RELATIVE GAIN ARRAY (RGA; BRISTOL MATRIX)

- It is the matrix Λ built with the relative gains of **all possible combinations** (pairings) between MVs and CVs
  - Each λ_{n1} element indicates how the gain of the potential loop $y_i \leftarrow u_j$ changes when the other loops are closed
  - Therefore, the array summarizes the (steady-state) characteristics of all possible CV ⟵ MV pairings

$$\begin{array}{cc} & \begin{array}{cccc} u_1 & u_2 & \cdots & u_n \end{array} \\ \Lambda = & \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_{n1} & \lambda_{n2} & \cdots & \lambda_{nn} \end{bmatrix} \end{array} \begin{array}{c} \\ \leftarrow y_1 \\ \leftarrow y_2 \\ \cdots \\ \leftarrow y_n \end{array}$$

- **Properties** (which can be demonstrated)
  - Each row and each column of Λ **sum up to** 1
  - The λ_{ij} values can be calculated in a simple way from the sole values of the open-loop gains ⟶ the calculation is very simple!
    - Schur product (element by element)

$$\Lambda = K_p \otimes \left(K_p^{-1}\right)^T$$

CALCULATION OF RGA FROM A 2X2 STEADY-STATE MODEL

$$\lambda_{ij} \triangleq \frac{\left(\frac{\partial y_i}{\partial u_j}\right)_{u_{i\neq j}}}{\left(\frac{\partial y_i}{\partial u_j}\right)_{y_{j\neq i}}}$$

$$\boxed{\Lambda = \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix}}$$

For a 2x2 system, calculating a single relative gain (e.g., λ_{11}) is sufficient

$$y_1 = K_{11}u_1 + K_{12}u_2$$
$$y_2 = K_{21}u_1 + K_{22}u_2$$
$$y = K_p u$$

Linearized steady-state model (in deviation variables)

How is $\lambda_{11} = \lambda$ calculated? ➡ $\lambda_{11} = \dfrac{\left(\frac{\partial y_1}{\partial u_1}\right)_{u_2=cons}}{\left(\frac{\partial y_1}{\partial u_1}\right)_{y_2=const.}}$

- Numerator: since u_2 remains constant (i.e., the 2° loop is open) ➡ $\left(\frac{\partial y_1}{\partial u_1}\right)_{u_2=const.} = K_{11}$
- Denominator: u_2 varies (the 2° loop is closed), and how this variation affects y_1 should be determined

$y_2 = const$ ➡ $y_2 = 0$ ➡ $u_2 = -\dfrac{K_{21}}{K_{22}}u_1$ ➡ $y_1 = K_{11}u_1 - K_{12}\left(\dfrac{K_{21}}{K_{22}}u_1\right)$ ➡

➡ $\left(\frac{\partial y_1}{\partial u_1}\right)_{y_2=const.} = K_{11} - \dfrac{K_{12}K_{21}}{K_{22}} = K_{11}\left(1 - \dfrac{K_{12}K_{21}}{K_{11}K_{22}}\right)$ ➡ $\boxed{\lambda = \dfrac{1}{1 - \dfrac{K_{12}K_{21}}{K_{11}K_{22}}}}$

PAIRING CVs AND MVs IN 2X2 SYSTEMS

$$\begin{array}{ccc} & u_1 & u_2 & \\ \Lambda = & \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix} & \begin{array}{c} y_1 \\ y_2 \end{array} \end{array}$$

$$\lambda_{ij} = \frac{K_{ij}^{open}}{K_{ij}^{closed}}$$

$$K_{ij}^{closed} = \frac{1}{\lambda_{ij}} K_{ij}^{open}$$

Is the $y_1 \leftarrow u_1$ pairing appropriate? ➡ One needs to determine the value of $\lambda_{11} = \lambda$

- $\lambda = 1$
  - The gain between y_1 and u_1 with loop 2-2 closed is equal to that with loop 2-2 open → opening or closing loop 2-2 has no effect on loop 1-1 → it is convenient to pair y_1 to u_1
- $\lambda = 0$
  - The gain between y_1 and u_1 with loop 2-2 open is zero → u_1 has not effect on y_1 → it is convenient to pair y_1 to u_2
- $0 < \lambda < 1$
  - The gain between y_1 e u_1 with loop 2-2 closed is greater than that with loop 2-2 open → coupling exists between the two loops, and it is maximum for $\lambda = 0.5$
- $\lambda > 1$
  - closing loop 2-2 makes the gain between y_1 and u_1 decrease → coupling exists between the two loops, and it is maximum for $\lambda \to \infty$
    - If $\lambda \gg 1$, when the loop 2-2 is closed the gain of loop 1-1 becomes much smaller than with loop 2-2 open → a controller with high gain $K_{c,11}$ is required → if, at a given time, loop 2-2 gets open, the large value of $K_{c,11}$ may drive loop 1-1 toward instability
- $\lambda < 0$
  - The gain between y_1 and u_1 with loop 2-2 closed has opposite sign with respect to that with loop 2-2 open → opening or closing loop 2-2 has a destabilizing effect on loop 1 (it makes it oscillate) → y_1 must not be paired to u_1

Adapted from Barolo M

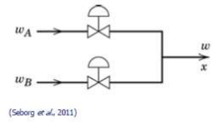
RECOMMENDATION FOR LOOP COUPLING

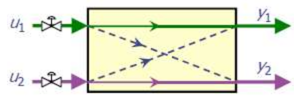
Recommendation
Pair the controlled and manipulated variables so that the corresponding relative gains are **positive** and **as close to one** as possible

- Integrity of a control system
  - A control system possesses integrity if, when one or more loops go out of service (e.g., due to failure or valve saturation), the rest of the closed-loop system remains stable with no need to **change sign** to the gain of any of the controllers remaining in operation
- If a loop (with integral action) pairs CVs and MVs with $\lambda_{ij} < 0$, then one of the following conditions is always met:
  - The multiloop system is unstable when all the controllers are in operation
  - Loop $y_i \leftarrow u_j$ becomes unstable when all the other controllers are set to manual mode
  - The multiloop system becomes unstable when the $y_i \leftarrow u_j$ controller is set to manual

Adapted from Barolo M

APPLICATIONS OF RGA: THE BLENDER REVISITED





- **Problem: controlling both w and x**
  - Consider to limiting situations: $x_{sp} \cong 1$ or $x_{sp} \cong 0$ (x is the mass fraction of A)
    - Which is the best pairing in both limiting cases?

$$\boxed{\begin{array}{l} y_1 = w;\, y_2 = x \\ u_1 = w_A;\ u_2 = w_B \end{array}}$$

$$\begin{array}{ccc} & w_A & w_B \\ \Lambda = & \begin{bmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{bmatrix} & \begin{array}{c} w \\ x \end{array} \end{array}$$

- For a 2x2 system: $\boxed{\lambda = \dfrac{1}{1 - \dfrac{K_{12}K_{21}}{K_{11}K_{22}}}}$ ⇒ Let us find the steady-state gains

Steady-state balances (total + species A)

$$\begin{cases} w_A + w_A = w \\ w_A \cdot 1 + w_B \cdot 0 = w \cdot x \end{cases} \Rightarrow x = \frac{w_A}{w_A + w_B}$$

Calculating the open-loop gains:

$$K_{11} = \left(\frac{\partial y_1}{\partial u_1}\right)_{u_2=cons} = \left(\frac{\partial w}{\partial w_A}\right)_{w_B=const} = 1; \quad K_{12} = \left(\frac{\partial w}{\partial w_B}\right)_{w_A=const} = 1$$

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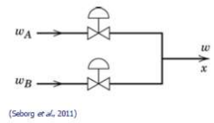
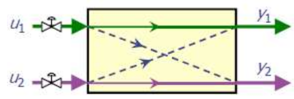
APPLICATIONS OF RGA: THE BLENDER REVISITED /cont'd





Calculating the open-loop gains:

$$K_{21} = \left(\frac{\partial x}{\partial w_A}\right)_{w_B=const} = \frac{1 \cdot (w_A + w_B) - w_A \cdot 1}{(w_A + w_B)^2} = \frac{w_B}{(w_A + w_B)^2} = \frac{(1-x)}{w}$$

$$K_{22} = \left(\frac{\partial x}{\partial w_B}\right)_{w_A=const} = \frac{1 \cdot (w_A + w_B) - w_B \cdot 1}{(w_A + w_B)^2} = \frac{w_A}{(w_A + w_B)^2} = \frac{-x}{w}$$

⇒ $\boxed{\lambda = \dfrac{1}{1 - \dfrac{K_{12}K_{21}}{K_{11}K_{22}}} = \cdots = x_{sp}}$

$$\boxed{\begin{array}{l} y_1 = w;\, y_2 = x \\ u_1 = w_A;\ u_2 = w_B \end{array}}$$

The resulting RGA is: $\Lambda = \begin{bmatrix} x_{sp} & 1 - x_{sp} \\ 1 - x_{sp} & x_{sp} \end{bmatrix}$

$x_{sp} = 0.05$

$$\Lambda = \begin{bmatrix} 0.05 & 0.95 \\ 0.95 & 0.05 \end{bmatrix}$$

$x_{sp} = 0.99$

$$\Lambda = \begin{bmatrix} 0.99 & 0.01 \\ 0.01 & 0.99 \end{bmatrix}$$

- Therefore, the suggested pairing depends on the desired outlet composition x_{sp}
  - If $x_{sp} < 0.5$, then pair $w \leftarrow w_B$ and $x \leftarrow w_A$
    - This is reasonable: if x is small, the mass fraction of A in the product is small, i.e., w_A is small → it is convenient to control the total flow w with the larger stream (w_B), and the composition x with the smaller stream (w_A).
  - If $x_{sp} > 0.5$, then pair $w \leftarrow w_A$ and $x \leftarrow w_B$

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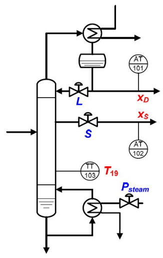
APPLICATIONS OF RGA: 3x3 EXAMPLE BINARY DISTILLATION COLUMN WITH A SIDESTREAM



$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} \dfrac{0.66e^{-2.6s}}{6.7s+1} & \dfrac{-0.61e^{-3.5s}}{8.64s+1} & \dfrac{-0.0049e^{-s}}{9.06s+1} \\ \dfrac{-33.68e^{-9.2s}}{8.15s+1} & \dfrac{46.2e^{-9.4s}}{10.9s+1} & \dfrac{0.87(11.61s+1)e^{-s}}{(3.89s+1)(18.8s+1)} \\ \dfrac{1.11e^{-6.5s}}{3.25s+1} & \dfrac{-2.36e^{-3}}{5s+1} & \dfrac{-0.012e^{-1.2s}}{7.09s+1} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

- y_1 = mole fr. EtOH in distillate = x_D
- y_2 = tray 19 temperature = T_{19}
- y_3 = mole fr. EtOH in sidestream = x_S
- u_1 = reflux flowrate = L
- u_2 = sidestream flowrate = S
- u_3 = reboiler steam pressure = P_{steam}

$$\boxed{\Lambda = K_p \otimes \left(K_p^{-1}\right)^T}$$

$$K_p = G_p(0) = \begin{bmatrix} 0.66 & -0.61 & -0.0049 \\ -33.68 & 46.2 & 0.87 \\ 1.11 & -2.36 & -0.012 \end{bmatrix} \Rightarrow K_p^{-1} = \begin{bmatrix} 2.9476 & 0.0083 & -0.5985 \\ 1.1044 & -0.0049 & -0.8047 \\ 55.466 & 1.7316 & 19.563 \end{bmatrix}$$

$$\Rightarrow \left(K_p^{-1}\right)^T = \begin{bmatrix} 2.9476 & 1.1044 & 55.466 \\ 0.0083 & -0.0049 & 1.7316 \\ -0.5985 & -0.8047 & 19.563 \end{bmatrix}$$

$$\Lambda = K_p \otimes \left(K_p^{-1}\right)^T = \begin{bmatrix} 1.945 & -0.6737 & -0.2718 \\ -0.2811 & -0.2254 & 1.507 \\ -0.6643 & 1.899 & -0.2348 \end{bmatrix}$$

$$\begin{matrix} y_1 \leftarrow u_1 \\ y_3 \leftarrow u_2 \\ y_2 \leftarrow u_3 \end{matrix}$$

Adapted from Barolo M

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BEWARE OF THE DYNAMICS!

- The conventional RGA approach only uses steady-state information
  - The process interactions are evaluated at steady state
  - Any consideration about the system dynamics is neglected

$$G_p(s) = \begin{array}{cc} U_1(s) & U_2(s) \end{array}$$

$$G_p(s) = \begin{bmatrix} \dfrac{2}{10s+1} & \dfrac{2}{s+1} \\ \dfrac{1}{s+1} & \dfrac{-4}{10s+1} \end{bmatrix} \begin{matrix} \leftarrow Y_1(s) \\ \leftarrow Y_2(s) \end{matrix}$$

$$K_p = \begin{bmatrix} 2 & 2 \\ 1 & -4 \end{bmatrix} \Rightarrow \lambda = \frac{1}{1 - \dfrac{K_{12}K_{21}}{K_{11}K_{22}}} = \frac{1}{1 - \dfrac{2 \times 1}{2 \times (-4)}} = 0.8$$

$$\Rightarrow \Lambda = \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 0.8 \end{bmatrix} \Rightarrow \begin{matrix} y_1 \leftarrow u_1 \\ y_2 \leftarrow u_2 \end{matrix}$$

Reasonably, on a steady-state perspective: the gains along the diagonal are on average greater than the off-diagonal ones

- However, the off-diagonal pairing responds much faster
  - When u_1 is used to correct y_1, its action must persist long due to the slow dynamics; instead, the effect of u_1 on y_2 is very fast
    - With the pairing 1-1, u_1 continues to perturb y_2 for a long time, forcing u_2 to react
  - If the $y_1 \leftarrow u_2$ pairing were used instead, when u_2 is manipulated, y_1 gets corrected immediately, and y_2 is perturbed much more slowly; u_1 can bring it back to the set-point rapidly, because its effect on y_2 is immediate

In pairing MVs to CVs, also consider the speed of response of the CVs to the MVs.
Select the pairing that has a «small» effective time constant and a «small» dead time

Adapted from Barolo M

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TUNING MULTILOOP PID CONTROLLERS

- Decentralized control
  - A (PID) loop is in place for each single variable to be controlled
    - In a (n x n) system, n PID loops are present, and each of them needs to be tuned individually

PROS

- The tuning task is easy to understand
- It requires tuning fewer parameters than a centralized (multivariable) controller
  - In multivariable controller, instead, the situation resembles one where each loop manipulates an MV accounting for the errors in all CVs
- The tolerance loop failure (integrity) can be verified easily

CONS

- Loop coupling makes the tuning of each single loop complicated (possibly, very much)

- Most frequently used tuning methods for decentralized controllers
  - Detuning
  - Modified relay auto-tuning

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DECENTRALIZED CONTROLLERS: DETUNING METHOD

- If some loops respond much faster than the others (at least ~5 times faster), then they are tuned first (with the other loops open)
  - The slow loops (e.g., temperature) are then tuned with the fast loops (e.g., flow) closed
- Otherwise, if no «very fast» loops exist:
  - Each loop is tuned with the other ones open
  - Loop coupling is attenuated by detuning each loop
    - Usually, considering the most interacting loops, their controller gains are reduced and their intergal time constants are increased (i.e., tuning is made more conservative), until the response is satisfactory.
  - BLT method
    - Each loop j is tuned with the Ziegler-Nichols parameters ($K_{c,ZN,j}; \tau_{I,ZN,j}$) as derived from $K_{cu,j}$ and $\omega_{u,j}$ (usually determined by the relay method)
    - Then, a single detuning parameter F is adjusted on field, and it is used for all loops

$$K_{c,j} = \frac{K_{c,ZN,j}}{F}; \quad \tau_{I,j} = F \cdot \tau_{I,ZN,j}$$

  - Typical values: $1.5 < F < 4.0$

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MULTIVARIABLE CONTROL

- In multivariable control (centralized control) the value of each MV depends on the error
  - Simplified example: 2x2 system, simple proportional control

$$u_1(t) = K_{c11}e_1(t) + K_{c12}e_2(t)$$
$$u_2(t) = K_{c21}e_1(t) + K_{c22}e_2(t)$$

  - In the control laws, model-based terms are included

- Two examples of multivariable **model-based controllers** are:
  - Decouplers
  - Predictive controllers (**MPC** - model predictive control; cfr. SEMD, 3° Ed, Chapter 20, available for download from the textbook website)

Adapted from Barolo M

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DECOUPLERS

- Decoupling control is a technique used to reduce the **control-loop interactions**
  - Decouplers are controllers that are added to a conventional multiloop configuration

- It is a **model-based** control technique
  - The model may be steady state or dynamic
  - With a perfect dynamic model, the interactions are totally removed

- Limitations
  - The model is never perfect → the interactions can never be eliminated completely
  - In the control law, **model inverse terms** appear, which may make the controller unrealizable
  - The tuning is more complex than in a simple multiloop system
  - **Completely** removing the interactions may actually **reduce the overall performance** of the control system
    - **Totally** removing the control-loop interactions means making the effects of process interactions not visible anymore
    - However, the process interactions can «help» process control: in fact, controlling one single CV might be sufficient to keep also the other CVs close to their set-points
    - Sometimes, it may be convenient to achieve one-way decoupling only

Adapted from Barolo M

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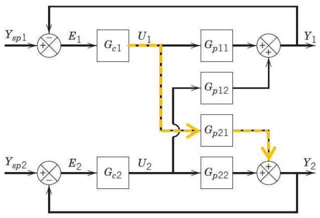
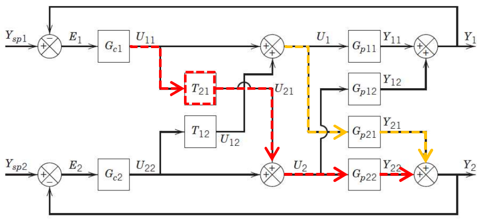
DECOUPLING IN A 2x2 SYSTEM

- Let us assume that the pairings is $y_1 \leftarrow u_1$ and $y_2 \leftarrow u_2$
- A rationale for decoupling loop 2-2 from loop 1-1 may be the following:
  - When the CO of G_{c1} changes, not only Y_1 but also Y_2 is affected; namely, Y_2 gest affected through G_{p21} (process interaction)
  - Basic idea: including in the system a dynamic element $T_{21}(s)$ that gets excited by the CO of G_{c1}, so that an effect is obtained on Y_2 that is equal and contrary to the «natural» one deriving from the process interactions
    - Therefore, these interactions can be completely compensated for





- To decouple the 1-1 loop from the 2-2 one, the rationale is the same
- 4 controllers result: 2 feedback ones (G_{c1}, G_{c2}) and 2 decouplers (T_{12}, T_{21})

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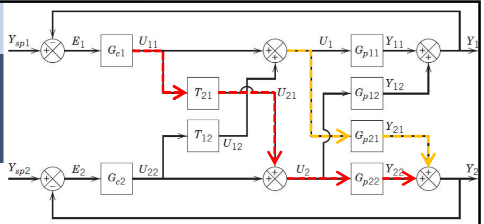
DESIGN OF A $T_{21}(s)$ DECOUPLER



- The two feedback COs are U_{11} and U_{22}
- The two signals that actually arrive to the process are: $\Rightarrow$ $U_1 = U_{11} + T_{12}U_{22}$, $U_2 = U_{22} + T_{21}U_{11}$
- We would like to eliminate the effect of G_{c1} from Y_2
- The Y_2 output is determined by:

$$Y_2 = Y_{21} + Y_{22} \Rightarrow Y_2 = G_{p21}U_1 + G_{p22}U_2 \Rightarrow$$

$$\Rightarrow Y_2 = G_{p21}(U_{11} + T_{12}U_{22}) + G_{p22}(U_{22} + T_{21}U_{11}) \Rightarrow Y_2 = (G_{p21} + G_{p22}T_{21})U_{11} + (G_{p21}T_{12} + G_{p22})U_{22} \Rightarrow$$

  - If it is required to remove the effect of controller G_{c1} from Y_2, the term containing U_{11} must vanish $\Rightarrow$

$$\Rightarrow G_{p21} + G_{p22}T_{21} = 0 \Rightarrow \boxed{T_{21} = -\frac{G_{p21}}{G_{p22}}} \qquad \boxed{T_{12} = -\frac{G_{p1}}{G_{p1}}}$$

- A sort of feedforward control $G_f = -\frac{G_d}{G_p}$
  - The «disturbance» is the off-diagonal term (2,1), which determines the process interaction
  - The «process» is the diagonal element (2,2) which determines the «primary» effect on Y_2 from U_{22}

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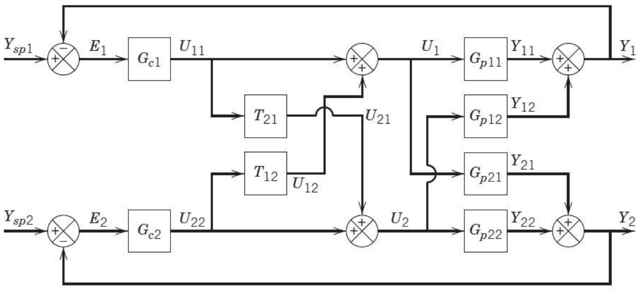
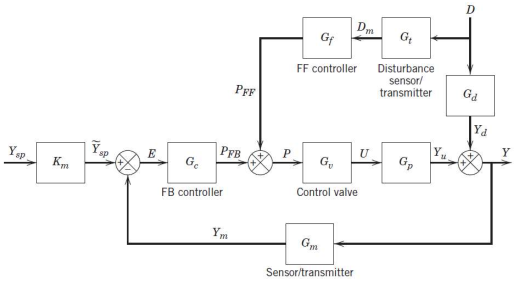
RELATION BETWEEN DECOUPLING AND FEEDFORWARD





$$T_{12}(s) = -\frac{G_{p12}(s)}{G_{p11}(s)}; \qquad T_{21} = -\frac{G_{p21}}{G_{p22}}$$

$$G_f(s) = -\frac{G_d(s)}{G_p(s)}$$

- Differently from a feedforward controller, the decoupler is an integral part of the feedback loop

Adapted from Barolo M