SECOND-ORDER SYSTEMS

$$U_{sin}(s) = \frac{A\omega}{s^2+\omega^2}$$

$$U_S(s) = \frac{M}{s}$$

- They are constituted by a 2°-order linear differential equation with $a_0 \neq 0$:

$$\frac{a_2}{a_0}\frac{d^2y(t)}{dt^2} + \frac{a_1}{a_0}\frac{dy(t)}{dt} + y(t) = \frac{b_0}{a_0}u(t) \quad \rightarrow \quad \boxed{\tau^2\frac{d^2y(t)}{dt^2} + 2\zeta\tau\frac{dy(t)}{dt} + y(t) = Ku(t)}$$

$$\boxed{G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}}$$

- K: steady-state gain of the system [output u.m./input u.m.]
- $\tau > 0$: natural period (characteristic time; effective time constant) of the system [time]
- ζ: damping factor: damping coefficient [dimensionless]

- Often (albeit not necessarily) two 1°order systems in series give rise to a SO sistem

SEMD 4° p.76, 3° p.81

$$U(s) \rightarrow \boxed{\frac{K_1}{\tau_1 s + 1}} \xrightarrow{X(s)} \boxed{\frac{K_2}{\tau_2 s + 1}} \rightarrow Y(s)$$

$$\boxed{\frac{Y(s)}{U(s)} = \frac{K_1K_2}{(\tau_1 s + 1)(\tau_2 s + 1)}}$$

$$\begin{cases} K = K_1K_2 \\ \tau^2 s^2 + 2\zeta\tau s + 1 = (\tau_1 s + 1)(\tau_2 s + 1) \end{cases}$$

$$\tau = \sqrt{\tau_1\tau_2}$$

$$\zeta = \frac{(\tau_1+\tau_2)/2}{\sqrt{\tau_1\tau_2}}$$

STEP RESPONSE OF SO SYSTEMS

$$u(t) = \begin{cases} 0 & t < 0 \\ M & t \geq 0 \end{cases}$$

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}; \quad Y(s) = G(s)U(s); \quad U(s) = \frac{M}{s} \quad \rightarrow \quad \boxed{Y(s) = \frac{KM}{s(\tau^2 s^2 + 2\zeta\tau s + 1)}}$$

$$y\big|_{t=0^+} = 0 \qquad y\big|_{t=\infty} = KM$$

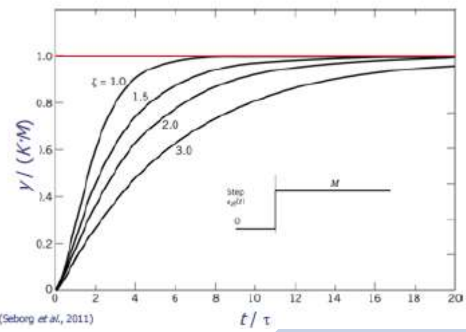
- The response shape depends on the value of the damping factor

Overdamped response ($\zeta > 1$)

$$\boxed{y(t) = KM\left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2}\right)}$$

Critically damped response ($\zeta = 1$)

$$\boxed{y(t) = KM\left[1 - \left(1 + \frac{t}{\tau}\right)e^{-t/\tau}\right]}$$



(Seborg et al., 2011)

- τ and ζ appear in hyperbolic trigonometric form of the functions
- The horizontal asymptote is never crossed (no overshoot)
- $\uparrow \zeta \rightarrow$ slower response (more sluggish)
- The initial speed of response is zero
- The response curve shows an inflection point (i.e., maximum speed of response) at t>0

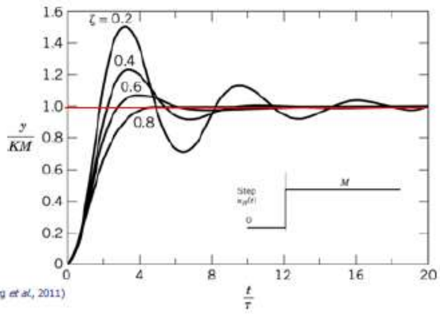
STEP RESPONSE OF SO SYSTEMS /cont'd

$$u(t) = \begin{cases} 0 & t < 0 \\ M & t \geq 0 \end{cases}$$

Underdamped response $(0 < \zeta < 1)$

$$y(t) = KM\left\{1 - e^{-\zeta t/\tau}\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\}$$

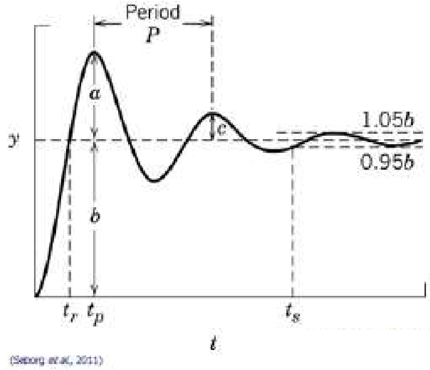
- ▻ An overshoot always exists
- ▻ $\downarrow \zeta \rightarrow \downarrow$ rise time, $\uparrow$ overshoot
- ▻ The initial speed of response is zero
- ▻ The response is «sluggish»
- ▻ The response curve shows an inflection point at t>0



- Notice these are open-loop responses!!
  - ▻ In closed-loop, several real systems respond to set-point changes (servo control) similarly to how a 2°-order underdamped system responds at open-loop to a step input change

CHARACTERIZING THE UNDERDAMPED RESPONSE

- Oscillations are not always a negative issue in a generic response
  - ▻ However, if oscillations are present, is important that their damping is sufficiently high
  - ▻ Quarter decay ratio: $1/4 \quad (\zeta = 0.22; OS = 50\%)$
- Rise time: t_r
- Time to first peak: t_p
- Settling time: t_s, time required for the response to reach and remain within a certain range in the vicinity of the final steady-state (e.g., ±5%, ±3%,…)
- Overshoot: OS=a/b
- Decay ratio: DR=c/a
- Period of oscillation: P
- How can we calculate ζ from measurements?



$$DR = OS^2 = \exp\left(\frac{-2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

From DR (measurable) one calculates both OS and ζ
See SEMD p.77

DAMPING IN SO SYSTEMS

- ζ provides information about the degree of oscillation of the open-loop response to an input perturbation

$$\frac{Y(s)}{U(s)} = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$ Characteristic polynomial

$$r_{1,2} = \frac{-\zeta \pm \sqrt{\zeta^2 - 1}}{\tau}$$

$$r_1 = p_1 = \frac{-\zeta + \sqrt{\zeta^2 - 1}}{\tau}$$

$$r_2 = p_2 = \frac{-\zeta - \sqrt{\zeta^2 - 1}}{\tau}$$

- Alternatively

$$\frac{Y(s)}{U(s)} = \frac{K}{(\tau_1 s + 1)(\tau_2 s + 1)}$$

$$p_1 = -\frac{1}{\tau_1};\; p_2 = -\frac{1}{\tau_2}$$

Damping factor	Roots of the chatacteristic equation (poles)	Response Type
$\zeta > 1$	Real and distinct	Overdamped (monotonic and stable)
$\zeta = 1$	Real and coincident	Critically damped (monotonic and stable)
$0 < \zeta < 1$	Complex and conjugated	Underdamped (oscillatory and stable)
$\zeta = 0$	Complex and conjugated	Undamped (oscillatory and permanent)
$-1 < \zeta < 0$	Complex and conjugated	Unstable (oscillatory and amplified)
$\zeta \leq -1$	Real and distinct	Runaway (monotonic and unstable)

RESPONSE OF SO SYSTEMS TO A SINUSOIDAL

$$u(t) = \begin{cases} 0 & t < 0 \\ A \sin \omega t & t \geq 0 \end{cases}$$

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}; \qquad Y(s) = G(s)U(s); \qquad U(s) = \frac{A\omega}{s^2 + \omega^2}$$

$$y(t) = \cdots + \frac{KA}{\sqrt{[1-(\omega\tau)^2]^2 + (2\zeta\omega\tau)^2}} \sin(\omega t + \phi) \qquad \phi = -\tan^{-1}\left[\frac{2\zeta\omega\tau}{1-(\omega\tau)^2}\right]$$

- An initial transient exists, which dies out after a few τ. After the transient, the response is purely sinusoidal
- The frequency is the same as the forcing function
- It lags behid the input perturbation by ϕ rad
- The amplitude is different from the FF and depends on:
  - The process: K, τ, ζ
  - The forcing function, namely its angular frequency ω and its amplitude A
- The normalized amplitude ratio (AR_N) is:

$$AR_N = \frac{1}{\sqrt{[1-(\omega\tau)^2]^2 + (2\zeta\omega\tau)^2}}$$

For $AR_N > 1$ → the response will be amplified
For $AR_N < 1$ → the response will be damped

- The AR_N quantifies the effect of the sole dynamic parameters of the system and of the forcing function

AMPLIFICATION OF OSCILLATIONS

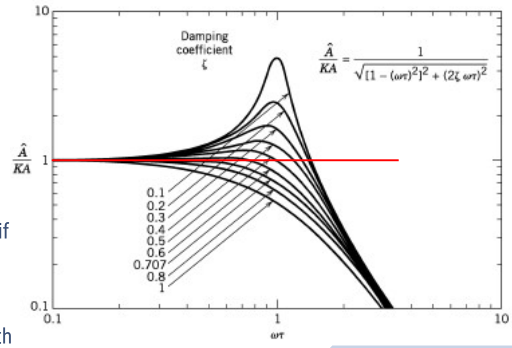
Tacoma Bridge

https://www.youtube.com/watch?v=3mclp9QmCGs&t=1s

- For a given τ, in systems with rather small damping ($0 < \zeta < \sim 0.707$), AR_N shows a peak for a certain value ω^* of ω
- In systems with greater damping ζ, AR_N monotonically decreases with increasing ω
- For a given process (τ, ζ assigned):

$$\frac{\hat{A}}{KA} = \frac{1}{\sqrt{[1-(\omega\tau)^2]^2 + (2\zeta\omega\tau)^2}}$$

$$\frac{\partial AR_N}{\partial \omega} = 0 \rightarrow \begin{cases} \omega^* = \dfrac{\sqrt{1-2\zeta^2}}{\tau} \\ AR_N\Big|_{max} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} \end{cases} \qquad \omega^* \epsilon\, \mathbb{R} \Leftrightarrow 0 < \zeta < \frac{1}{\sqrt{2}} \cong 0.707$$



- An overdamped system ($\zeta > 1$) always dampens the input oscillations
- An underdamped system ($\zeta < 1$) may amplify them, possibly to a very large extent (if the input frequency is close to the resonance one)
- The frequency ω^* at which a peak in AR_N is found is called resonance frequency
  - A process with little damping (ζ «small») exhibits an oscillatory response with high amplitude (logarithmic scale!) if it is perturbed by a periodic signal with frequency close to the resonance one

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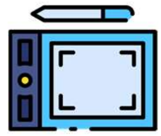
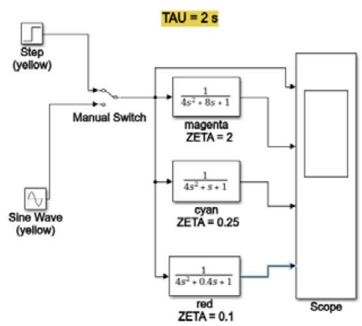
EXAMPLES



- Temperature measurement in a CSTR (SEMD, 4° Example 5.6 p.79, 3° 5.6 p.85)

An engineer uses a temperature sensor mounted in a thermowell to measure the temperature in a CSTR. The temperature sensor/transmitter combination operates approximately as a first-order system with time constant equal to 3 s. The thermowell behaves like a first-order system with time constant of 10 s. The engineer notes that the measured reactor temperature has been cycling approximately sinusoidally between 180 and 183 °C with a period of 30 s for at least several minutes. What can be concluded concerning the actual temperature in the reactor?

- SO systems simulation



MATLAB & SIMULINK

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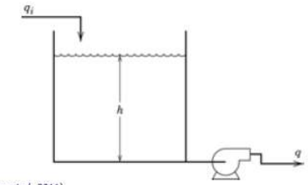
INTEGRATING PROCESSES (pure integrators or pure capacitances)

- The output flow does not depend from the level h, but only from the pump speed
- If the volumetric inlet (q_i) and outlet (q) flows are not «balanced», the vessel either runs empty or overflows

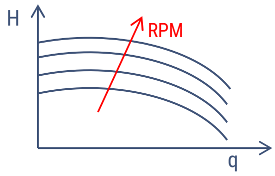
$$\frac{d[\rho A h(t)]}{dt} = \rho q_i(t) - \rho q(t)$$



(Seborg *et al.*, 2011)

- Starting from steady state, in terms of deviation variables we have: $A\frac{dh'(t)}{dt} = q_i'(t) - q'(t)$

$$sAH'(s) = Q_i'(s) - Q'(s) \quad \rightarrow \quad H'(s) = \frac{\frac{1}{A}}{s}Q_i'(s) - \frac{\frac{1}{A}}{s}Q'(s) \quad \rightarrow \quad \boxed{G_1(s) = \frac{H'(s)}{Q_i'(s)} = \frac{K'}{s}}$$



  - Notice. The outlet flow q, as well as the inlet flow q_i, is an input in this case!
- The TF of each input contains the (1/s) factor
  - This factor is called an integrator
    - It arises when the output depends on time through the integral of an input
    - In fact, the integration of the initial balance gives as result: $h(t) - \bar{h} = \frac{1}{A}\int_0^t [q_i(t^*) - q(t^*)]dt^*$

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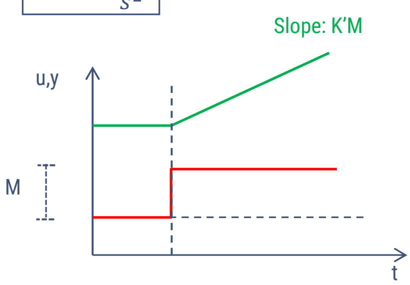
STEP RESPONSE OF PURE INTEGRATORS

$$u(t) = \begin{cases} 0 & t < 0 \\ M & t \geq 0 \end{cases}$$

$$G(s) = \frac{K'}{s}; \quad Y(s) = G(s)U(s); \quad U(s) = \frac{M}{s} \quad \rightarrow \quad \boxed{Y(s) = \frac{K'M}{s^2}}$$

Upon inverse transformation: $\boxed{y(t) = K'Mt}$



- Integrating processes do not have a steady-state gain
- They are not self-regulating
  - When forced by a step perturbation, they do not settle to a new steady state
  - They are open-loop unstable
- Typically, batch processes are integrative systems
- Notice that in an integrative system, the output derivative does not depend on the output, but only on the input
- This is the most remarkable «formal» difference from 1°-order systems

integrator: $\tau\frac{dy(t)}{dt} = K'u(t)$

FO system: $\tau\frac{dy(t)}{dt} + y(t) = Ku(t)$

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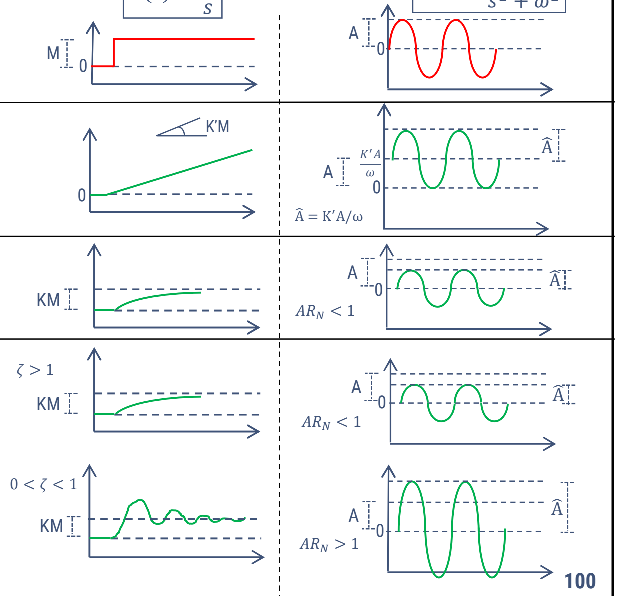
SUMMARY OF ELEMENTARY DYNAMIC RESPONSE

SYSTEM	GENERAL EQUATION	TRANSFER FUNCTION
Integrator	$\tau \frac{dy(t)}{dt} = K'u(t)$	$G(s) = \frac{K'}{s}$
First Order	$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$	$G(s) = \frac{K}{\tau s + 1}$
Second Order	$\tau^2 \frac{d^2y(t)}{dt^2} + 2\zeta\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$	$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$

FORCING FUNCTIONS

$U(s) = \frac{M}{s}$ $\qquad$ $U(s) = \frac{A\omega}{s^2 + \omega^2}$



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QUESTIONS #2

1. What happened to the Tacoma Bridge?
2. In everyday life, can you make an example of a underdamped system?
3. Can you demonstrate mathematically that two FO functions in series give rise to an overdamped SO system?
4. Is it possible for a FO system to amplify the amplitude of an oscillatory perturbation?

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