Notes of the course

Advanced Geometry 3

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Lesson 1. Affine algebraic sets and Zariski topology.

The aim of this course is to introduce the notion of algebraic variety in the classical sense, over a field K.

Roughly speaking, algebraic varieties are sets of solutions of a system of algebraic equations, i.e. equations given by polynomials. The natural space where to look at these solutions seems to be the affine space, but one realizes that the projective ambient is more convenient. On one hand the projective space extends the affine space and includes it naturally, on the other the projective ambient allows to prove more general and complete results.

After introducing the notions of affine and projective varieties, we will study the notion of dimensions. Then we will introduce two kinds of transformations of algebraic varieties: regular and rational maps. They give rise to two types of equivalence or isomorphism: biregular isomorphism and birational equivalence, and therefore to two classification problems.

In this course we will see many examples of varieties, and of regular and rational maps. In particular we will see some classes of varieties related to the notion of tensor (without symmetries, symmetric, skew-symmetric); they are much studied because of many recent applications in fields as control theory, signal transmission, etc. We will see also examples of rational and unirational varieties, and we will have a taste of the modern classification problems. We will then study the notions of tangent space, and of smoothness.

Classical algebraic geometry is the basis and gives the motivations for modern algebraic geometry: from schemes, introduced by Grothendieck in the sixties of last century, to the stacks, due to Mumford and Artin. All these notions are strongly based on commutative algebra, i.e. the theory of commutative rings, in particular polynomial rings and their quotients, local rings, and homological algebra.

The reference books I've chosen, all of which have become classics, have different flavours: the book of Šafarevič is complete and precise, and contains almost all algebraic notions needed; Harris' book has a more geometric flavour, proofs are not complete but there are many many examples and ideas; Harthshorne's book, the "Bible" of algebraic geometry since its appearance, treats classical varieties quickly in the first chapter, then moves to modern language, but always with an eye to classical problems.

1. Affine and projective spaces.

In this first section, we begin by fixing the ambient in which we will work: the affine and the projective space over any field K. In particular we recall some basic facts about the projective space.

Let K be a field. For us the *affine space* of dimension n over K will simply be the set K^n : on it, the additive group of K^n acts naturally by translation. The affine space will be denoted by $\mathbb{A}^n_K$ or simply $\mathbb{A}^n$. So the points of $\mathbb{A}^n_K$ are $n-$tuples $(a_1, \dots, a_n)$, where $a_i \in K$ for $i = 1, \dots, n$.

Let V be a $K-$vector space of dimension $n+1$. Let $V^* = V \setminus \{0\}$ be the subset of non–zero vectors. The following relation in V^* is an equivalence relation (relation of proportionality):

$v \sim v'$ if and only if $\exists \lambda \neq 0, \lambda \in K$ such that $v' = \lambda v$.

The quotient set $V^*/\sim$ is called the *projective space* associated to V and is denoted by $\mathbb{P}(V)$. The points of $\mathbb{P}(V)$ are the lines of V (through the origin) deprived of the origin. In particular, $\mathbb{P}(K^{n+1})$ is denoted by $\mathbb{P}^n_K$ (or simply $\mathbb{P}^n$) and called the *numerical projective n-space*. By definition, the dimension of $\mathbb{P}(V)$ is equal to $\dim V - 1$.

There is a canonical surjection $p : V^* \to \mathbb{P}(V)$ which takes a vector v to its equivalence class $[v]$. If $(x_0, \dots, x_n) \in (K^{n+1})^*$, then the corresponding point of $\mathbb{P}^n$ is denoted by $[x_0, \dots, x_n]$. So $[x_0, \dots, x_n]$=$[x'_0, \dots, x'_n]$ if and only if $\exists \lambda \in K^*$ such that $x'_0 = \lambda x_0, \dots, x'_n = \lambda x_n$.

If a basis $e_0, \dots, e_n$ of V is fixed, then a system of *homogeneous coordinates* is introduced in V, in the following way: if $v = x_0 e_0 + \dots + x_n e_n$, then $x_0, \dots, x_n$ are called homogeneous coordinates of the corresponding point P =$[v]$= $p(v)$ in $\mathbb{P}(V)$. We also write $P[x_0, \dots, x_n]$. Note that homogeneous coordinates of a point P are not uniquely determined by P, but are defined only up to multiplication by a non–zero constant. If $\dim V = n + 1$, a system of homogeneous coordinates allows to define a *bijection*

$$\mathbb{P}(V) \longrightarrow \mathbb{P}^n$$

$$P = [v] \longrightarrow [x_0, \dots, x_n]$$

where $v = x_0 e_0 + \dots + x_n e_n$.

The points $E_0[1, 0, \dots, 0], \dots, E_n[0, 0, \dots, 1]$ are the fundamental points, and $U[1, \dots, 1]$ the unit point of the given system of coordinates.

A *projective* (or *linear*) *subspace* of $\mathbb{P}(V)$ is a subset of the form $\mathbb{P}(W)$, where $W \subset V$ is a subspace.

If W, U are vector subspaces of V, the following *Grassmann relation* holds:

$$\dim U + \dim W = \dim(U \cap W) + \dim(U + W).$$

From this relation, observing that $\mathbb{P}(U \cap W) = \mathbb{P}(U) \cap \mathbb{P}(W)$, we get in $\mathbb{P}(V)$:

$$\dim \mathbb{P}(U) + \dim \mathbb{P}(W) = \dim(\mathbb{P}(U) \cap \mathbb{P}(W)) + \dim \mathbb{P}(U + W).$$

Note that $\mathbb{P}(U+W)$ is the minimal linear subspace of $\mathbb{P}(V)$ containing both $\mathbb{P}(U)$ and $\mathbb{P}(W)$: it is denoted $\mathbb{P}(U) + \mathbb{P}(W)$.

1.1. Example. Let $V = K^3$, $\mathbb{P}(V) = \mathbb{P}^2$, $U, W \subset K^3$ subspaces of dimension 2. Then $\mathbb{P}(U), \mathbb{P}(V)$ are lines in the projective plane. There are two cases:
(i) $U = W = U + W = U \cap W$;
(ii) $U \neq W$, $\dim U \cap W = 1$, $U + W = K^3$.
In case (i) the two lines in $\mathbb{P}^2$ coincide; in case (ii) $\mathbb{P}(U) \cap \mathbb{P}(W) = \mathbb{P}(U \cap W) = [v]$, if $v \neq 0$ is a vector generating $U \cap W$. Observe that *never* $\mathbb{P}(U) \cap \mathbb{P}(W) = \emptyset$.

What are the possible reciprocal positions in $\mathbb{P}^3$ for two lines? For two planes? For a line and a plane?

Let $T \subset \mathbb{P}(V)$ be a non–empty set. The linear span $\langle T \rangle$ of T is the intersection of the projective subspaces of $\mathbb{P}(V)$ containing T, i.e. the minimum subspace containing T.

For example, assume that $T = \{P_1, \dots, P_t\}$ is a finite set, and that $v_1, \dots, v_t$ are vectors such that $P_1 = [v_1], \dots, P_t = [v_t]$. Then $\langle P_1, \dots, P_t \rangle = \mathbb{P}(W)$, where W is the vector subspace of V generated by $v_1, \dots, v_t$.

So $\dim\langle P_1, \dots, P_t \rangle \leq t - 1$ and equality holds if and only if $v_1, \dots, v_t$ are linearly independent; in this case, also the points $P_1, \dots, P_t$ are called *linearly independent*. In particular, for $t = 2$, two points are linearly independent if they generate a line, for $t = 3$, three points are linearly independent if they generate a plane, etc. It is clear that, if $P_1, \dots, P_t$ are linearly independent, then $t \leq n + 1$, and any subset of $\{P_1, \dots, P_t\}$ is formed by linearly independent points.

$P_1, \dots, P_t$ are said to be *in general position* if either $t \leq n + 1$ and they are linearly independent or $t > n + 1$ and any $n + 1$ points among them are linearly independent.

1.2. Proposition. *The fundamental points $E_0, \dots, E_n$ and the unit point U of a system of homogeneous coordinates on $\mathbb{P}^n$ are $n + 2$ points in general position. Conversely, if $P_0, \dots, P_n, P_{n+1}$ are $n + 2$ points in general position, then there exists a system of homogeneous coordinates in which $P_0, \dots, P_n$ are the fundamental points and P_{n+1} is the unit point.*

Proof. The proof is linear algebra.

If $e_0, \ldots, e_n$ is a basis, then clearly the $n+1$ vectors $e_0, \ldots, \hat{e_i}, \ldots, e_n, e_0 + \ldots + e_n$ are linearly independent: this proves the first claim. To prove the second claim, we fix vectors $v_0, \ldots, v_{n+1}$ such that $P_i = [v_i]$ for all i. So $v_0, \ldots, v_n$ is a basis and there exist $\lambda_0, \ldots, \lambda_n$ in K such that $v_{n+1} = \lambda_0 v_0 + \ldots + \lambda_n v_n$. The assumption of general position easily implies that $\lambda_0, \ldots, \lambda_n$ are all different from 0, hence $\lambda_0 v_0, \ldots, \lambda_n v_n$ is a new basis such $[\lambda_i v_i] = P_i$ and P_{n+1} is the corresponding unit point. □

b) Embedding of the affine in the projective space

Let $H_0 = \langle E_1, \ldots, E_n \rangle, H_1 = \langle E_0, E_2, \ldots, E_n \rangle, \ldots, H_n = \langle E_0, \ldots, E_{n-1} \rangle$ be $n+1$ hyperplanes in $\mathbb{P}^n$ (subspaces of codmension 1). Note that H_i is simply defined by the equation $x_i = 0$. These hyperplanes are called the *fundamental hyperplanes*.

Let $U_i = \mathbb{P}^n \setminus H_i = \{P[x_0, \ldots, x_n] \mid x_i \neq 0\}$. Note that $\mathbb{P}^n = U_0 \cup U_1 \cup \ldots \cup U_n$, because no point in $\mathbb{P}^n$ has all coordinates equal to zero.

There is a map $\phi_0 : U_0 \longrightarrow \mathbb{A}^n (= K^n)$ defined by

$$\phi_0([x_0, \ldots, x_n]) = (\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}).$$

ϕ_0 is bijective and the inverse map is $j_0 : \mathbb{A}^n \longrightarrow U_0$ such that $j_0(y_1, \ldots, y_n) = [1, y_1, \ldots, y_n]$.

So ϕ_0 and j_0 establish a bijection between the affine space $\mathbb{A}^n$ and the subset U_0 of the projective space $\mathbb{P}^n$. There are other similar maps ϕ_i and j_i for $i = 1, \ldots, n$. So $\mathbb{P}^n$ is covered by $n+1$ subsets, each one in natural bijection with $\mathbb{A}^n$.

There is a natural way of thinking of $\mathbb{P}^n$ as a completion of $\mathbb{A}^n$; this is done by identifying $\mathbb{A}^n$ with U_i via ϕ_i, and by interpreting the points of $H_i (= \mathbb{P}^n \setminus U_i)$ as *points at infinity* of $\mathbb{A}^n$, or directions in $\mathbb{A}^n$. We do this explicity for $i = 0$. First of all we identify $\mathbb{A}^n$ with U_0 via ϕ_0 and j_0. So if $P[a_0, \ldots, a_n] \in \mathbb{P}^n$, either $a_0 \neq 0$ and $P \in \mathbb{A}^n$, or $a_0 = 0$ and $P[0, a_1, \ldots, a_n] \notin \mathbb{A}^n$. Then we consider in $\mathbb{A}^n$ the line L, passing through $O(0, \ldots, 0)$ and of direction given by the vector $(a_1, \ldots, a_n)$. Parametric equations for L are the following:

$$\begin{cases} x_1 = a_1 t \\ x_2 = a_2 t \\ \ldots \\ x_n = a_n t \end{cases}$$

with $t \in K$. The points of L are identified with points of U_0 (via j_0) with homogeneous coordinates $x_0, \ldots, x_n$ given by:

$$\begin{cases} x_0 = 1 \\ x_1 = a_1 t \\ x_2 = a_2 t \\ \ldots \end{cases}$$

or equivalently, if $t \neq 0$, by:

$$\begin{cases} x_0 = \frac{1}{t} \\ x_1 = a_1 \\ x_2 = a_2 \\ \dots \end{cases}.$$

Now, roughly speaking, if t *tends to infinity*, this point goes to $P[0, a_1, \dots, a_n]$. Clearly this is not a rigorous argument, but just a hint to the intuition.

In this way $\mathbb{P}^n$ can be interpreted as $\mathbb{A}^n$ with the points at infinity added, each point at infinity corresponding to one direction in $\mathbb{A}^n$.

Exercise 1.

Let V be a vector space of finite dimension over a field K. Let $\check{V}$ denote the dual of V, i.e. the space of linear forms (or functionals) on V. Prove that $\mathbb{P}(\check{V})$ can be put in bijection with the set of the hyperplanes of $\mathbb{P}(V)$ (hint: the kernel of a non-zero linear form on V is a subvector space of V of codimension one).

2. Algebraic sets.

Roughly speaking, algebraic subsets of the affine or of the projective space are sets of solutions of systems of algebraic equations, i.e. common roots of sets of polynomials.

Examples of algebraic sets are: linear subspaces of both the affine and the projective space, plane algebraic curves, quadrics, graphics of polynomials functions, ...

Algebraic geometry is the branch of mathematics which studies algebraic sets (and their generalizations). Our first aim is *to give a formal definition of algebraic sets.*

a) Affine algebraic sets

Let $K[x_1, \dots, x_n]$ be the polynomial ring in n variables over the field K. If $P(a_1, \dots, a_n) \in \mathbb{A}^n$, and $F = F(x_1, \dots, x_n) \in K[x_1, \dots, x_n]$, we can consider the value of F at P, i.e. $F(P) = F(a_1, \dots, a_n) \in K$. We say that P is a *zero of* F if $F(P) = 0$.

For example the points $P_1(1, 0)$, $P_2(-1, 0)$, $P_3(0, 1)$ are zeroes of $F = x_1^2 + x_2^2 - 1$ over any field. If $G = x_1^2 + x_2^2 + 1$ then G has no zeroes in $\mathbb{A}^2_{\mathbb{R}}$, but it does have zeroes in $\mathbb{A}^2_{\mathbb{C}}$.

2.1. Definition. A subset X of $\mathbb{A}^n_K$ is an *affine algebraic set*, or an *affine variety*, if X is the set of common zeroes of a family of polynomials of $K[x_1, \dots, x_n]$.

Remark. In some texts the term "variety" is reserved to the affine algebraic sets which are irreducible.

X is an affine algebraic set means that there exists a subset $S \subset K[x_1, \ldots, x_n]$ such that

$$X = \{P \in \mathbb{A}^n \mid F(P) = 0 \ \forall \ F \in S\}.$$

In this case X is called the zero set of S and is denoted $V(S)$ (or in some books $Z(S)$, e.g. this is the notation of Hartshorne's book). In particular, if $S = \{F\}$, then $V(S)$ will be simply denoted by $V(F)$.

2.2. Examples and remarks.

1. $S = K[x_1, \ldots, x_n]$: then $V(S) = \emptyset$, because S contains non–zero constants.
2. $S = \{0\}$: then $V(S) = \mathbb{A}^n$.
3. $S = \{xy - 1\}$: then $V(xy - 1)$ is the hyperbola.
4. If $S \subset T$, then $V(S) \supset V(T)$.

Let $S \subset K[x_1, \ldots, x_n]$ be a set of polynomials, let $\alpha := \langle S \rangle$ be the ideal generated by S. Recall that $\alpha = \{$finite sums of products of the form HF where $F \in S$, $H \in K[x_1, \ldots, x_n]\}$.

2.3. Proposition. $V(S) = V(\alpha)$.

Proof. If $P \in V(\alpha)$, then $F(P) = 0$ for any $F \in \alpha$; in particular for any $F \in S$ because $S \subset \alpha$.

Conversely, if $P \in V(S)$, let $G = \sum_i H_i F_i$ be a polynomial of α ($F_i \in S \ \forall \ i$). Then $G(P) = (\sum H_i F_i)(P) = \sum H_i(P) F_i(P) = 0$. □

The above Proposition is important in view of the following:

Hilbert's Basis Theorem. *If R is a Noetherian ring, then the polynomial ring $R[x]$ is Noetherian.*

Proof. Assume by contradiction that $R[x]$ is not Noetherian. Let $I \subset R[x]$ be a not finitely generated ideal. Let $f_1 \in I$ be a non-zero polynomial of minimum degree. We define by induction as follows a sequence $\{f_k\}_{k \in \mathbb{N}}$ of polynomials: if f_k ($k \geq 1$) has already been chosen, let f_{k+1} be a polynomial of minimum degree in $I \setminus \langle f_1, \ldots, f_k \rangle$. Let n_k be the degree of f_k and a_k be its leading coefficient. Note that, by the very choice of f_k, the chain of the degrees is increasing: $n_1 \leq n_2 \leq \ldots$.

We will prove now that $\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \ldots$ is a chain of ideals, that does not become stationary: this will give the required contradiction. Indeed, if $\langle a_1, \ldots, a_r \rangle = \langle a_1, \ldots, a_r, a_{r+1} \rangle$, then $a_{r+1} = \sum_{i=1}^r b_i a_i$, for suitable $b_i \in R$. In this case, we consider the element $g := f_{r+1} - \sum_{i=1}^r b_i x^{n_{r+1} - n_i} f_i$: g belongs to I, but $g \notin \langle f_1, \ldots, f_r \rangle$, and its degree is strictly lower than the degree of f_{r+1}: contradiction. □

2.4. Corollary. *Any affine algebraic set $X \subset \mathbb{A}^n$ is the zero set of a finite number of polynomials, i.e. there exist $F_1, \ldots, F_r \in K[x_1, \ldots, x_n]$ such that $X = V(F_1, \ldots, F_r)$.*

□

Note that $V(F_1,\dots,F_r) = V(F_1)\cap\dots\cap V(F_r)$, so every algebraic set is a finite intersection of algebraic sets of the form $V(F)$, i.e. zeroes of a unique polynomial F. If $F = 0$, then $V(0) = \mathbb{A}^n$; if $F = c \in K \setminus \{0\}$, then $V(c) = \emptyset$; if $deg\ F > 0$, then $V(F)$ is called a *hypersurface*.

2.5. Proposition. *The affine algebraic sets of $\mathbb{A}^n$ satisfy the axioms of the closed sets of a topology, called the Zariski topology.*

Proof. It is enough to check that finite unions and arbitrary intersections of algebraic sets are again algebraic sets.

Let $V(\alpha), V(\beta)$ be two algebraic sets, with α, β ideals of $K[x_1,\dots,x_n]$. Then $V(\alpha)\cup V(\beta) = V(\alpha\cap\beta) = V(\alpha\beta)$, where $\alpha\beta$ is the product ideal, defined by:

$$\alpha\beta = \{\sum_{\text{fin}} a_i b_i \mid a_i \in \alpha, b_i \in \beta\}.$$

Indeed: $\alpha\beta \subset \alpha\cap\beta$ so $V(\alpha\cap\beta) \subset V(\alpha\beta)$, and both $\alpha\cap\beta \subset \alpha$ and $\alpha\cap\beta \subset \beta$ so $V(\alpha)\cup V(\beta) \subset V(\alpha\cap\beta)$. Assume now that $P \in V(\alpha\beta)$ and $P \notin V(\alpha)$: hence $\exists F \in \alpha$ such that $F(P) \neq 0$; on the other hand, if $G \in \beta$ then $FG \in \alpha\beta$ so $(FG)(P) = 0 = F(P)G(P)$, which implies $G(P) = 0$.

Let $V(\alpha_i), i \in I$, be a family of algebraic sets, $\alpha_i \subset K[x_1,\dots,x_n]$. Then $\cap_{i\in I} V(\alpha_i) = V(\sum_{i\in I}\alpha_i)$, where $\sum_{i\in I}\alpha_i$ is the sum ideal of α_i's. In fact $\alpha_i \subset \sum_{i\in I}\alpha_i$ $\forall i$, hence $V(\sum_i \alpha_i) \subset V(\alpha_i)$ $\forall i$ and $V(\sum_i\alpha_i) \subset \cap_i V(\alpha_i)$. Conversely, if $P \in V(\alpha_i)$ $\forall i$, and $F \in \sum_i \alpha_i$, then $F = \sum_i F_i$; therefore $F(P) = \sum F_i(P) = 0$.

□

2.6. Examples.

1. The Zariski topology of the affine line $\mathbb{A}^1$.

Let us recall that the polynomial ring $K[x]$ in one variable is a PID (principal ideal domain), so every ideal $I \subset K[x]$ is of the form $I = \langle F\rangle$. Hence every closed subset of $\mathbb{A}^1$ is of the form $X = V(F)$, the set of zeroes of a unique polynomial $F(x)$. If $F = 0$, then $V(F) = \mathbb{A}^1$, if $F = c \in K^*$, then $V(F) = \emptyset$, if $\deg F = d > 0$, then F can be decomposed in linear factors in polynomial ring over the algebraic closure of K; it follows that $V(F)$ has at most d points.

We conclude that the closed sets in the Zariski topology of $\mathbb{A}^1$ are: $\mathbb{A}^1$, $\emptyset$ and the finite sets.

2. If $K = \mathbb{R}$ or $\mathbb{C}$, then the Zariski topology and the Euclidean topology on $\mathbb{A}^n$ can be compared, and it results that the Zariski topology is coarser. Indeed every open set in the Zariski topology is open also in the usual topology. Let $X = V(F_1,\dots,F_r)$ be a closed set in the Zariski topology, and $U := \mathbb{A}^n \setminus X$; if $P \in U$, then $\exists\ F_i$ such that $F_i(P) \neq 0$, so there exists an open neighbourhood of P in the usual topology in which F_i does not vanish.

Conversely, there exist closed sets in the usual topology which are not Zariski closed, for example the balls. The first case, of an interval in the real affine line, follows from part 1.