P controllers for FO «processes»: servo response

P controller
$$G_c(s) = K_c$$

$D(s) \to G_d(s)$

$Y_{sp}(s) \to \oplus \to G_c(s) \to U(s) \to \dfrac{K_p}{\tau_p s + 1} \to \oplus \to Y(s)$

■ Overall process: FO system (at open loop)

▷ G_p includes the overall loop dynamics and gains (except the controller ones)

$$Y = \dfrac{G_d}{1 + G_{OL}} D + \dfrac{K_m G_c G_v G_p}{1 + G_{OL}} Y_{sp}$$

(crossed out)

$$CLTF_{sp} = \dfrac{Y(s)}{Y_{sp}(s)} = \dfrac{K_m G_c G_v G_p}{1 + G_c G_v G_p G_m} = \dfrac{K_c \dfrac{K_p}{\tau_p s + 1}}{1 + K_c \dfrac{K_p}{\tau_p s + 1}} = \dfrac{\dfrac{K_c K_p}{\tau_p s + 1}}{\dfrac{\tau_p s + 1 + K_c K_p}{\tau_p s + 1}} = \dfrac{K_c K_p}{\tau_p s + (1 + K_c K_p)} = \dfrac{\dfrac{K_c K_p}{1 + K_c K_p}}{\dfrac{\tau_p}{1 + K_c K_p} s + 1}$$

$$CLTF_{sp}: \quad \boxed{\dfrac{Y(s)}{Y_{sp}(s)} = \dfrac{K_1}{\tau_1 s + 1}} \quad \begin{cases} K_1 = \dfrac{K_c K_p}{1 + K_c K_p} = \dfrac{K_{OL}}{1 + K_{OL}} < 1 \\[2mm] \tau_1 = \dfrac{\tau_p}{1 + K_c K_p} = \dfrac{\tau_p}{1 + K_{OL}} < \tau_p \end{cases}$$

$K_{OL} > 0$: gain of the open loop

Remarks:
- ❖ Il the loop is closed with a P controller, the closed-loop response is still a FO response
- ❖ Since $K_1 < 1$ (hence ≠ 1), y can never attain $y_{sp} \to$ offset
- ❖ Since $\tau_1 < \tau_p$, the closed-loop response is faster than the open-loop one

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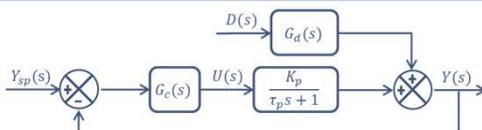## P controllers for FO «processes»: servo response /cont'd

$D(s) \to G_d(s)$

$Y_{sp}(s) \to \oplus \to G_c(s) \to U(s) \to \dfrac{K_p}{\tau_p s + 1} \to \oplus \to Y(s)$

$$CLTF_{sp}: \quad \boxed{\dfrac{Y(s)}{Y_{sp}(s)} = \dfrac{K_1}{\tau_1 s + 1}} \quad \begin{cases} K_1 = \dfrac{K_{OL}}{1 + K_{OL}} < 1 \\[2mm] \tau_1 = \dfrac{\tau_p}{1 + K_{OL}} < \tau_p \end{cases}$$

P controller
$$G_c(s) = K_c$$

Servo problem: $Y_{sp} = \dfrac{1}{s}$ (unit step)
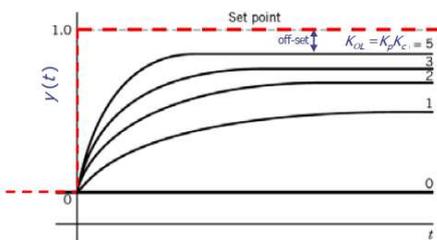
(Smith and Corripio, 2006)

Off-set:

$$y_{sp} - y(\infty) \to y_{sp} - \lim_{s \to 0}\left[s\left(\dfrac{K_1}{s(\tau_1 s + 1)} \right) \right] = 1 - K_1 = \boxed{\dfrac{1}{1 + K_{OL}}}$$

- ❖ If Kc > 0 increases, **the dynamics is made faster** and the off-set decreases
- ❖ What happens if $K_{OL} < 0$?

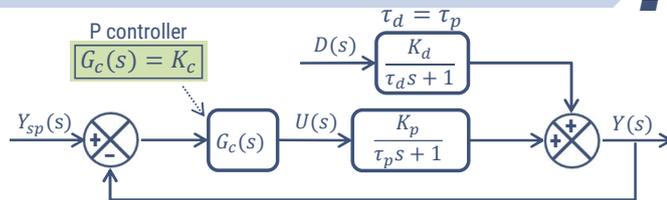## P controllers for FO «processes»: disturbance rejection

P controller
$$G_c(s) = K_c$$

$$\tau_d = \tau_p$$

$D(s)$ → $\dfrac{K_d}{\tau_d s + 1}$

$Y_{sp}(s)$ → (+/−) → $G_c(s)$ → $U(s)$ → $\dfrac{K_p}{\tau_p s + 1}$ → (+) → $Y(s)$

■ Overall process: FO system (at open loop)

▹ G_p includes the overall loop dynamics and gains (except the controller ones)

$$Y = \frac{G_d}{1 + G_{OL}} D + \frac{K_m G_c G_v G_p}{1 + G_{OL}} Y_{sp}$$

$$CLTF_{sp} = \frac{Y(s)}{D(s)} = \frac{G_d}{1 + G_c G_v G_p G_m} = \frac{\dfrac{K_d}{\tau_p s + 1}}{1 + K_c \dfrac{K_p}{\tau_p s + 1}} = \frac{\dfrac{K_d}{\tau_p s + 1}}{\dfrac{\tau_p s + 1 + K_c K_p}{\tau_p s + 1}} = \frac{K_d}{\tau_p s + (1 + K_c K_p)} = \frac{\dfrac{K_d}{1 + K_c K_p}}{\dfrac{\tau_p}{1 + K_c K_p} s + 1}$$

$$CLTF_d: \quad \boxed{\frac{Y(s)}{D(s)} = \frac{K_2}{\tau_1 s + 1}} \begin{cases} K_2 = \dfrac{K_d}{1 + K_c K_p} = \dfrac{K_d}{1 + K_{OL}} > 0 \\[2mm] \tau_1 = \dfrac{\tau_p}{1 + K_c K_p} = \dfrac{\tau_p}{1 + K_{OL}} < \tau_p \end{cases}$$

$K_{OL} > 0$: gain of the open loop

Remarks:

❖ Il the loop is closed with a P controller, the closed-loop response is still a **FO response**

❖ Since $K_2 > 0$ (hence ≠ 0), y can be **never** restored back to y_{sp} at steady state → **offset**

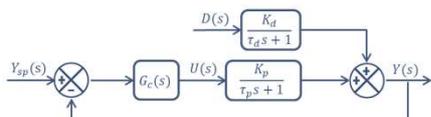❖ Since $\tau_1 < \tau_p$, the closed-loop response is **faster** than the open-loop one

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P controllers for FO «processes»: disturbance rejection /cont'd

$D(s)$ → $\dfrac{K_d}{\tau_d s + 1}$

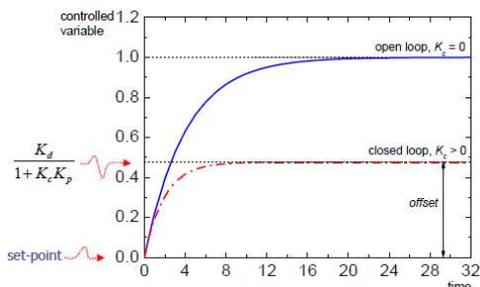$Y_{sp}(s)$ → $G_c(s)$ → $U(s)$ → $\dfrac{K_p}{\tau_p s + 1}$ → $Y(s)$

$$CLTF_d: \quad \boxed{\frac{Y(s)}{D(s)} = \frac{K_2}{\tau_1 s + 1}} \begin{cases} K_2 = \dfrac{K_d}{1 + K_{OL}} > 0 \\[2mm] \tau_1 = \dfrac{\tau_p}{1 + K_{OL}} < \tau_p \end{cases}$$

P controller
$$G_c(s) = K_c$$

Disturbance rejection: $D(s) = \dfrac{1}{s}$ (unit step)

Off-set:

$$y_{sp} - y(\infty) \rightarrow y_{sp} - \lim_{s \to 0}\left[s\left(\frac{K_2}{s(\tau_1 s + 1)}\right)\right] = 0 - K_2 = \boxed{\frac{-K_d}{1 + K_c K_p}}$$

❖ If Kc > 0 increases, **the dynamics is made faster** and the off-set decreases

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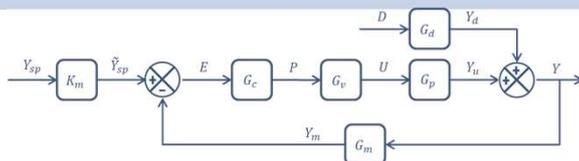## GENERALIZING CLTFs UNDER P CONTROL

P controller
$$G_c(s) = K_c$$

If the only system dynamics are those of the process and of the disturbance, and if they all are FO ($K_p; K_d; \tau_p$), then the CLFTs for a P-only controller are (for a step change of amplitude M):

Servo control:

$$CLTF_{sp}: \boxed{\frac{Y(s)}{Y_{sp}(s)} = \frac{K_1}{\tau_1 s + 1}} \begin{cases} K_1 = \dfrac{K_{OL}}{1 + K_{OL}} < 1 \\ \tau_1 = \dfrac{\tau_p}{1 + K_{OL}} < \tau_p \end{cases}$$

Offset: $\dfrac{M}{1 + K_{OL}}$

The dynamics is still first-order, but it is made faster

Regulatory control:

$$CLTF_d: \boxed{\frac{Y(s)}{D(s)} = \frac{K_2}{\tau_1 s + 1}} \begin{cases} K_2 = \dfrac{K_d}{1 + K_{OL}} > 0 \\ \tau_1 = \dfrac{\tau_p}{1 + K_{OL}} < \tau_p \end{cases}$$

Offset: $\dfrac{-K_d M}{1 + K_{OL}}$

Adapted from Barolo M.

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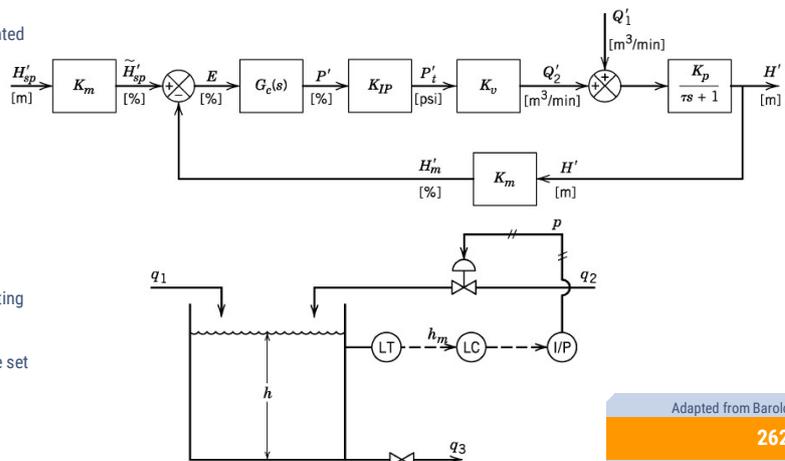
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EXAMPLES

Examples: SEMD 4° Examples 11.2 + 11.3 p.183-184; 3° Examples 11.2 + 11.3 p.191-192

Consider the level control system shown in the figure, implemented with a computer whose inputs and outputs are calibrated in terms of full range (100%). The tank is 1 m in diameter, while the valve on the exit line acts as a linear resistance with R = 6.37 min/m². The level transmitter has a span of 0.5 m and an output range of 0 to 100%. The valve characteristic f of the equal percentage control valve is related to the fraction of lift ℓ by the relation $f(\ell) = (30)^{\ell-1}$. The air-to-open control valve receives a 3 to 15 psi signal from an I/P transducer, which, in turn, receives a 0 to 100% signal from the computer-implemented proportional-only controller. When the control valve is fully open ($f(\ell) = 1$), the flow rate through the valve is 0.2 m³/min. At the nominal operating condition, the control valve is half-open ($f(\ell) = 0.5$).

Calculate the closed-loop responses to a unit step change in the set point for three values of the controller gain: Kc = 1, 2, and 5.

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PI controllers for FO «processes»: servo response

PI controller $G_c(s) = K_c\left(1 + \dfrac{1}{\tau_I s}\right)$

- Overall process: FO system (at open loop)
  - G_p includes the overall loop dynamics and gains (except the controller ones)

$$Y = \frac{G_d}{1 + G_{OL}}D + \frac{K_m G_c G_v G_p}{1 + G_{OL}}Y_{sp}$$

$$CLTF_{sp} = \frac{Y(s)}{Y_{sp}(s)} = \frac{K_m G_c G_v G_p}{1 + G_c G_v G_p G_m} = \frac{\left(K_c \frac{\tau_I s + 1}{\tau_I s}\right)\frac{K_p}{\tau_p s + 1}}{1 + \left(K_c \frac{\tau_I s + 1}{\tau_I s}\right)\frac{K_p}{\tau_p s + 1}} = \frac{\frac{K_c K_p(\tau_I s + 1)}{\tau_I s(\tau_p s + 1)}}{\frac{(\tau_I \tau_p s^2 + \tau_I s) + K_c K_p(\tau_I s + 1)}{\tau_I s(\tau_p s + 1)}} = \frac{K_c K_p(\tau_I s + 1)}{\tau_I \tau_p s^2 + \tau_I(1 + K_c K_p)s + K_c K_p}$$

Remarks:
- ❖ If the loop is closed with a PI controller, the closed-loop response has a FO numerator and a SO denominator → the dynamics of the closed-loop system becomes SO with numerator dynamics
- ❖ The net effect of P-only control was instead to only change the system parameters, while keeping FO the response
- ❖ A PI controller, instead, adds a pole and a zero to the closed-loop system response (for a FO system)

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PI controllers for FO «processes»: servo response /cont'd

PI controller

$G_c(s) = K_c\left(1 + \dfrac{1}{\tau_I s}\right)$

$$CLTF_{sp} = \frac{Y(s)}{Y_{sp}(s)} = \frac{K_c K_p(\tau_I s + 1)}{\tau_I \tau_p s^2 + \tau_I(1 + K_c K_p)s + K_c K_p} = \frac{\tau_I s + 1}{\frac{\tau_I \tau_p}{K_c K_p}s^2 + \frac{\tau_I(1 + K_c K_p)}{K_c K_p}s + 1} = \frac{\tau_I s + 1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$\tau = \sqrt{\frac{\tau_I \tau_p}{K_c K_p}} = \sqrt{\frac{\tau_I \tau_p}{K_{OL}}} \qquad 2\zeta\tau = \frac{\tau_I(1 + K_c K_p)}{K_c K_p} \to \zeta = \frac{1}{2}(1 + K_c K_p)\sqrt{\frac{\tau_I}{\tau_p K_c K_p}} = \frac{1}{2}\left(\frac{1 + K_{OL}}{\sqrt{K_{OL}}}\right)\sqrt{\frac{\tau_I}{\tau_p}}$$

- For $Y_{sp} = M/s$, the response is of this type: $y(t) = A_0 + A_1 e^{p_1 t} + A_2 e^{p_2 t}$
  - The numerical values of the roots p_i depend on the **process** parameters **and** on the **controller** parameters

Remarks:
- ❖ Since the CLTF has one zero, the closed-loop response *may* display an overshoot
- ❖ If the two roots p_1 and p_2 are **real** and **negative**, the response exponentially attains a final value A_0
- ❖ If the two roots p_1 and p_2 are **complex conjugate**, and with **negative real parts**, the response will be **oscillating**, but the oscillations eventually die out (underdamped response)
- ❖ If at least **one** of the roots is **real** and **positive** (or the roots are complex with positive real part), then the response will not settle to a final value, but will diverge (possibly with oscillations)

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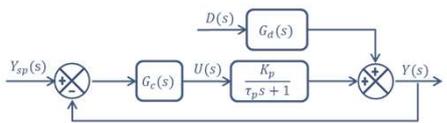© Andrea Mio, University of Trieste

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PI controllers for FO «processes» - Removing the offset

$$G_c(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$$

$$CLTF_{sp} = \frac{Y(s)}{Y_{sp}(s)} = \frac{\tau_I s + 1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$Y_{sp}(s) = \frac{M}{s}$$

What will be the final response value?

$$y(\infty) = \lim_{s\to 0}[sY(s)] = \lim_{s\to 0}\left[s\left(\frac{M}{s} \times \frac{\tau_I s + 1}{\tau^2 s^2 + 2\zeta\tau s + 1}\right)\right] = M$$

The final value is equal to the new set-point M

■ The integral action can eliminate the offset in servo control problems
  ▷ For disturbance rejection problems, the result is similar

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## PI controllers for FO «processes» - Removing the offset /cont'd

PI controller

$$G_c(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$$

■ The integral action can completely compensate for load (or set-point) changes, provided that the change is sustained
  ▷ For example, the integral action cannot compensate for a ramp-like change (of a disturbance or set-point) during the ramp evolution
    ❖ But when the transient has died out, the change can be completely compensated

$$CLTF_{sp} = \frac{Y(s)}{Y_{sp}(s)} = \frac{\tau_I s + 1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

$$\tau = \sqrt{\frac{\tau_I \tau_p}{K_c K_p}} = \sqrt{\frac{\tau_I \tau_p}{K_{OL}}}$$
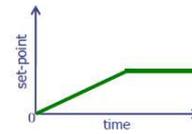
$$\zeta = \frac{1}{2}(1 + K_c K_p)\sqrt{\frac{\tau_I}{\tau_p K_c K_p}} = \frac{1}{2}\left(\frac{1 + K_{OL}}{\sqrt{K_{OL}}}\right)\sqrt{\frac{\tau_I}{\tau_p}}$$

■ Effect of K_c and τ_I
  ▷ If K_c ↑ or τ_I ↓ the characteristic time τ ↓: the response is sped up
  ▷ If τ_I ↓, the damping factor ζ ↓: the response becomes more oscillatory (i.e., less stable)
  ▷ It may seem that the same happens also if K_c ↓
    ❖ But this is due to the fact that the valve and transmitter dynamics have been neglected; if they are accounted for, the response becomes more oscillatory when K_c ↑ (as always happens in practice)

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GENERALIZING CLTFs UNDER PI CONTROL
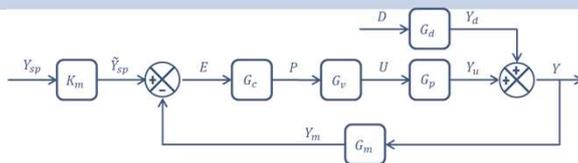
PI controller

$$G_c(s) = K_c\left(1 + \frac{1}{\tau_I s}\right)$$

- Similar consideration hold true also for systems of order greater than 1
- The integral action modifies the order of the system response
- For any dynamics system $G_p(s)$ **with steady-state gain K_p**, the following holds true:
  - ▻ The closed-loop response under **proportional-only** feedback control always shows offset
  - ▻ The **integral** action can eliminate the steady-state offset from the closed-loop response, provided that the disturbance or set-point change are sustained
  - ▻ **Increasing the controller gain K_c** makes the response more oscillatory

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STABILITY OF A CONTROLLED SYSTEM

An unconstrained linear system is said to be stable if the output response is bounded for all bounded inputs. Otherwise, it is said to be unstable

- Most physical systems are **open-loop stable** (self-regulating systems)
  - ▻ They attain a new steady state when a sustained input change is applied
  - ▻ They are «inherently» stable, i.e., they are stable when they are not part of a feedback control loop
- Some real processes are **open-loop unstable**
  - ▻ To be operated, they need a feedback control loop that stabilizes them
    - ❖ Example: some levels (if pure integrators); exothermic reactors with cooling jacket (generated heat vs removed heat)
- Even if a process is **stable at open loop**, not necessarily does it remain stable when the control loop is **closed**

Adapted from Barolo M.

❖ Unconstrained system or «sistema non vincolato»: there are no physical limits on the input and output variables

❖ Bounded signal or «segnale confinato»: the signal remains within upper and lower limits for all values of time
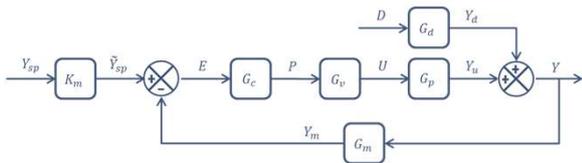
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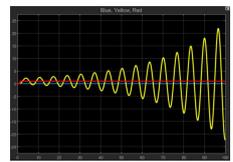
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EFFECT OF THE CONTROLLER ON THE RESPONSE
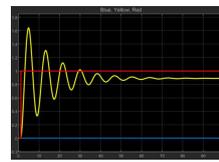
Example: SEMD 4° Ex.11.4 p. 187; 3° Ex. 11.4 p. 195

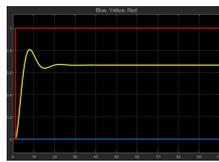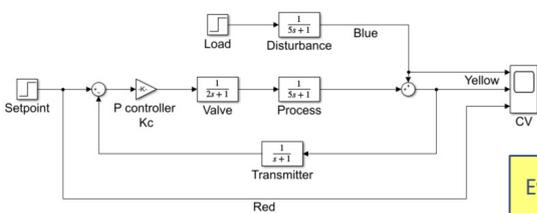$$G_p = G_d = \frac{1}{5s+1} \qquad G_c = K_c$$

$$G_v = \frac{1}{2s+1} \qquad G_m = \frac{1}{s+1}$$

$K_c = 2 \qquad K_c = 8 \qquad K_c = 15$

Even if **P-only** control is used, there exists a limiting value of K_{cu} of the controller gain (ultimate gain o «guadagno critico») beyond which the response becomes **unstable**
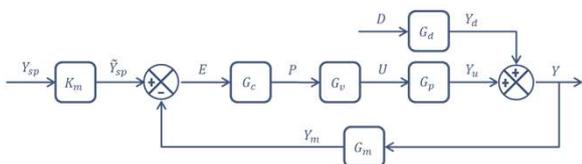
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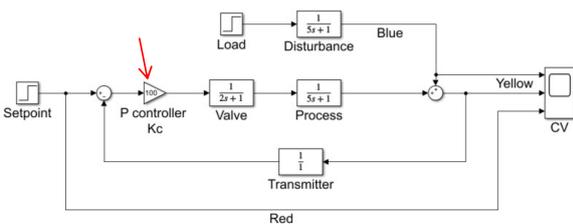

EFFECT OF THE CONTROLLER ON THE RESPONSE /cont'd

Example: SEMD 4° Ex.11.4 p. 187; 3° Ex. 11.4 p. 195

$$G_p = G_d = \frac{1}{5s+1} \qquad G_c = K_c$$

$$G_v = \frac{1}{2s+1} \qquad G_m = 1 \neq \frac{1}{s+1}$$

$K_c = 100$

It looks like that a limiting value for the controller gain does **not** exist for **all** processes

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SIGNAL AMPLIFICATION WITHIN THE LOOP

- For some systems, the signal variation reinforce (amplify) when travel around the loop, **even if P-only control is used**
  - ▻ The amplifications may be such that the **oscillations** become unbounded even for a **bounded** input change
    - ❖ In reality, as a result of physical constraints (example: valve opening/closure; structural resistance of an equipment) the oscillations amplitude will be bounded and/or there will be structural failure of the equipment
- To design the process control system:
  - ▻ We must know whether or not an ultimate gain value exists for a given system
  - ▻ When it exists, we must be able to calculate it

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CHARACTERISTIC EQUATION
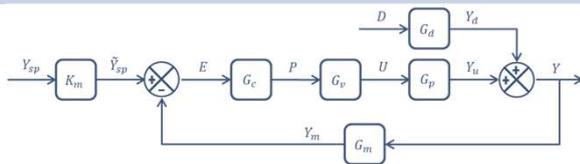
$$1 + G_{OL} = 0$$

$$G_{OL} \triangleq G_c G_v G_p G_m$$

$$Y = \frac{G_d}{1 + G_{OL}} D + \frac{K_m G_c G_v G_p}{1 + G_{OL}} Y_{sp}$$

As an example, consider set-point changes only: $D(s) = 0$

If no repeated poles exist, the CLTF can be factored as:

$$\frac{Y(s)}{Y_{sp}(s)} = \frac{K_m G_c G_v G_p}{1 + G_{OL}} = \frac{b_m (s - z_1)(s - z_2) \dots (s - z_m)}{a_n (s - p_1)(s - p_2) \dots (s - p_n)} \rightarrow$$

$$Y(s) = \frac{numerator\ terms}{a_n (s - p_1)(s - p_2) \dots (s - p_n)(input\ terms)} \rightarrow$$

The input is bounded

$$Y(s) = \frac{A_1}{(s - p_1)} + \frac{A_2}{(s - p_2)} + \dots + \frac{A_n}{(s - p_n)} + (addendum\ from\ input\ terms)$$

Notice.
The poles p_k are the solutions of the characteristic equation of the closed-loop system: $1 + G_{OL} = 0$

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UNFORCED RESPONSE AND FORCED RESPONSE

$$Y(s) = \frac{A_1}{(s - p_1)} + \frac{A_2}{(s - p_2)} + \cdots + \frac{A_n}{(s - p_n)} + (addendum\ from\ input\ terms)$$

$$y(t) = \underbrace{A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t}}_{Unforced\ response} + \underbrace{(addendum\ from\ input\ terms)}_{Forced\ response}$$

- Each term of the **unforced response** includes a root **(pole)** of the characteristic equation (closed-root response)
  - ▷ Since the values of the A_k coefficients depend on the actual forcing input (and on the zeroes of the CLTF), the response **shape** $y(t)$ does **depend on the input**
  - ▷ However, the speed at which $y(t)$ decays or diverges is determined by the roots of the closed-loop characteristic equation only

$$\boxed{1 + G_{OL} = 0}$$

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STABILITY CONDITIONS

- Closed-loop response:

$$y(t) = A_1 e^{p_1 t} + A_2 e^{p_2 t} + \cdots + A_n e^{p_n t} + (addendum\ from\ input\ terms)$$

- **Real poles p_k**
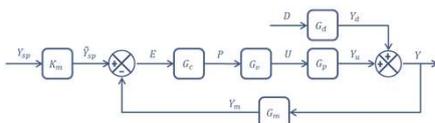 - ▷ Positive
    - ❖ If for any closed-loop pole p_k it results that $p_k > 0 \rightarrow e^{p_k t}$ becomes unbounded $\rightarrow y(t)$ becomes unbounded $\rightarrow$ **the overall response is unstable**
- **Complex poles p_k**
  - ▷ Positive real part
    - ❖ For **any** pole $p_k = \sigma \pm j\omega$, it follows that $e^{p_k t} = e^{(\sigma \pm j\omega)t} = e^{\sigma t} \cdot e^{\pm j\omega t}$
    - ❖ If $\sigma > 0 \rightarrow e^{\sigma t}$ becomes unbounded $\rightarrow$ **the overall response is unbounded** (and also oscillating as a result of $e^{\pm j\omega t}$)

A feedback control system is stable <u>if and only if</u> **all** roots of the closed-loop characteristic equation are **negative** (if real) or have negative real parts (if complex). Otherwise, the system is unstable
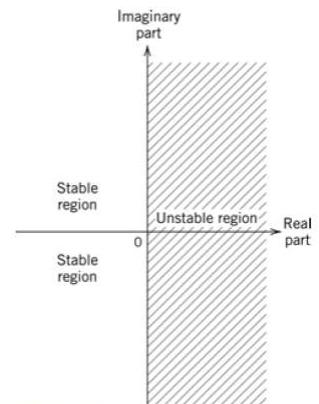
olo M.

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REPRESENTATION IN THE COMPLEX PLANE

$$1 + G_{OL} = 0$$

- The solutions of the closed-loop characteristic equation that lie in the right-half plane give rise to instability conditions in the controlled system
  - ▻ Notice: the poles location depends on G_c!
- The root positions with respect to the imaginary axis indicate how rapid the transient response will be
  - ▻ The (real) solutions $s = a < 0$ corresponds to a closed-loop time constant $\tau = -1/a$
    - ▻ The farther left the root with respect to the imaginary axis, the faster the closed-loop response (small τ)
    - ▻ With reference to time, the dominant responses (slow) are those corresponding to poles close to the imaginary axis.
- The farther away the (complex) solutions from the real axis, the more oscillatory the transient response

(Seborg et al., 2011)

Adapted from Barolo M.

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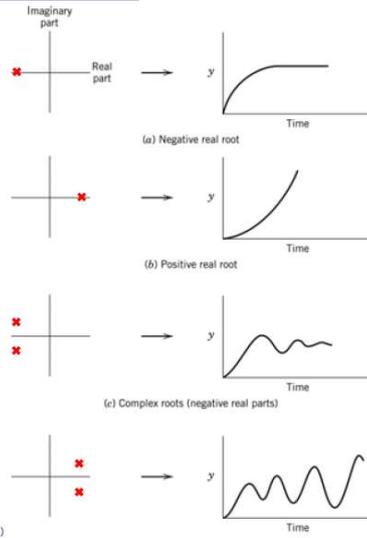© Andrea Mio, University of Trieste

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ROOT LOCATIONS AND CLOSED-LOOP RESPONSE

$$1 + G_{OL} = 0$$

Note that the dynamic elements that are **outside the loop** have **no effect on the stability** of the controlled system!

Final remarks:

- The stability criterion also holds for:
  - ▻ Load disturbances
  - ▻ Repeated poles
- If nonlinear elements (e.g., dead times) are included within G_{OL}, they need to be approximated (e.g., Padè)
  - ▻ Alternatively, a rigorous stability condition can be derived via frequency analysis (SEMD, Chapter 14)

(Seborg et al., 2011)

(a) Negative real root
(b) Positive real root
(c) Complex roots (negative real parts)
(d) Complex roots (positive real parts)

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