1 Background on Sobolev spaces

1.1 Weak derivatives

Let Ω be an open subset of $\mathbb{R}^n$. Let $u \in C^k(\Omega)$ and φ a generic **infinitely differentiable function with compact support**, that is, $\varphi \in C_c^\infty(\Omega)$. The integration by parts formula leads to

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega \varphi D^\alpha u dx \tag{1}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index of order k, that is, $|\alpha| = \alpha_1 + ... + \alpha_n = k$, and where

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

The left hand side of relation (1) still makes sense if we only assume u to be **locally integrable** $u \in L^1_{loc}(\Omega)$, that is, its integral is finite on every compact subset of its domain of definition (clearly $C^k(\Omega) \subset L^1_{loc}(\Omega)$). This is true because $D^\alpha \varphi$ has a compact support. Indeed there is an equivalent definition: $u \in L^1_{loc}(\Omega)$ if

$$\int_\Omega |u\psi| \, dx < \infty$$

for each $\psi \in C_c^\infty(\Omega)$.

The extension of the left hand side of (1) from $C^k(\Omega)$ to $L^1_{loc}(\Omega)$ allow to define the weak derivative. If there exists a locally integrable function v, such that

$$\int_\Omega u D^\alpha \varphi dx = (-1)^{|\alpha|} \int_\Omega \varphi v dx, \quad \varphi \in C_c^\infty(\Omega), \tag{2}$$

then v is called the **weak α-th partial derivative** of u, and we write $v = D^\alpha u$. If there exists a weak α-th partial derivative of u, then it is uniquely defined almost everywhere, and thus it is uniquely determined as an element of $L^1_{loc}(\Omega)$. This means that if $w = v$ a.e. then w is also a weak α-th partial derivative of u. On the other hand, if $u \in C^k(\Omega)$, then the classical and the weak derivative coincide.

For example, the function

$$u(x) = \begin{cases} 1 + x & -1 < x < 0 \\ 2 & x = 0 \\ 1 - x & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases} \quad \in L^1_{loc}(\mathbb{R})$$

is not continuous at 0, and not differentiable at $-1, 0, 1$. Anyway the function

$$v(x) = \begin{cases} 1 & -1 < x < 0 \\ -1 & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

satisfies the definition of weak derivative of u.

1.2 Sobolev spaces in one dimension

In the one-dimensional case the **Sobolev space** $W^{k,p}(\Omega)$ for $1 \leq p \leq \infty$, $\Omega \subset \mathbb{R}$, is the subset of $L^p(\Omega)$ containing the functions f such that $f^{(i)} \in L^p(\Omega)$ for $i = 0, ..., k$, where the $f^{(i)}$ denotes the i-th weak derivative, that is,

$$W^{k,p}(\Omega) = \left\{ f : \int_\Omega \left| f^{(i)} \right|^p dx < \infty, i = 0, ..., k \right\}.$$

Clearly $W^{0,p}(\Omega) = L^p(\Omega)$.

Let $p < \infty$. Denoting by $\|\cdot\|_p$ the L^p norm, that is

$$\|f\|_p = \left(\int_\Omega |f|^p \, dx \right)^{1/p},$$

by definition the Sobolev space $W^{k,p}(\Omega)$ admits the natural norm

$$\|f\|_{W^{k,p}} = \left(\sum_{i=0}^k \left\| f^{(i)} \right\|_p^p \right)^{1/p} = \left(\sum_{i=0}^k \int_\Omega \left| f^{(i)} \right|^p dx \right)^{1/p} \tag{3}$$

For $p = \infty$, the natural norm is

$$\|f\|_{W^{k,\infty}} = \max_{i=0,...,k} \left\| f^{(i)} \right\|_\infty$$

Using these norm, $W^{k,p}(\Omega)$ assumes the stucture of Banach space.

In the case of $p = 2$ the notation

$$H^k(\Omega) = W^{k,2}(\Omega)$$

is commonly used. We then have $H^0(\Omega) = W^{0,2}(\Omega) = L^2(\Omega)$. It can be shown that the functional

$$\langle u, v \rangle_{H^k} = \sum_{i=0}^k \left\langle u^{(i)}, v^{(i)} \right\rangle_{L^2} = \sum_{i=0}^k \int_\Omega u^{(i)} v^{(i)} dx$$

defines an inner product so that H^k assumes the structure of Hilbert space. The induced norm is

$$\|u\|_{H^k} = \left(\sum_{i=0}^k \left\| u^{(i)} \right\|_2^2 \right)^{1/2} = \left(\sum_{i=0}^k \int_\Omega \left| u^{(i)} \right|^2 dx \right)^{1/2}$$

1.3 Multidimensional Sobolev spaces

Let $k \in \mathbb{N}$ and $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ is defined to be the set of all functions f on Ω such that for every multi-index α with $|a| \leq k$ the partial derivative

$$f^{(\alpha)} = D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$$

exists in the weak sense and belongs to $L^p(\Omega)$, that is, $\left\|f^{(\alpha)}\right\|_p < \infty$. Formally,

$$W^{k,p}(\Omega) = \left\{ f \in L^p(\Omega) : f^{(\alpha)} \in L^p(\Omega), \forall \left|a\right| \leq k \right\}$$

The functional

$$\|f\|_{W^{k,p}} = \begin{cases} \left(\sum_{|a|\leq k} \left\|f^{(\alpha)}\right\|_p^p\right)^{1/p} & 1 \leq p < \infty \\ \max_{|a|\leq k} \left\|f^{(\alpha)}\right\|_\infty & p = \infty \end{cases}$$

defines the most commonly used norm for these spaces. With respect to this norm $W^{k,p}(\Omega)$ is a Banach space. Conventionally, $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$ and it is a Hilbert space with inner product

$$\langle u, v\rangle_{H^k} = \sum\nolimits_{|a|\leq k} \int_\Omega D^\alpha u D^\alpha v dx$$

and norm

$$\|u\|_{H^k} = \left(\langle u, u\rangle_{H^k}\right)^{1/2} = \left(\sum\nolimits_{|a|\leq k} \int_\Omega \left|D^\alpha u\right|^2 dx\right)^{1/2}$$

In what follows we also make use of the seminorm

$$\lfloor u \rfloor_{H^k} = \left(\sum\nolimits_{|a|= k} \int_\Omega \left|D^\alpha u\right|^2 dx\right)^{1/2} \tag{4}$$

in which we only consider the higher order derivatives.

Theorem 1 *Let $p < \infty$ and assume that Ω is open. For each $f \in W^{k,p}(\Omega)$ there exists a sequence of functions $f_m \in C^\infty(\Omega)$ such that*

$$\|f_m - f\|_{W^{k,p}} \to 0$$

The above result states that $C^\infty(\Omega)$ is dense in $W^{k,p}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p}}$.

1.4 Functions vanishing at the boundary

The Sobolev space $H^1(\Omega) = W^{1,2}(\Omega)$ is a Hilbert space that has a very important subspace in the context of the solution of partial differential equations. $H_0^1(\Omega)$ is defined as the closure in $H^1(\Omega)$ of the set of the infinitely differentiable functions compactly supported in Ω, denoted by $C_c^\infty(\Omega)$. The closure is intended with respect to the norm

$$\|f\|_{H^1} = \left(\|f\|_2^2 + \|\nabla f\|_2^2\right)^{1/2}$$

where

$$\|\nabla f\|_2^2 = \int_\Omega |\nabla f|^2 = \int_\Omega \sum\nolimits_{i=0}^n \left(\frac{\partial f}{\partial x_i}\right)^2 dx$$

Definition 2 *(informal) Ω is a Lipschitz domain (or domain with Lipschitz boundary) if its boudary can be locally interpreted as the graph of a Lipschitz continuous function.*

When Ω is a Lipschitz domain, $H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$ that vanish at the boundary in the sense of traces that we describe below. For $n=1$, if $\Omega=(a,b)$ is bounded, then $H^1(a,b)\subset C^0([a,b])$ and $H_0^1(a,b)$ consists of continuous functions on $[a,b]$ of the form

$$f(x)=\int_a^x f'(t)dt,$$

where $f'\in L^2(a,b)$ is the weak derivative such that

$$\int_a^b f'(t)dt=0,$$

so that $f(a)=f(b)=0$.

Theorem 3 ***(Poincaré inequality)*** *If Ω is a bounded then there exists a constant $C=C(\Omega)$ such that*

$$\int_\Omega |f|^2 \le C^2 \int_\Omega |\nabla f|^2\,, \quad f\in H_0^1(\Omega) \tag{5}$$

that is

$$\|f\|_2 \le C\,\|\nabla f\|_2$$

Corollary 4 *If Ω is bounded then the Poincaré inequality allows to show that the seminorm (see (4))*

$$\lfloor f\rfloor_{H^1} := \|\nabla f\|_2$$

is actually a norm on $H_0^1(\Omega)$, equivalent to $\|\cdot\|_{H^1}$.

Proof. By (5) we have that

$$C^2\,\|\nabla f\|_2^2+\|\nabla f\|_2^2 \ge \|f\|_2^2+\|\nabla f\|_2^2$$

(simply adding $\|\nabla f\|_2^2$ to both sides), which means

$$\|f\|_{H^1} \le (1+C^2)^{1/2}\,\lfloor f\rfloor_{H^1}$$

On the other side, the inequality $\lfloor f\rfloor_{H^1} \le \|f\|_{H^1}$ is obvious so that we have proved the equivalence and the fact that $\lfloor\cdot\rfloor_{H^1}$ is actually a norm. ■

1.5 Further regularity results

We have already seen that for $n = 1$, if $\Omega = (a, b)$ is bounded, then $H^1(a, b) \subset C^0([a, b])$, because we can write

$$f(x) = \int_a^x f'(t)dt, \quad f \in H^1(a, b)$$

where f' is the weak derivative. This result cannot be extended in dimension $n \geq 2$.

Proposition 5 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\overline{\Omega}$ its closure. Then*

$$H^m(\Omega) \subset C^0(\overline{\Omega}) \quad \textit{for } m > n/2$$

In particular $H^2(\Omega) \subset C^0(\overline{\Omega})$ for $n \leq 3$.

Proposition 6 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $\overline{\Omega}$ its closure. Then for any $k \geq 0$*

$$H^m(\Omega) \subset C^k(\overline{\Omega}) \quad \textit{for } m > k + n/2$$

Example: in two dimensions $H^3(\Omega) \subset C^1(\overline{\Omega})$

1.6 Trace operator

When working with a continuous function v on $\overline{\Omega}$, where Ω is bounded and open in $\mathbb{R}^n$, with boundary $\partial\Omega$, the restriction $v_{|\partial\Omega}$ is well defined and continuous. In this sense, the meaning of $v_{|\partial\Omega} = 0$ is clear. On the other side, if $v \in H^1(\Omega)$ then we do not know its behavior on the boundary. So for any $x \in \partial\Omega$, $v(x)$ should be defined as the limit of $v(x_n)$ for $x_n \to x$. Unfortunately, since v may be not continuous, it may happen that for some x we obtain different values of $v(x)$ approaching x from different directions. In order to solve this ambiguity we need the definition of trace.

Theorem 7 ***(Trace theorem)*** *Let Ω be a bounded Lipschitz domain. Let $C^\infty(\overline{\Omega})$ the set of the infinitely differentiable functions in $\overline{\Omega}$. Then $C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$ and the operator*

$$\gamma_0 : v \in C^\infty(\overline{\Omega}) \to \gamma_0 v = v_{|\partial\Omega} \in C^0(\partial\Omega)$$

can be continuously extended to a linear operator

$$\gamma_0 : H^1(\Omega) \to L^2(\partial\Omega).$$

γ_0 is called **trace operator** and $\gamma_0 v = v_{|\partial\Omega}$ is the **trace**. The above theorem can be used to prove the following results.

Proposition 8 *Let Ω be a bounded Lipschitz domain. Then*

$$H_0^1(\Omega) = Ker(\gamma_0)$$

In other words, for bounded Lipschitz domains we can write

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v_{|\partial\Omega} = 0\}$$

where $v_{|\partial\Omega}$ is intended in the sense of the trace. We remark that in one dimension $H^1(a,b) \subset C^0([a,b])$, so that we don't need the concept of trace.

In the applications sometimes we may have conditions only on subsets of the boundary. In this view, if Γ is a portion of $\partial\Omega$ of measure greater than 0, it is possible to consider the trace

$$\gamma_\Gamma : H^1(\Omega) \to L^2(\Gamma)$$

and the space

$$H_\Gamma^1(\Omega) = \{v \in H^1(\Omega) : v_{|\Gamma} = \gamma_\Gamma v = 0\}$$

Poincaré inequality holds also in $H_\Gamma^1(\Omega)$. The space of functions that are traces on $\Gamma \subseteq \partial\Omega$ of functions in $H^1(\Omega)$ is denoted by $H^{1/2}(\Gamma)$.

1.7 Green formula

Proposition 9 ***(Green formula)*** *Let Ω be a bounded Lipschitz domain and let $\mathbf{n} = \mathbf{n}(x)$, $x \in \partial\Omega$ be the unit vector orthogonal to $\partial\Omega$ in x, directed outwards. Then*

$$\int_\Omega \frac{\partial u}{\partial x_i} v dx = -\int_\Omega u \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} uv\mathbf{n}_i d\gamma, \quad u, v \in H^1(\Omega)$$

where $d\gamma$ denotes the arclength element.

The result can be proved by observing that the formula holds for $u, v \in C^\infty(\overline{\Omega})$ and then using the density stated by the above theorem. It is closely related to (2).

Let

$$\frac{\partial u}{\partial \mathbf{n}} = \sum_{i=1}^n \frac{\partial u}{\partial x_i} \mathbf{n}_i$$

be the derivative along $\mathbf{n}$, and assume that u is regular enough. Then we also have

$$\int_\Omega \frac{\partial^2 u}{\partial x_i^2} v dx = -\int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_{\partial\Omega} \frac{\partial u}{\partial x_i} v\mathbf{n}_i d\gamma, \quad v \in H^1(\Omega). \tag{6}$$

This relation allows to obtain the **Green formula for the Laplacian operator**

$$\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$$

that is

$$\int_\Omega (-\Delta u) v dx = \int_\Omega \nabla u \cdot \nabla v \ dx - \int_{\partial\Omega} \frac{\partial u}{\partial \mathbf{n}} v d\gamma$$

that follows from (6) by taking the sum.