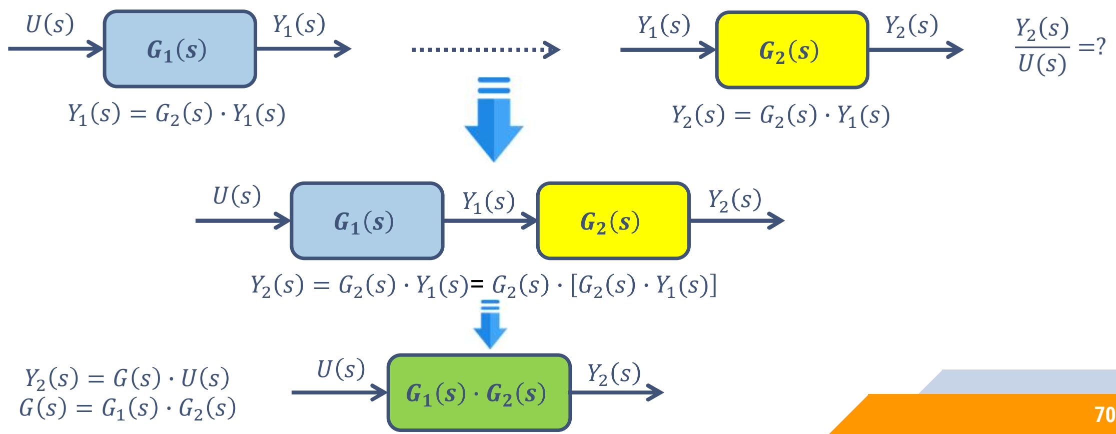
- In real («physical») dynamic systems, n≥m always results
  - Physical realizability condition.
- Why does this hold true in a real physical system?
  - If m>n were possible, then, for example, the following would be possible: $a_0 y = b_1 \frac{du}{dt} + b_0 u$ (n=0; m=1>n)
  - A step change in u (e.g., a valve is suddenly opened) would result in an impulse change in y (i.e., y would change infinitely fast as the step occurs: $\frac{du}{dt} \rightarrow \infty$
  - This is usually impossible to achieve in real systems
  - Every real system responds with «inertia» to any input change («at most», n=m may be found)
    - The relative order quantifies the degree of inertia of an output with respect to an input change
  - Few exceptions exists (e.g., explosions!!!!)
- When m>0 is found, the system is said to have numerator dynamics
  - This means that a polynomial of degree at least 1 appears as the TF numerator

69

69

ONE MORE PROPERTY OF TFs

- Besides the additive property, TFs also have a multiplicative property



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70

QUESTIONS

1. How does the value of K change, if the forcing function changes?
2. Can you think of a real case when it is not possible to identify the gain?
3. How will the output appear, if the K value is negative?
4. Can you make an example of a case where the K value is negative?
5. Suppose you are an operator, responsible for the performance of a chemical process. Your boss asks you to evaluate the gain of a dynamic control system using a step forcing function. Will you use a high or a low amplitude?
6. Would you rather have a high or a low time constant for a chemical reactor? Consider different point of view.

LINEAR SYSTEMS AND NONLINEAR SYSTEMS

- The Laplace transformation can be applied only to a linear diffrential equations with constant coefficients (linear systems)

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \cdots + b_1 \frac{du(t)}{dt} + b_0 u(t)$$

  - The output dynamics (i.e., the y derivatives) depends linearly on the output itself, on the input u and on the input derivatives
- What does «linear system» mean on a physical basis?
  1. For a linear system, the superposition principle holds true

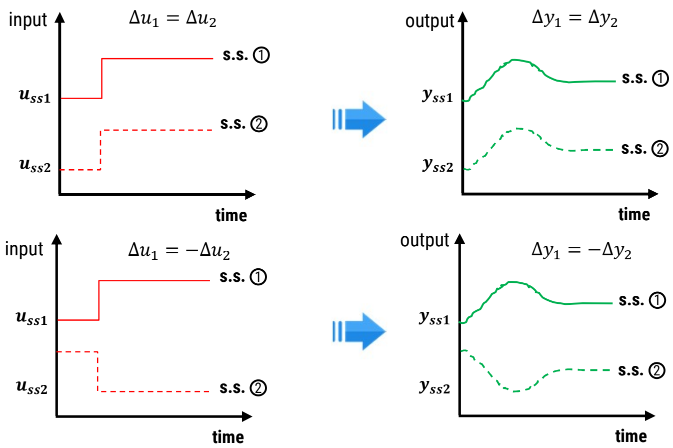
$$u_1(t) \to y_1(t)$$

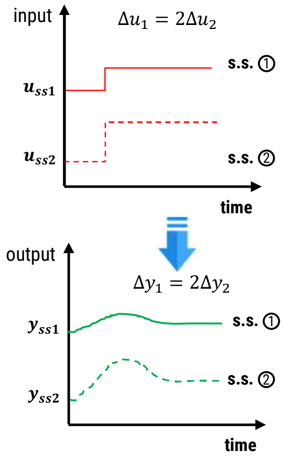
$$\to (u_1 + u_2) \to (y_1 + y_2)$$

$$u_2(t) \to y_2(t)$$

LINEAR SYSTEMS AND NONLINEAR SYSTEMS /cont'd

2. The dynamic response is independent of the operating conditions





NONLINEAR SYSTEMS

- It is very uncommon (actually, never) that a real process appears linear: $\frac{a_1}{a_0}\frac{dy(t)}{dt} + y(t) = \frac{b_0}{a_0}u(t)$
  - Usually, the dynamic response is represented by: $\frac{dy(t)}{dt} = f[y(t); u(t)]$

  where function f include terms that are nonlinear with respect to the output y and/or to the independent input u (and possibly to the derivatives of u)

  - Examples of nonlinearity related to typical outputs (temperature T and composition x)
    - A kinetic constant k depends on temperature T in a nonlinear way: $k = g[T(t)] = k_0 e^{\frac{E_a}{[RT(t)]}}$
    - Vapor pressure P^{sat} depends on temperature T in a nonlinear way: $P^{sat} = g[T(t)] = \exp\left(A - \frac{B}{[T(t)+C]}\right)$
    - The vapor-phase equilibrium composition y* depends on the liquid-phase composition in a nonlinear way even for trivial ideal systems:

$$y* = g[x(t)] = \frac{\alpha x(t)}{1 + (\alpha - 1)x(t)}$$

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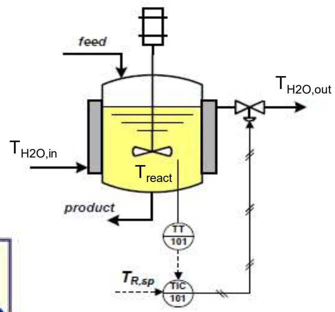
75

75

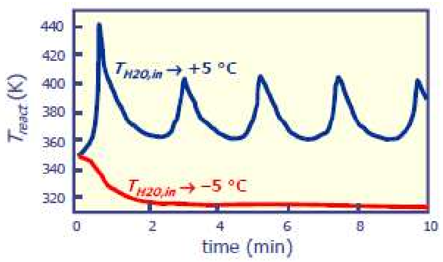
AN EXAMPLE OF A NONLINEAR RESPONSE

- A jacketed CSTR, wherein a first-order exothermic irreversible reaction occurs: $A \rightarrow B, \quad \Delta H_r < 0$
- Response of $y=T_{react}$ to symmetric (increase/decrease) step changes on $u=T_{H2O,in}$



The response is strongly nonlinear!!!

- The nonlinearity extent depends on the reaction kinetics, operating conditions, reactor geometry
- The system gain (and time constant) significantly change with operating conditions



76

76

LAPLACE TRANSFORM & NONLINEAR SYSTEMS

- Example
  - Linear system: $\frac{dy\prime(t)}{dt} = ay'(t) + bu'(t)$

    $\rightarrow \mathcal{L}\left[\frac{dy'(t)}{dt}\right] = \mathcal{L}[ay'(t) + bu'(t)] = \mathcal{L}[ay'(t)] + \mathcal{L}[bu'(t)] = aY'(s) + bU'(s)$
  - Nonlinear system: $\frac{dy\prime(t)}{dt} = ay'(t)u'(t)$ this product gives rise to a nonlinear term

    $\rightarrow \mathcal{L}\left[\frac{dy'(t)}{dt}\right] = \mathcal{L}[ay'(t)u'(t)] \neq \mathcal{L}[ay'(t)] \cdot \mathcal{L}[u'(t)] = \ ????$

The Laplace Transform cannot be calculated for nonlinear terms!!!

How to determine the TF of a nonlinear system?

77

77

LINEARIZATION OF NONLINEAR SYSTEMS

$$\frac{dy(t)}{dt} = f[y(t); u(t)]$$

- To be able to determine the TF for these systems, one needs to approximate f(y,u) with a linear expression
  - A Taylor series approximation around a nominal steady state is done, truncating it to the 1° order terms

$$f[y(t); u(t)] \cong f(\bar{y}; \bar{u}) + \left.\frac{\partial f}{\partial y}\right|_{\bar{y};\bar{u}} \cdot (y - \bar{y}) + \left.\frac{\partial f}{\partial u}\right|_{\bar{y};\bar{u}} \cdot (u - \bar{u})$$

  - At steady state, we have: $f(\bar{y}; \bar{u}) = 0$
  - By using deviation variables y' and u' we obtain:

$$\boxed{\frac{dy'}{dt} \cong \left.\frac{\partial f}{\partial y}\right|_{s.s.} \cdot y' + \left.\frac{\partial f}{\partial u}\right|_{s.s} \cdot u'}$$

Both y' and u' now appears as summation terms, i.e. linearly!!!
(s.s.=nominal steady state)

  - This operation is called system linearization.
    It must be extended to all independent input and all outputs appearing in f

78

78

INTERPRETING THE LINEARIZATION

$$\frac{dy(t)}{dt} = f[y(t); u(t)]$$

- Simple example:

$$\frac{dy(t)}{dt} = f[y(t)] \rightarrow \frac{dy'}{dt} \cong \left.\frac{\partial f}{\partial y}\right|_{\bar{y}} y'$$

- The parameters (gain, time constant) of the TF obtained after linearization are affected by the slope of function f, not by the value of f itself
  - The slope changes with the linearization point
  - For nonlinear systems, the gain and the time constant depend on the nominal steady state!
- If the actual conditions are not far from the nominal steady-state ones (i.e., if $y(t) \cong \bar{y}$), the linear approximation is usually satisfactory for a real system