LONGITUDINAL BEAM DYNAMICS

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Material from Joel LeDuff's Course at the CERN Accelerator School held at Jyvaskyla, Finland the 7-18 September 1992 (CERN 94-01) has been used as well.

LESSON I

Introduction

Fields & forces

Acceleration by time-varying fields

Relativistic equations

Fields and force

Equation of motion for a particle of charge *q*

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = q\left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$

$\vec{p} = m\vec{v}$	Momentum
$\vec{v}$	Velocity
$\vec{E}$	Electric field
$\vec{B}$	Magnetic field

The fields must satisfy *Maxwell's equations*

The integral forms, in vacuum, are recalled below:

1. Gauss's law

   (electrostatic)

$$\int_S \vec{E}\cdot \mathrm{d}\vec{s} = \frac{1}{e_\circ}\int_V r \;\mathrm{d}V$$

2. No free magnetic poles

   (magnetostatic)

$$\int_S \vec{B}\cdot \mathrm{d}\vec{s} = 0$$

3. Ampere's law
   (modified by Gauss)

   (electric varying)

$$\int_L \vec{B}\cdot \mathrm{d}\vec{l} = m_\circ \int_S \vec{j}\cdot \mathrm{d}\vec{s} + \frac{1}{c^2}\int_S \frac{\partial \vec{E}}{\partial t}\cdot \mathrm{d}\vec{s}$$

4. Faraday's law

   (magnetic varying)

$$\int_L \vec{E}\cdot \mathrm{d}\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot \mathrm{d}\vec{s}$$

Maxwell's equations

The differential forms, in vacuum, are recalled below:

1. Gauss's law

$$\nabla \cdot \vec{E} = \frac{1}{e_{\circ}} r(\vec{r}, t)$$

2. No free magnetic poles

$$\nabla \cdot \vec{B} = 0$$

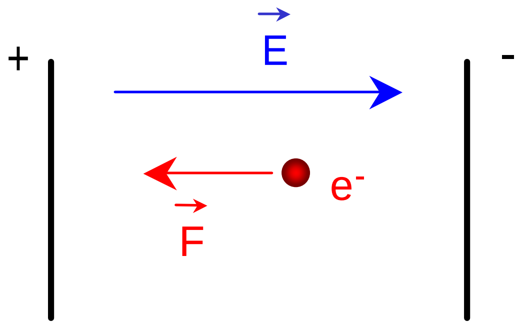
3. Ampere's law
   (modified by Gauss)

$$\nabla \times \vec{B} = m_{\circ} \, \vec{j}(\vec{r}, t) + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

4. Faraday's law

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Constant electric field



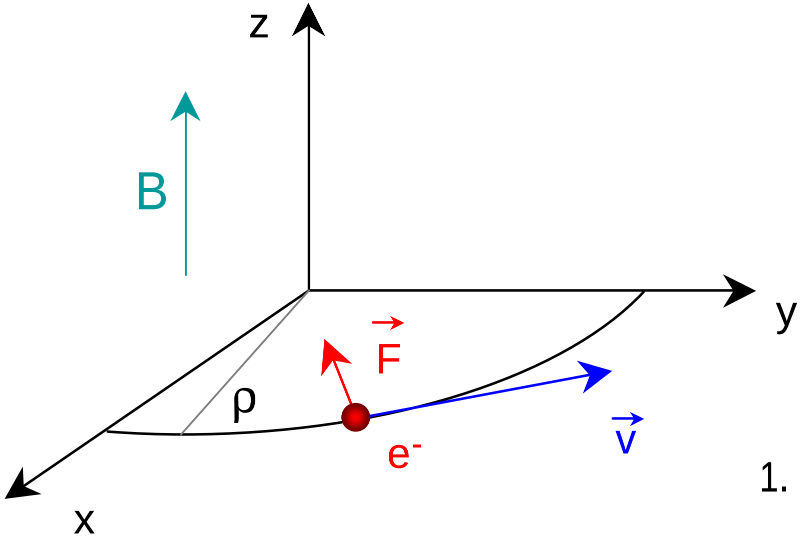
$$\frac{d\vec{p}}{dt} = -e\,\vec{E}$$

1. Direction of the force always parallel to the field

2. Trajectory can be modified, velocity also Þ momentum and energy can be modified

This force can be used to accelerate and decelerate particles

Constant magnetic field



$$\frac{d\vec{p}}{dt} = \vec{F} = -e\left(\vec{v} \wedge \vec{B}\right)$$

1. Direction always perpendicular to the velocity
2. Trajectory can be modified, but not the velocity

This force cannot modify the energy

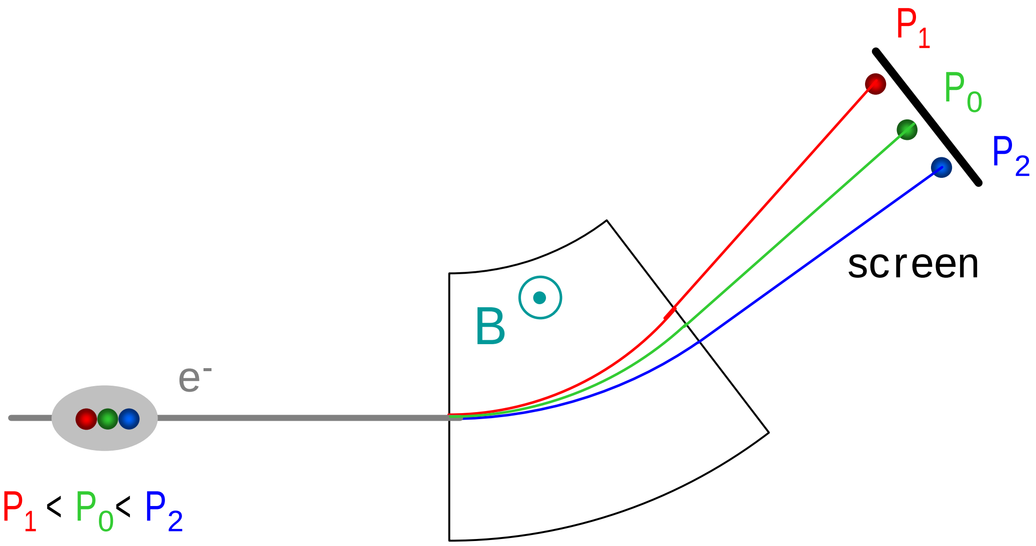
$$e\,v\,B = \frac{m\,v^2}{r}$$

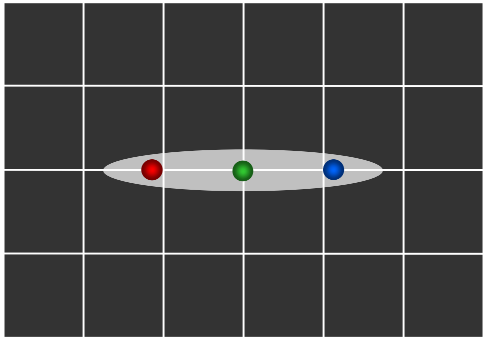
rigidity $$B\;r = \frac{p}{e}$$

angular frequency $$w = \frac{e}{m}B$$

Application: spectrometer

$$B\,r = \frac{p}{e} \;\Rightarrow\; r = \frac{p}{e\,B}$$





$$B\,r\;[\mathrm{Tm}] \approx \frac{p\;[\mathrm{GeV/c}]}{0.3}$$

Larmor formula

An accelerating charge radiates a power *P* given by:

$$P = \frac{2}{3}\frac{r_e}{m_0 c}\left\{ \dot{p}_{//}^{\,2} + \gamma^2 \dot{p}_{\perp}^{\,2} \right\}$$

Acceleration in the direction of the particle motion

Acceleration perpendicular to the particle motion

"Synchrotron radiation"

Energy lost on a trajectory *L*

$$W = \int_L \frac{P}{v}\,\mathrm{d}s$$

For electrons in a constant magnetic field:

$$W\,[\mathrm{eV/turn}] = 88 \cdot 10^3 \frac{E^2[\mathrm{GeV}]}{r[\mathrm{m}]}$$

Comparison of magnetic and electric forces

$$|\vec{B}| = 1\,\mathrm{T}$$

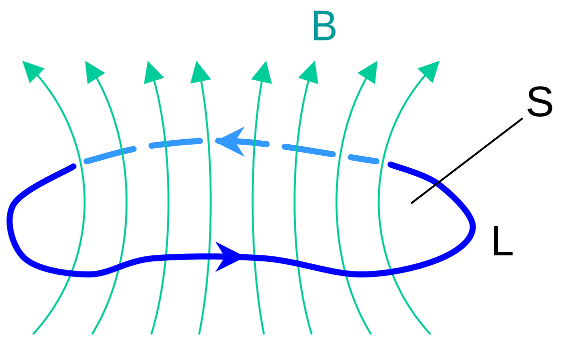
$$|\vec{E}| = 10\,\mathrm{MV/m}$$

$$\frac{F_{MAGN}}{F_{ELEC}} = \frac{e\,v\,B}{e\,E} = b\,c\,\frac{B}{E} \cong 3\cdot 10^{8}\,\frac{1}{10^{7}}\,b = 30\,b$$

Acceleration by time-varying magnetic field

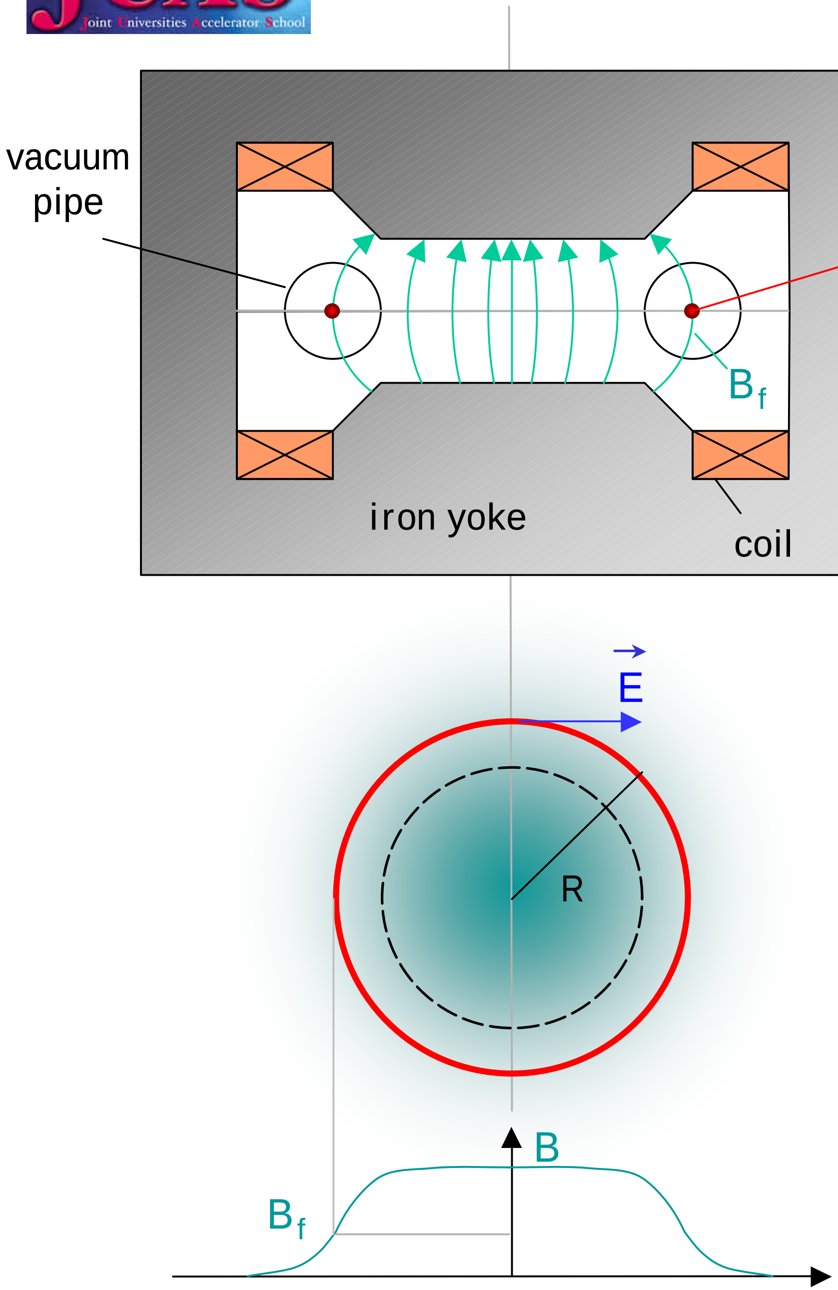
A variable magnetic field produces an electric field (Faraday's Law):

$$\int_L \vec{E} \cdot \mathrm{d}\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{s} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$



It is the Betatron concept

The varying magnetic field is used to guide particles on a circular trajectory as well as for acceleration



Betatron

$$\oint_L \vec{E}\cdot d\vec{l} = 2\pi R E = -\frac{d\Phi}{dt} = -\pi R^2 \frac{dB_{ave}}{dt}$$

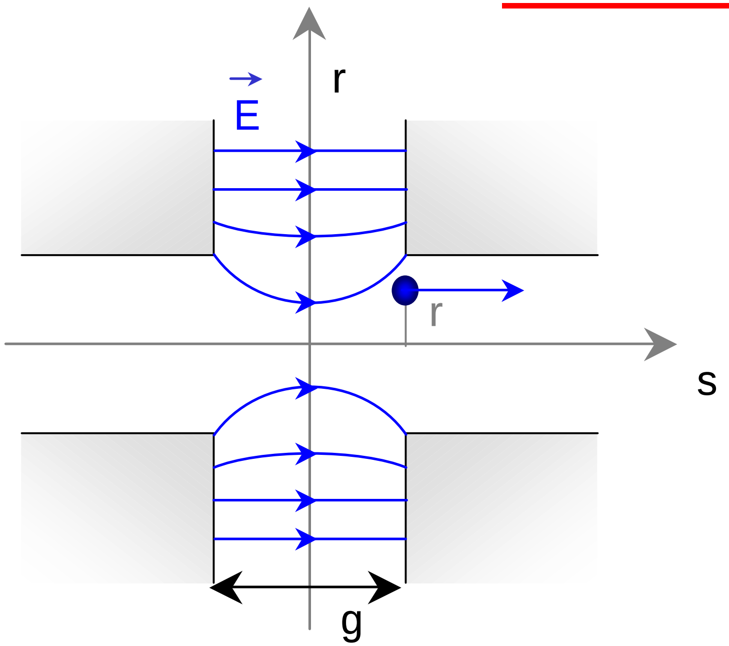
$$\frac{dp}{dt} = eE = \frac{1}{2} e R \frac{dB_{ave}}{dt}$$

$$B r = \frac{p}{e} \quad \Rightarrow \quad \frac{dp}{dt} = e R \frac{dB_f}{dt}$$

$$\Downarrow$$

$$B_f = \frac{1}{2} B_{ave} + const.$$

Acceleration by time-varying electric field



- Let V_{RF} be the amplitude of the RF voltage across the gap g
- The particle crosses the gap at a distance r
- The energy gain is:

$$\Delta E = e \int_{-g/2}^{g/2} \vec{E}(s,r,t)\, \mathrm{d}\vec{s}$$

[MeV] [MV/m]

In the cavity gap, the electric field is supposed to be:

$$E(s,r,t) = E_1(s,r) \cdot E_2(t)$$

In general, $E_2(t)$ is a sinusoidal time variation with angular frequency ω_{RF}

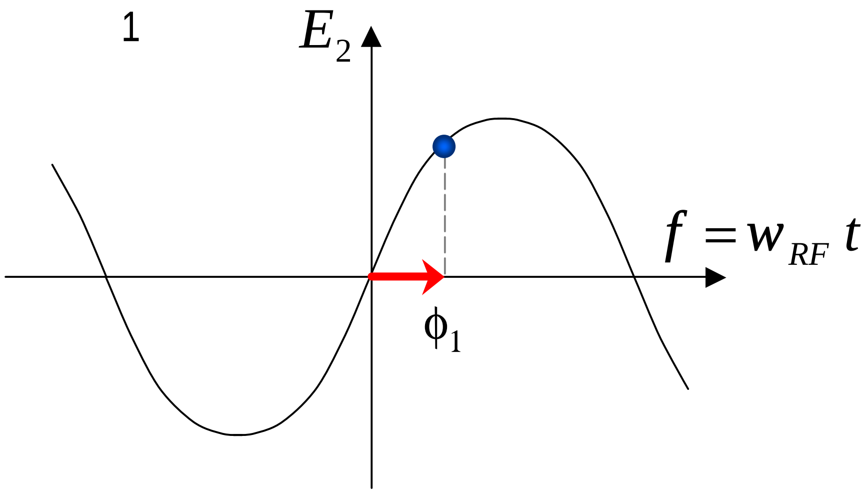
$$E_2(t) = E_\circ \sin \Phi(t) \qquad \text{where} \qquad \Phi(t) = \int_{t_0}^{t} w_{RF}\, \mathrm{d}t + \Phi_0$$

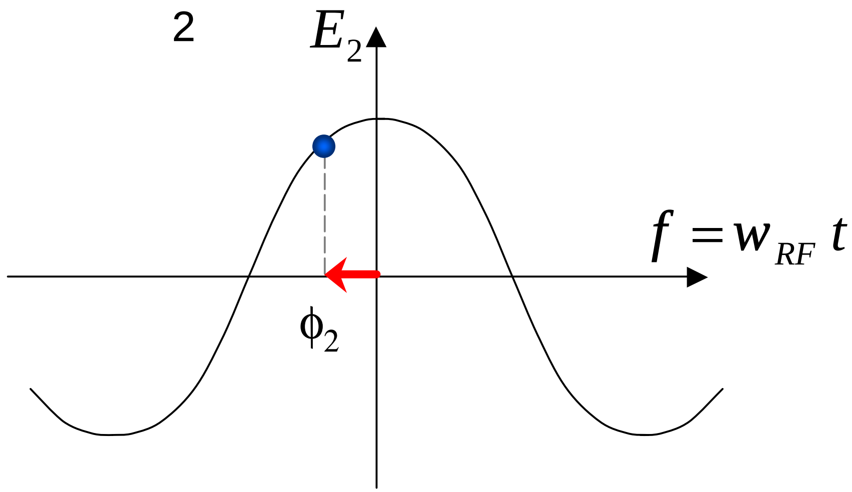
Convention

1. For circular accelerators, the origin of time is taken at the zero crossing of the RF voltage with positive slope

2. For linear accelerators, the origin of time is taken at the positive crest of the RF voltage

Time t= 0 chosen such that:





$$E_2(t) = E_\circ \sin(\omega_{RF}\, t) \qquad\qquad E_2(t) = E_\circ \cos(\omega_{RF}\, t)$$

Relativistic Equations

$$E = mc^2$$

normalized velocity

$$b = \frac{v}{c} = \sqrt{1 - \frac{1}{g^2}}$$

energy

$$E = E_{kin} + E_0$$

total — kinetic — rest

total energy / rest energy

$$g = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - b^2}}$$

momentum

$$p = mv = b\frac{E}{c} = b\, g\, m_0\, c$$

energy	momentum	mass
eV	eV/c	eV/c^2

$$p^2c^2 = E^2 - E_0^2$$

$$g = 1 + \frac{E_{kin}}{E_0}$$

$$p\,[\mathrm{GeV/c}] \cong 0.3\; B\,[\mathrm{T}]\; r\,[\mathrm{m}]$$

First derivatives

$$\mathrm{d}b = b^{-1} g^{-1}\,\mathrm{d}g$$

$$\mathrm{d}(cp) = E_0\, g^{3}\,\mathrm{d}b$$

$$\mathrm{d}g = \text{ß}\left(1-\text{ß}^{2}\right)^{-3/2}\mathrm{d}b$$

Logarithmic derivatives

$$\frac{\mathrm{d}b}{b} = (b\, g)^{-2}\frac{\mathrm{d}g}{g}$$

$$\frac{\mathrm{d}p}{p} = \frac{g^{2}}{g^{2}-1}\frac{\mathrm{d}E}{E} = \frac{g}{g+1}\frac{\mathrm{d}E_{kin}}{E_{kin}}$$

$$\frac{\mathrm{d}g}{g} = \left(g^{2}-1\right)\frac{\mathrm{d}b}{\text{ß}}$$

LESSON II

An overview of particle acceleration

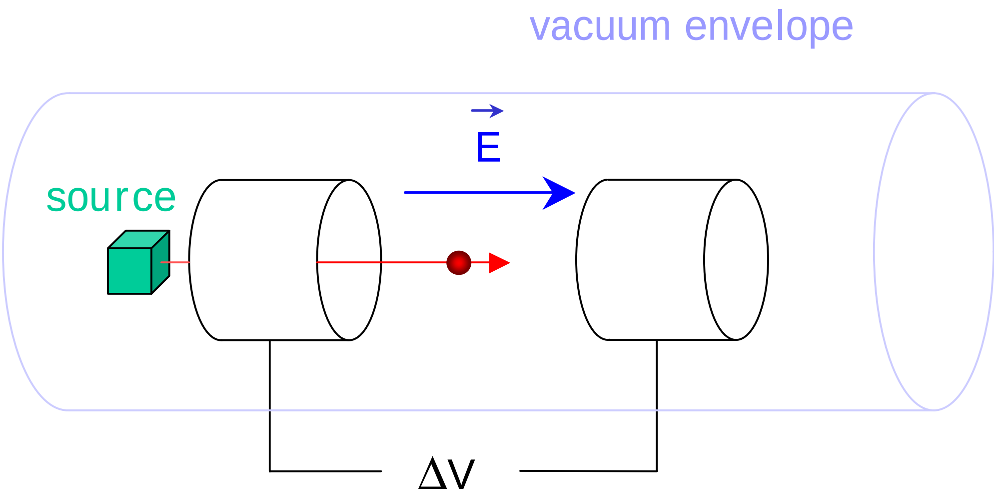
Transit time factor

Main RF parameters

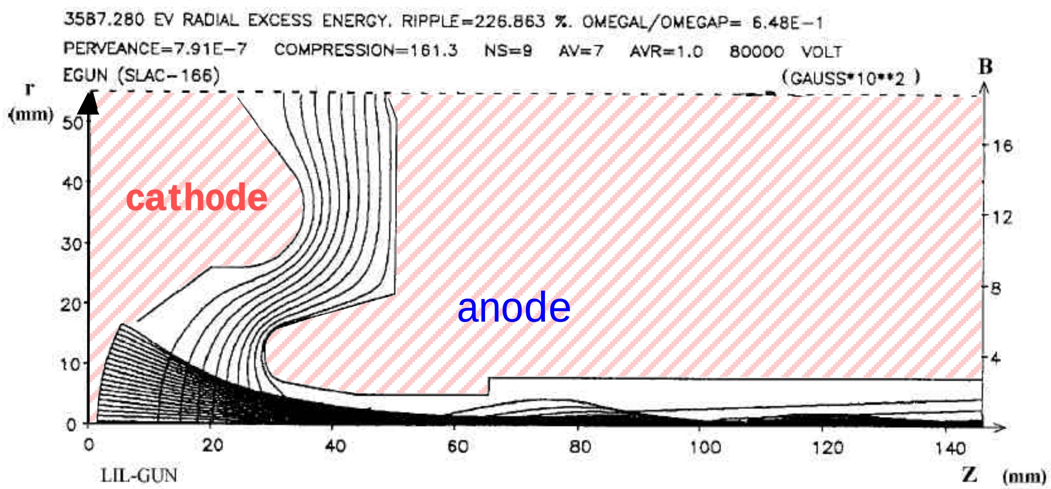
Momentum compaction

Transition energy

Electrostatic accelerators



- The potential difference between two electrodes is used to accelerate particles
- Limited in energy by the maximum high voltage (~ 10 MV)
- Present applications: x-ray tubes, low energy ions, electron sources (thermionic guns)

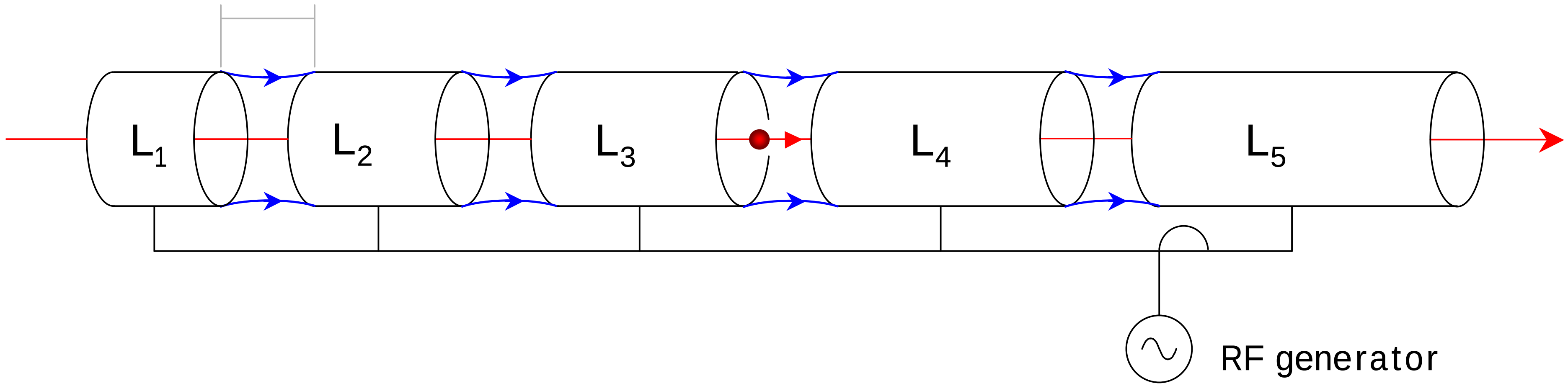


Electric field potential and beam trajectories inside an electron gun (LEP Injector Linac at CERN), computed with the code E-GUN

Alvarez structure

g

Used for protons, ions (50 – 200 MeV, *f* ~ 200 MHz)



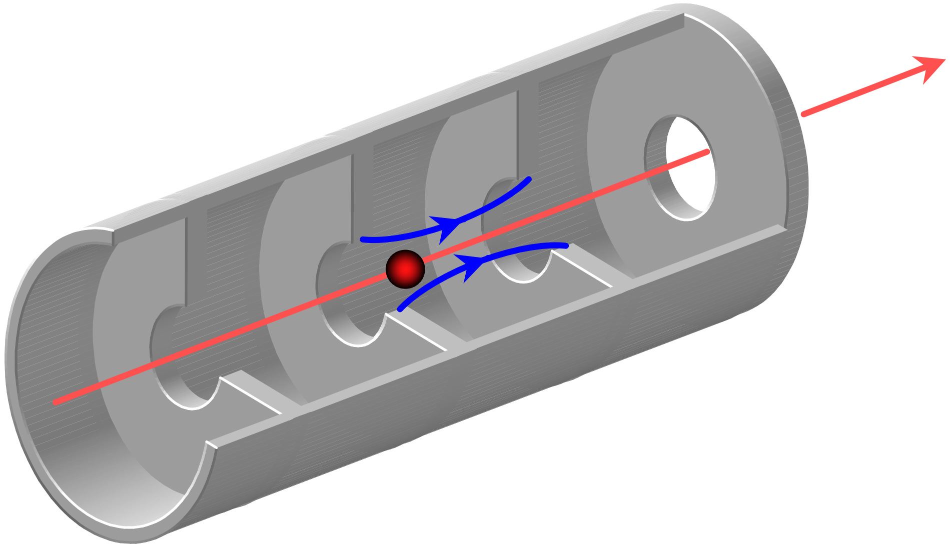
$$g << L$$

Synchronism condition

$$L = v_s T_{RF} = \beta_s \lambda_{RF}$$

$$\omega_{RF} = 2\pi \frac{v_s}{L}$$

Electron Linac



Electrons are light Þ fast acceleration
Þ b @ 1 already at an energy of a few MeV

Uniform disk-loaded waveguide, travelling wave

(up to 50 GeV, *f* ~ 3 GHz - S-band)

$$E(z,t) = E_0 \, e^{i(w t - k z)}$$ Electric field

Wave number $k = \frac{2p}{l_{RF}}$ Phase velocity $v_{ph} = \frac{w}{k}$ Group velocity $v_g = \frac{dw}{dk}$

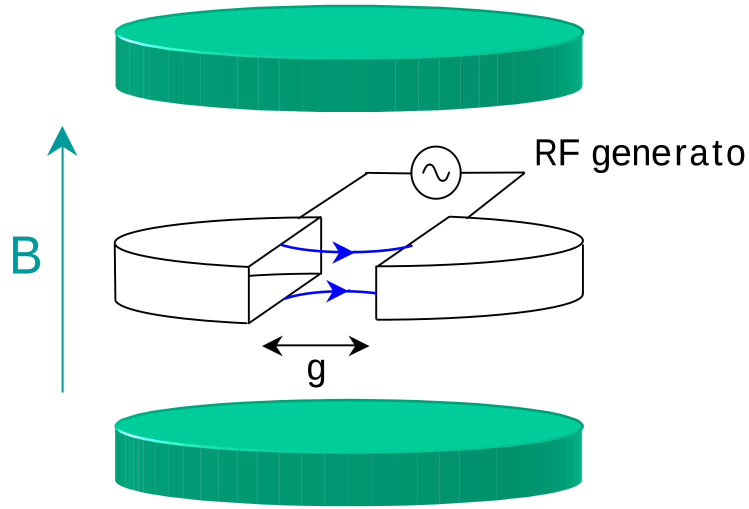
Synchronism condition

$$v_{el} = \frac{w}{k} = v_{ph}$$

Cyclotron

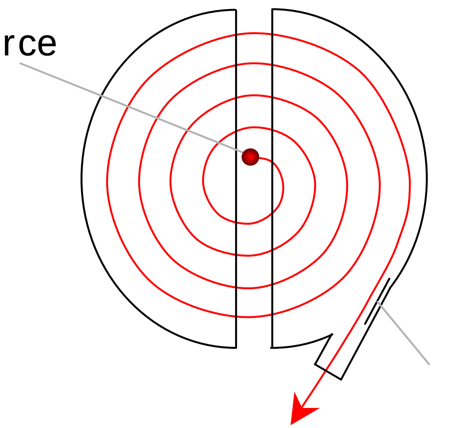
Used for protons, ions

B = constant
ω_{RF} = constant



Synchronism condition

$$w_s = w_{RF}$$
$$2p\ r = v_s T_{RF}$$



Cyclotron frequency

$$w = \frac{q\,B}{m_0\,g}$$

1. g increases with the energy
   Þ no exact synchronism

2. if **v** << **c** Þ g @ **1**

Synchrocyclotron

Same as cyclotron, except a modulation of ω_{RF}

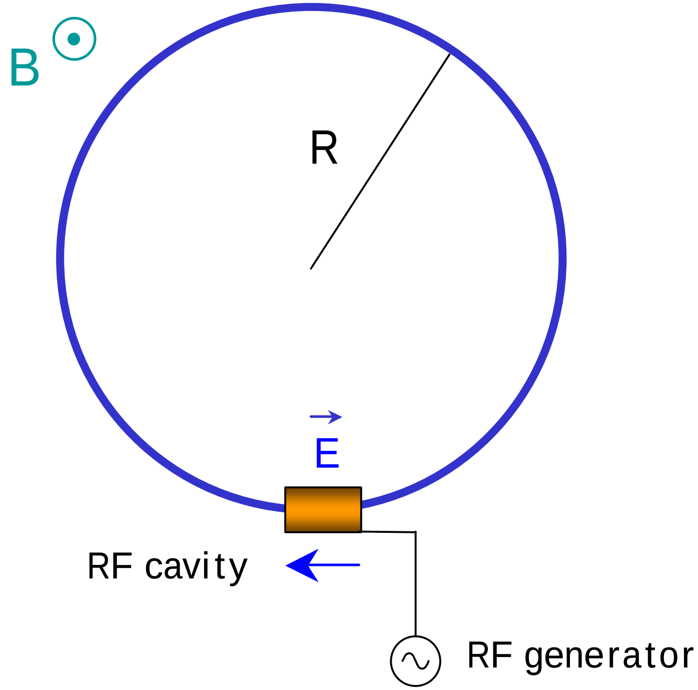
B	= constant	
$\gamma\ \omega_{RF}$	= constant	ω_{RF} decreases with time

The condition:

$$w_s(t) = w_{RF}(t) = \frac{q\,B}{m_0\,g\,(t)}$$

Allows to go beyond the non-relativistic energies

Synchrotron



Synchronism condition

$$T_s = h\,T_{RF}$$

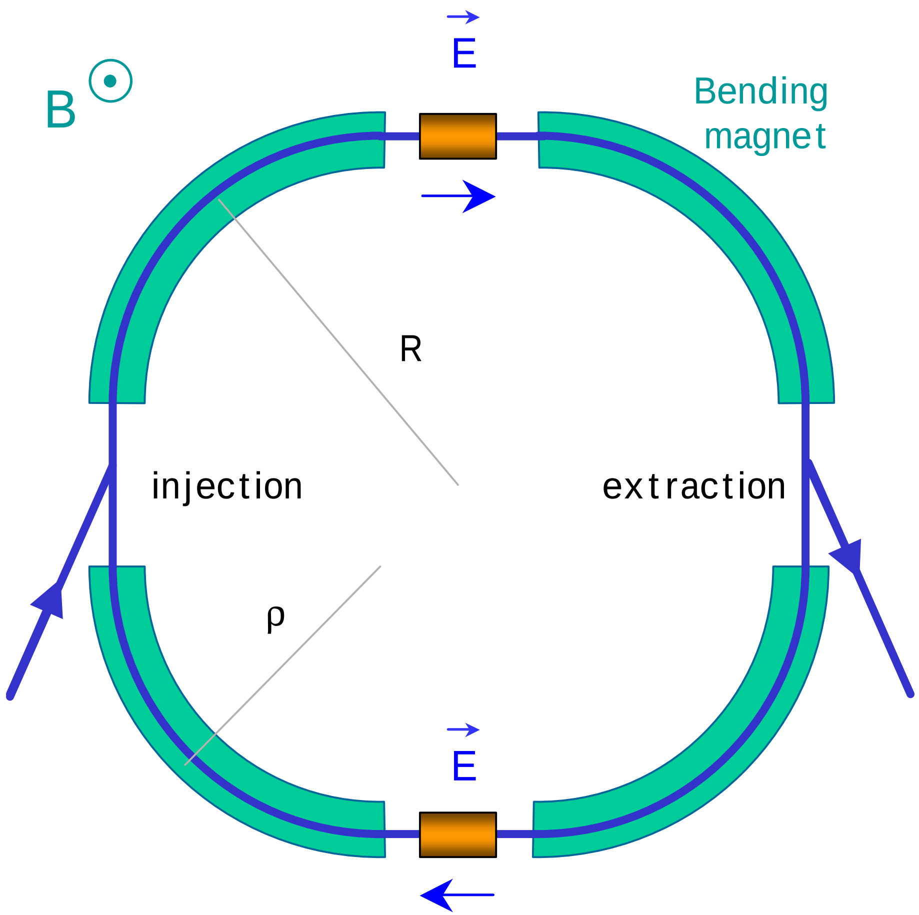
$$\frac{2\pi R}{v_s} = h\,T_{RF}$$

h integer,
harmonic number

1. ω_{RF} and ω increase with energy

2. To keep particles on the closed orbit, B should increase with time

Synchrotron



- In reality, the orbit in a synchrotron is not a circle, straight sections are added for RF cavities, injection and extraction, etc..
- Usually the beam is pre-accelerated in a linac (or a smaller synchrotron) before injection
- The bending radius ρ does not coincide to the machine radius $R = L/2\pi$

Parameters for circular accelerators

The basic principles, for the common circular accelerators, are based on the two relations:

1. The Lorentz equation: the orbit radius can be espressed as:

$$R = \frac{\gamma \; v \, m_0}{eB}$$

2. The synchronicity condition: The revolution frequency can be expressed as:

$$f = \frac{e\,B}{2\pi \,\gamma \; m_0}$$

According to the parameter we want to keep constant or let vary, one has different acceleration principles. They are summarized in the table below:

Machine	**Energy (γ)**	**Velocity**	**Field**	**Orbit**	**Frequency**
Cyclotron	~ 1	var.	const.	~ v	const.
Synchrocyclotron	var.	var.	B(r)	~ p	B(r)/γ(t)
Proton/Ion synchrotron	var.	var.	~ p	R	~ v
Electron synchrotron	var.	const.	~ p	R	const.

Transit time factor

RF acceleration in a gap *g*

$$E(s,r,t) = E_1(s,r) \cdot E_2(t)$$

Simplified model

$$E_1(s,r) = \frac{V_{RF}}{g} = \text{const.}$$

$$E_2(t) = \sin(w_{RF}\, t + f_0)$$

At t = 0, s = 0 and v ≠ 0, parallel to the electric field

Energy gain:

$$\Delta E = e \int_{-g/2}^{g/2} E(s,r,t)\,\mathrm{d}s \quad \Rightarrow \quad \Delta E = e\, V_{RF}\, T_a \sin f_0$$

where

$$T_a = \frac{\sin \dfrac{w_{RF}\, g}{2v}}{\dfrac{w_{RF}\, g}{2v}}$$

T_a is called transit time factor

- $T_a < 1$
- $T_a \rightarrow 1$ if $g \rightarrow 0$

Transit time factor II

In the general case, the transit time factor is given by:

$$T_a = \frac{\int_{-\infty}^{+\infty} E_1(s,r)\cos\left(w_{RF}\,\frac{s}{v}\right)\mathrm{d}s}{\int_{-\infty}^{+\infty} E_1(s,r)\,\mathrm{d}s}$$

It is the ratio of the peak energy gained by a particle with velocity *v* to the peak energy gained by a particle with infinite velocity.

Main RF parameters

I. Voltage, phase, frequency

In order to accelerate particles, longitudinal fields must be generated in the direction of the desired acceleration

$$E(s,t) = E_1(s) \cdot E_2(t) \qquad E_2(t) = E_0 \sin\left[\int_{t_0}^{t} w_{RF}\, dt + f_0\right]$$

$$w_{RF} = 2p\ f_{rev} \qquad \Delta E = e\,V_{RF}\ T_a \sin f_0$$

Such electric fields are generated in RF cavities characterized by the voltage amplitude, the frequency and the phase

II. Harmonic number

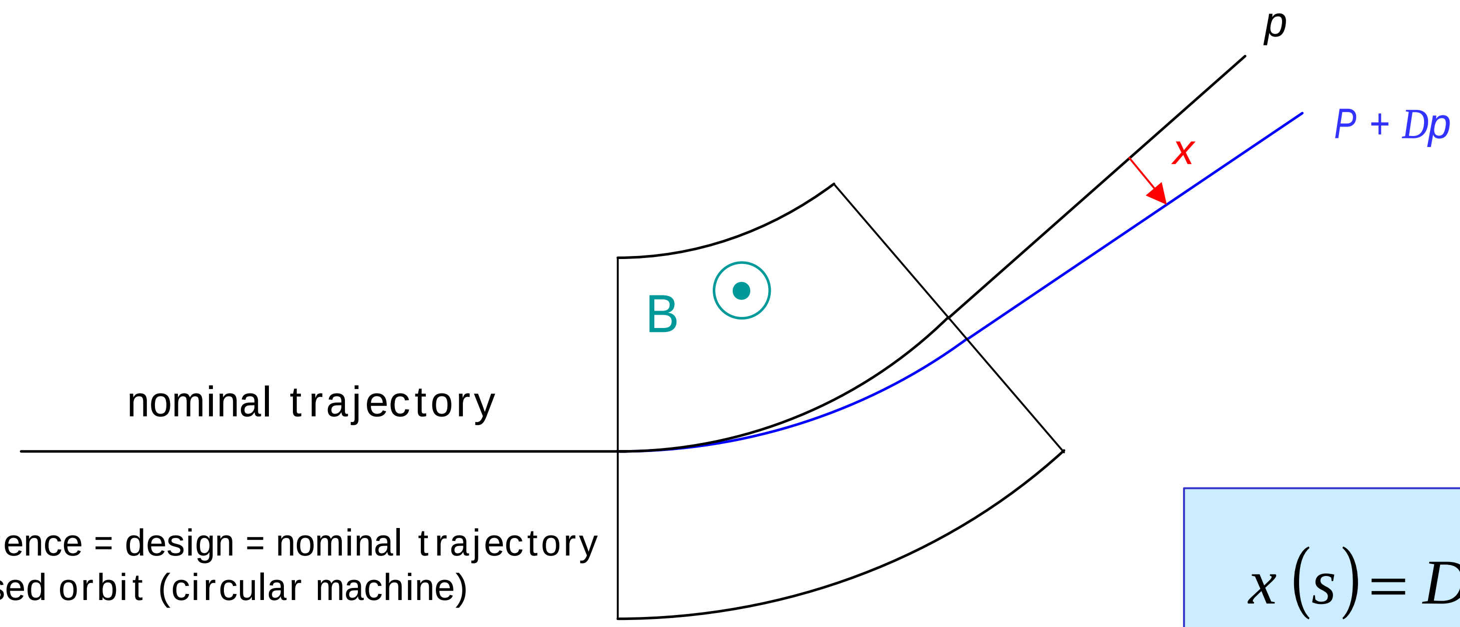
$$T_{rev} = h\,T_{RF} \quad \Rightarrow \quad f_{RF} = h\ f_{rev}$$

f_{rev} = revolution frequency
f_{RF} = frequency of the RF
h = harmonic number

harmonic number in different machines:

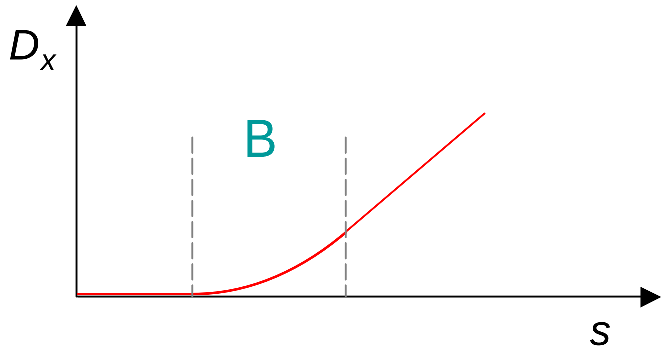
AA	EPA	PS	SPS
1	8	20	4620

Dispersion



reference = design = nominal trajectory
= closed orbit (circular machine)

$$x(s) = D_x(s)\,\frac{\Delta p}{p}$$



Momentum compaction factor in a transport system

In a particle transport system, a nominal trajectory is defined for the nominal momentum p.

For a particle with a momentum $p + \Delta p$ the trajectory length can be different from the length L of the nominal trajectory.

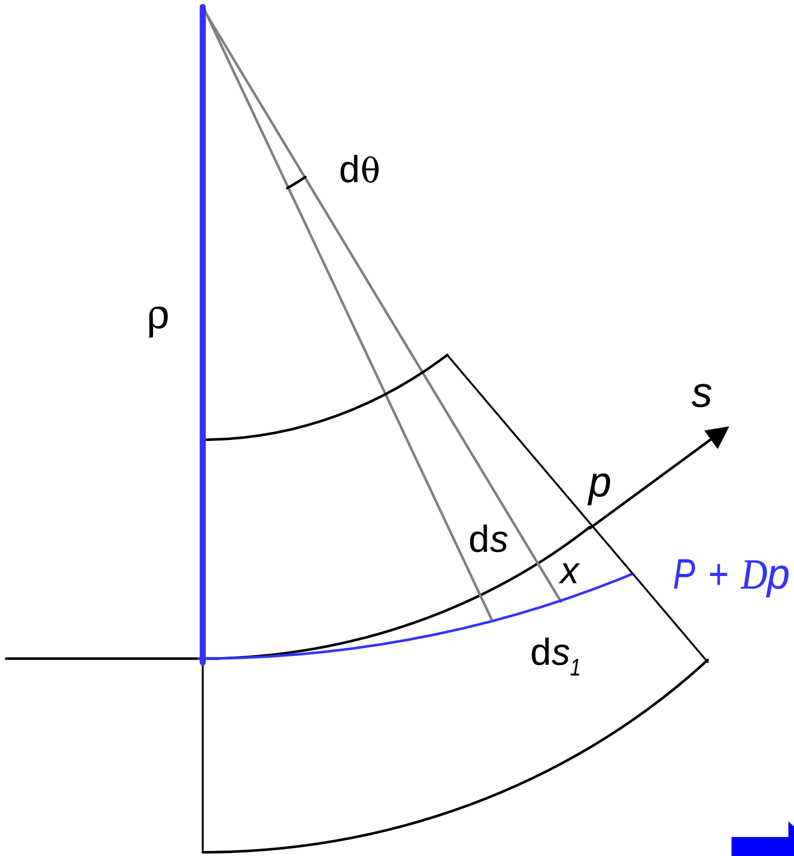
The momentum compaction factor is defined by the ratio:

$$\alpha_p = \frac{dL/L}{dp/p}$$

Therefore, for small momentum deviation, to first order it is:

$$\frac{\Delta L}{L} = \alpha_p \frac{\Delta p}{p}$$

Example: costant magnetic field



$$ds = r\ \mathrm{d}q$$

$$ds_1 = (r + x)\,\mathrm{d}q$$

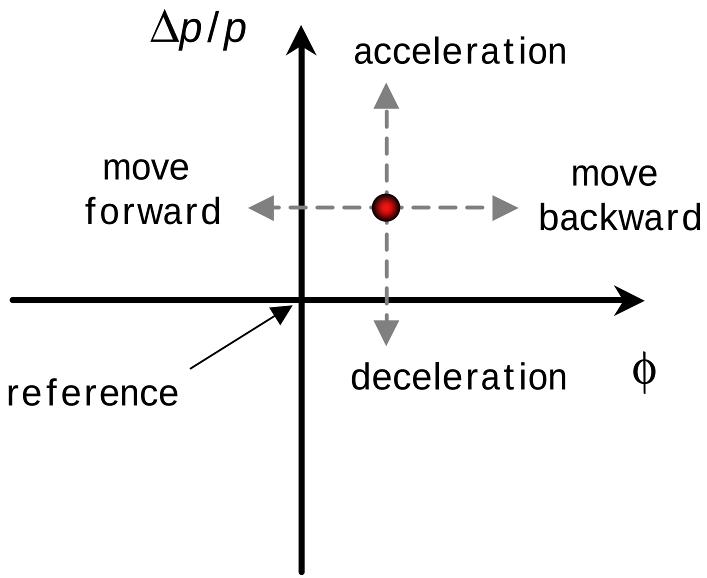
$$\frac{ds_1 - ds}{ds} = \frac{(r + x)\,\mathrm{d}q - r\ \mathrm{d}q}{r\ \mathrm{d}q} = \frac{x}{r} = \frac{D_x}{r}\frac{\mathrm{d}p}{p}$$

By definition of dispersion D_x

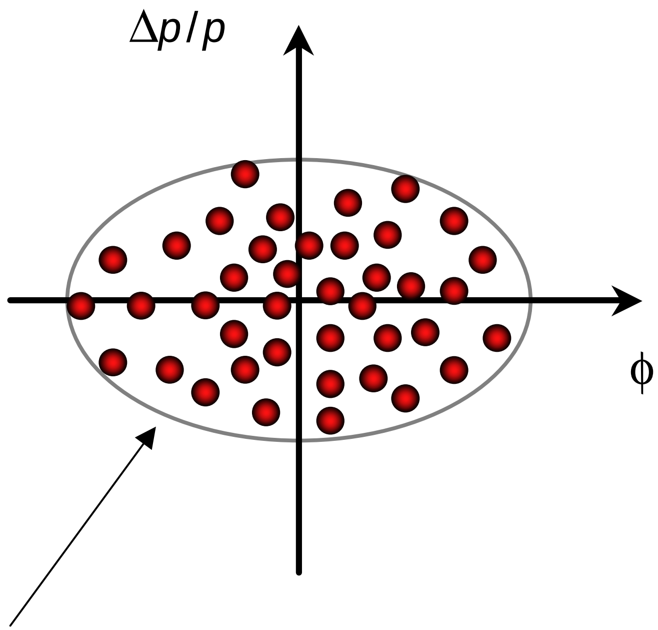
$$a_p = \frac{1}{L}\int_0^L \frac{D_x(s)}{r(s)}\,\mathrm{d}s$$

To first order, only the bending magnets contribute to a change of the trajectory length
(r = ∞ in the straight sections)

Longitudinal phase space



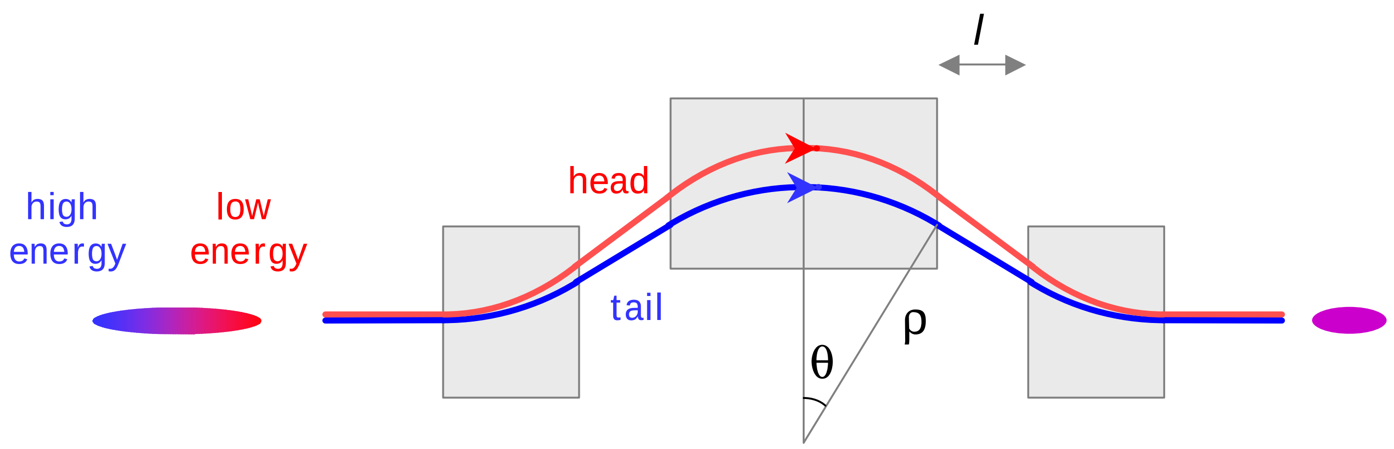
The particle trajectory in the phase space ($\Delta p/p$, ϕ) describes its longitudinal motion.



Emittance: phase space area including all the particles

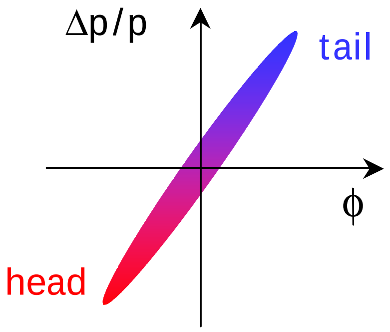
NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

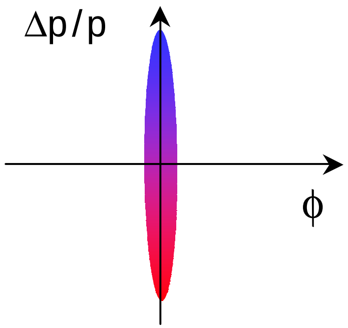
Bunch compressor



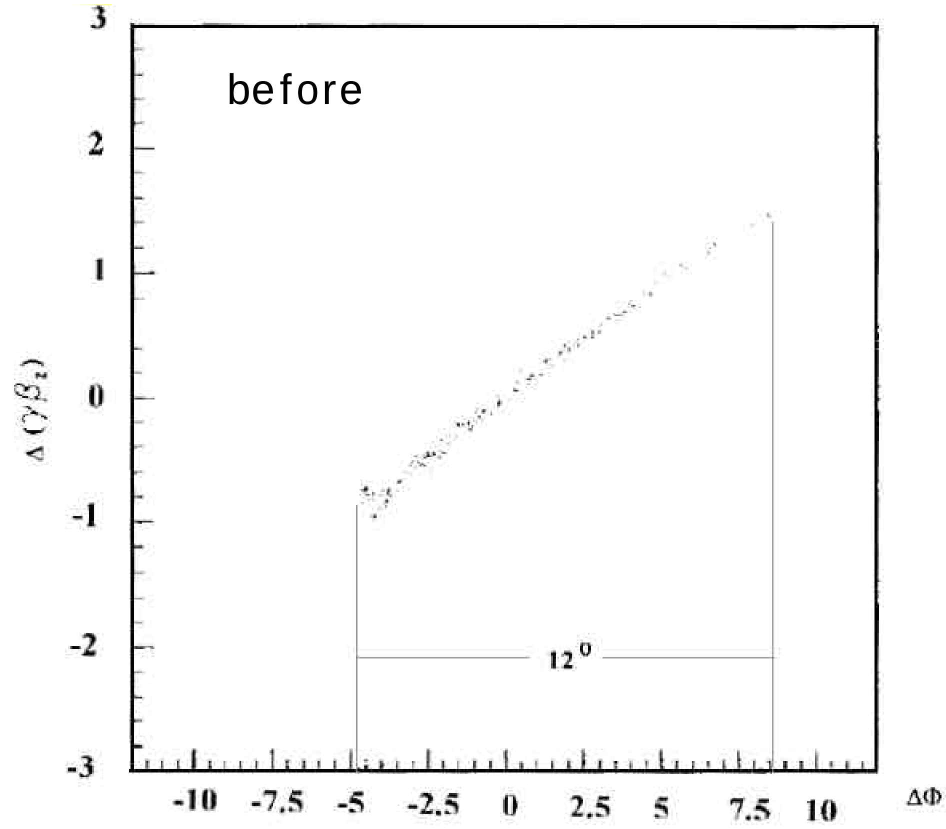
$$\Delta L = \left[4\rho \frac{\tan\theta - \theta}{\sin\theta} + 2l\tan^2\theta \right] \frac{\Delta p}{p}$$

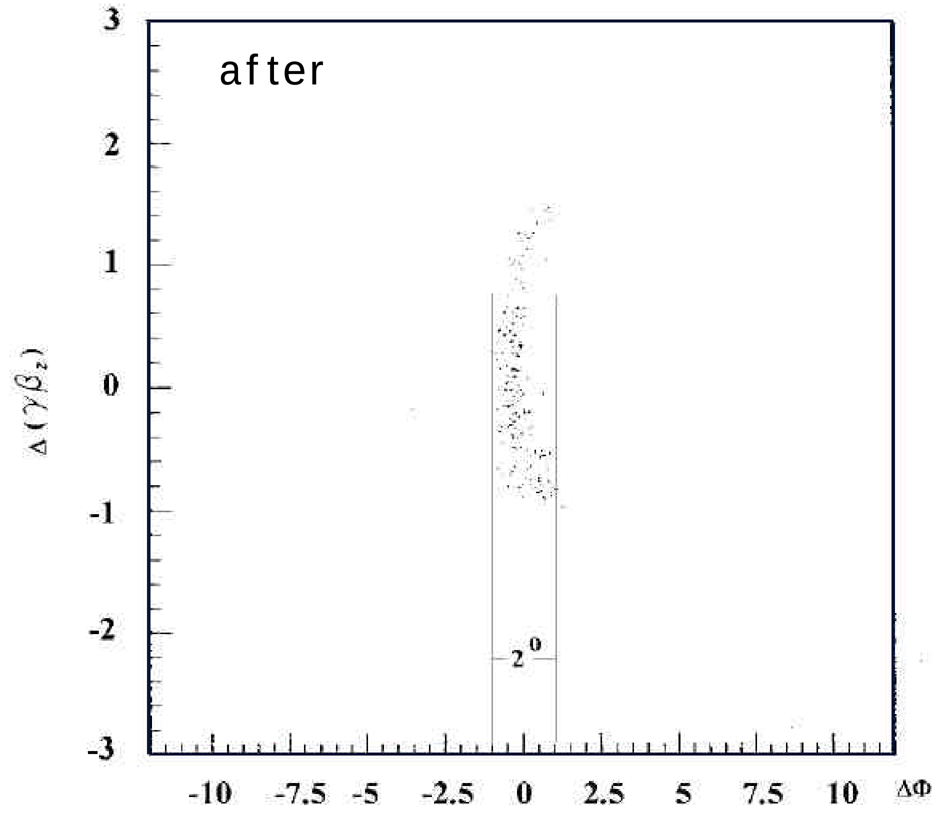
$$L = 4\rho\theta + 2\frac{l}{\cos\theta}$$





Bunch compression





Longitudinal phase space evolution for a bunch compressor (PARMELA code simulations)

Momentum compaction in a ring

In a circular accelerator, a nominal closed orbit is defined for the nominal momentum *p*.

For a particle with a momentum deviation Δp produces an orbit length variation ΔC with:

$$\frac{\Delta C}{C} = a_p \frac{\Delta p}{p}$$

$$C = 2p\ R$$

circumference — (average) radius of the closed orbit

The momentum compaction factor is defined by the ratio:

$$a_p = \frac{dC/C}{dp/p} = \frac{dR/R}{dp/p}$$

and

$$a_p = \frac{1}{C}\oint_C \frac{D_x(s)}{r(s)}\,\mathrm{d}s$$

N.B.: in most circular machines, a_p is positive $\Rightarrow$ higher momentum means longer circumference

Momentum Compaction as a function of energy

$$E = \frac{p\,c}{b} \quad \Rightarrow \quad \frac{\mathrm{d}E}{E} = b^{2}\,\frac{dp}{p}$$

$$a_{p} = b^{2}\,\frac{E}{R}\,\frac{\mathrm{d}R}{\mathrm{d}E}$$

Momentum Compaction as a function of magnetic field

Definition of average magnetic field

$$<B> = \frac{1}{2\pi R}\int_C B_\phi \,\mathrm{d}s = \frac{1}{2\pi R}\left(\int_{straights} B_\phi \,\mathrm{d}s + \int_{magnets} B_\phi \,\mathrm{d}s\right)$$

$$\int_{straights} B_\phi \,\mathrm{d}s = 0 \qquad \int_{magnets} B_\phi \,\mathrm{d}s = 2\pi\rho B_\phi$$

$$<B> = \frac{B_\phi \rho}{R}$$

$$B_\phi \rho = \frac{p}{e}$$

$$<B>R = \frac{p}{e} \quad \Rightarrow \quad \frac{\mathrm{d}<B>}{<B>} + \frac{\mathrm{d}R}{R} = \frac{\mathrm{d}p}{p}$$

$$\alpha_p = 1 - \frac{\mathrm{d}<B>}{<B>} \bigg/ \frac{\mathrm{d}p}{p}$$

Transition energy

Proton (ion) circular machine with a_p positive

1. Momentum larger than the nominal ($p + \Delta p$) Þ longer orbit ($C+\Delta C$)
2. Momentum larger than the nominal ($p + \Delta p$) Þ higher velocity ($v + \Delta v$)

What happens to the revolution frequency $f = v/C$?

- At low energy, v increases faster than C with momentum
- At high energy $v \lesssim c$ and remains almost constant

There is an energy for which the velocity variation is compensated by the trajectory variation Þ transition energy

Below transition:	higher energy	Þ	higher revolution frequency
Above transition:	higher energy	Þ	lower revolution frequency

Transition energy – quantitative approach

We define a parameter h (revolution frequency spread per unit of momentum spread):

$$h = \frac{\mathrm{d}f / f}{\mathrm{d}p / p} = \frac{\mathrm{d}w / w}{\mathrm{d}p / p}$$

$$f = \frac{v}{C} \quad \Rightarrow \quad \frac{\mathrm{d}f}{f} = \frac{\mathrm{d}b}{b} - \frac{\mathrm{d}C}{C}$$

from $p = \frac{m_0 c b}{\sqrt{1-b^2}}$ $\Rightarrow$ $\frac{\mathrm{d}b}{b} = \frac{1}{g^2}\frac{\mathrm{d}p}{p}$

definition of momentum compaction factor: $\frac{\mathrm{d}C}{C} = a_p \frac{\mathrm{d}p}{p}$

$$\frac{\mathrm{d}f}{f} = \left(\frac{1}{g^2} - a_p\right)\frac{\mathrm{d}p}{p}$$

Transition energy – quantitative approach

$$\eta = \frac{1}{\gamma^2} - \alpha_p$$

The transition energy is the energy that corresponds to $\eta = 0$
(α_p is fixed, and γ variable)

$$\gamma_{tr} = \sqrt{\frac{1}{\alpha_p}}$$

The parameter η can also be written as

$$\eta = \frac{1}{\gamma^2} - \frac{1}{\gamma_{tr}^2}$$

- At low energy $\eta > 0$
- At high energy $\eta < 0$

N.B.: for electrons, $\gamma \gg \gamma_{tr} \Rightarrow \eta < 0$
for linacs $\alpha_p = 0 \Rightarrow \eta > 0$

LESSON III

Equations related to synchrotrons

Synchronous particle

Synchrotron oscillations

Principle of phase stability

Equations related to synchrotrons

$$\frac{\mathrm{d}p}{p} = g_{tr}^{\;2}\frac{\mathrm{d}R}{R} + \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = g^{2}\frac{\mathrm{d}f}{f} + g^{2}\frac{\mathrm{d}R}{R}$$

$$\frac{\mathrm{d}B}{B} = g_{tr}^{\;2}\frac{\mathrm{d}f}{f} + \left[1-\left(\frac{g_{tr}}{g}\right)^{2}\right]\frac{\mathrm{d}p}{p}$$

$$\frac{\mathrm{d}B}{B} = g^{2}\frac{\mathrm{d}f}{f} + \left(g^{2} - g_{tr}^{\;2}\right)\frac{\mathrm{d}R}{R}$$

$p\,[\mathrm{MeV/c}]$	momentum
$R\,[\mathrm{m}]$	orbit radius
$B\,[\mathrm{T}]$	magnetic field
$f\,[\mathrm{Hz}]$	frequency
g_{tr}	transition energy

I - Constant radius

$$dR = 0$$

Beam maintained on the same orbit when energy varies

$$\frac{dp}{p} = \frac{dB}{B}$$

$$\frac{dp}{p} = g^2 \frac{df}{f}$$

If p increases

B increases

f increases

II - Constant energy

$$\mathrm{d}p = 0$$

$$V_{RF} = 0$$

Beam debunches

$$\frac{\mathrm{d}p}{p} = 0 = g_{tr}^{\ 2}\frac{\mathrm{d}R}{R} + \frac{\mathrm{d}B}{B}$$

$$\frac{\mathrm{d}p}{p} = 0 = g^{\,2}\frac{\mathrm{d}f}{f} + g^{\,2}\frac{\mathrm{d}R}{R}$$

If *B* increases

R decreases

f increases

III – Magnetic flat-top

$$\mathrm{d}B = 0$$

Beam bunched with constant magnetic field

$$\frac{\mathrm{d}p}{p} = \gamma_{tr}^{\,2} \frac{\mathrm{d}R}{R} \qquad \frac{\mathrm{d}B}{B} = 0 = \gamma_{tr}^{\,2} \frac{\mathrm{d}f}{f} + \left[1 - \left(\frac{\gamma_{tr}}{\gamma}\right)^2\right] \frac{\mathrm{d}p}{p}$$

$$\frac{\mathrm{d}B}{B} = 0 = \gamma^2 \frac{\mathrm{d}f}{f} + \left(\gamma^2 - \gamma_{tr}^{\,2}\right) \frac{\mathrm{d}R}{R}$$

If p increases
R increases
f increase $\gamma < \gamma_{tr}$
decreases $\gamma > \gamma_{tr}$

Four conditions - resume

Beam	**Parameter**	**Variations**
Debunched	$\Delta p = 0$	$B \Uparrow, R \Downarrow, f \Uparrow$
Fixed orbit	$\Delta R = 0$	$B \Uparrow, p \Uparrow, f \Uparrow$
Magnetic flat-top	$\Delta B = 0$	$p \Uparrow, R \Uparrow, f \Uparrow (h > 0)$ $f \Downarrow (h < 0)$
External oscillator	$\Delta f = 0$	$B \Uparrow, p \Downarrow, R \Downarrow (h > 0)$ $p \Uparrow, R \Uparrow (h < 0)$

p momentum

R orbit radius

B magnetic field

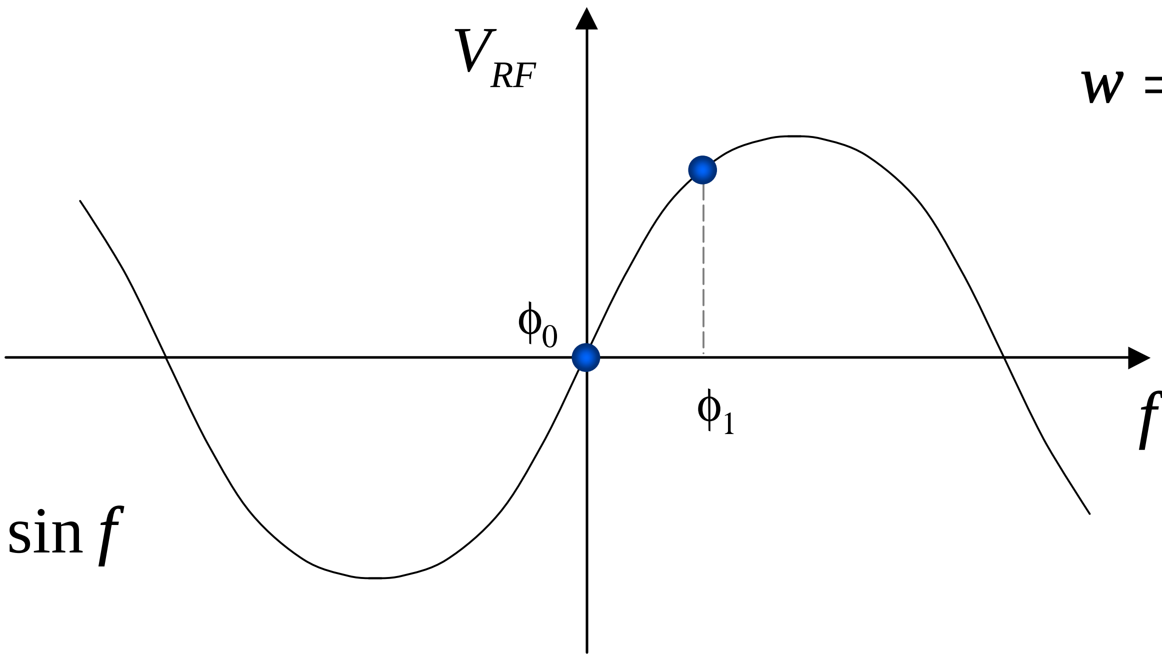
f frequency

Synchronous particle

Simple case (no accel.): B = const. $\qquad g < g_{tr}$

Synchronous particle: particle that sees always the same phase (at each turn) in the RF cavity

$$w = \frac{e B}{g \; m_0} = \frac{w_{RF}}{h}$$



$$f = w_{RF}\, t$$

$$\Delta E = \hat{V}_{RF} \sin f$$

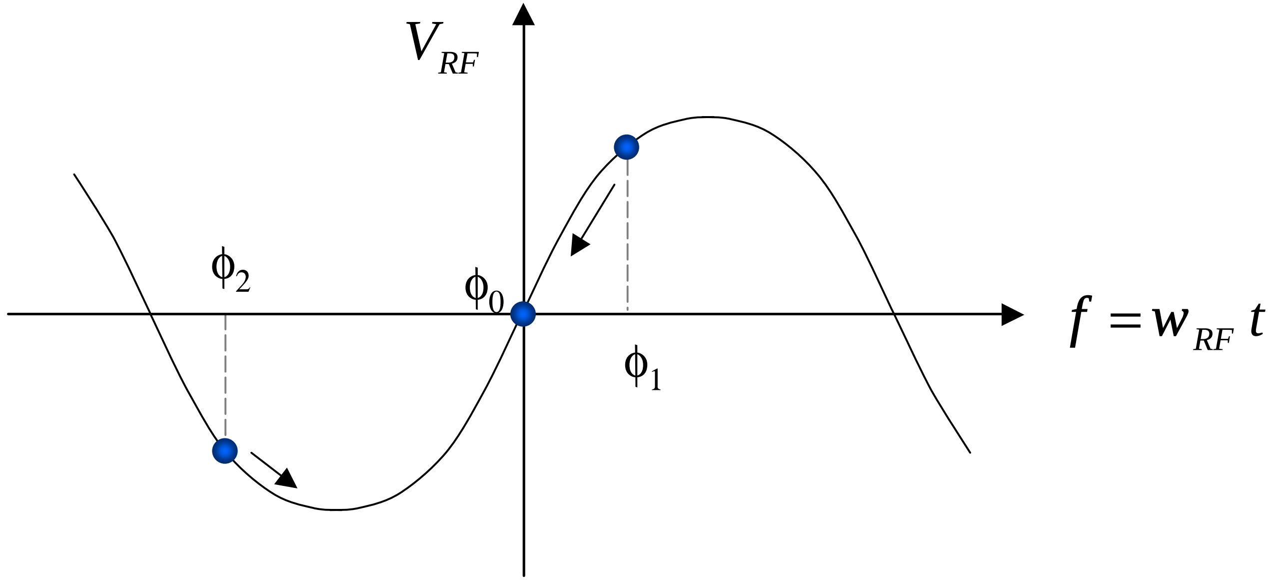
In order to keep the resonant condition, the particle must keep a constant energy

The phase of the synchronous particle must therefore be $f_0 = 0$ (circular machines convention)

Let's see what happens for a particle with the same energy and a different phase (e.g., f_1)

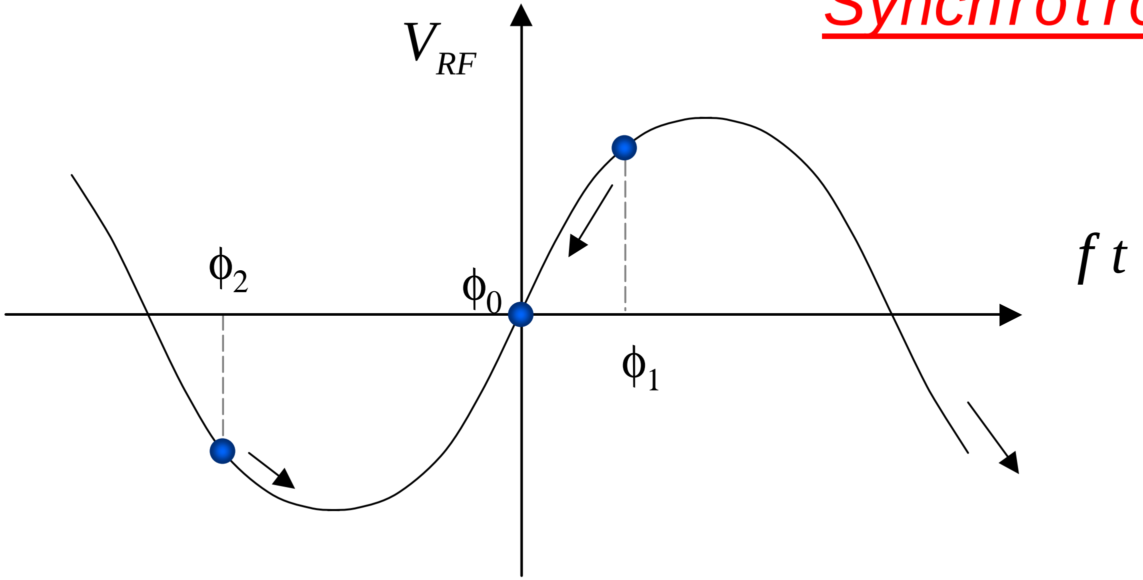
Synchrotron oscillations

f_1 - The particle is accelerated
- Below transition, an increase in energy means an increase in revolution frequency
- The particle arrives earlier – tends toward f_0

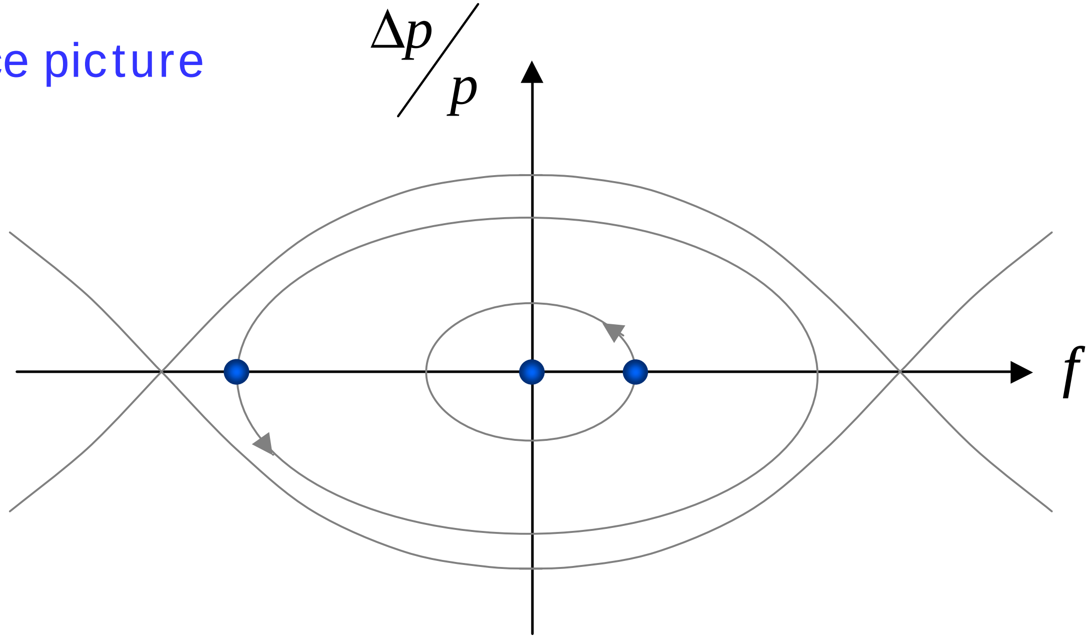


f_2 - The particle is decelerated
- decrease in energy - decrease in revolution frequency
- The particle arrives later – tends toward f_0

Synchrotron oscillations

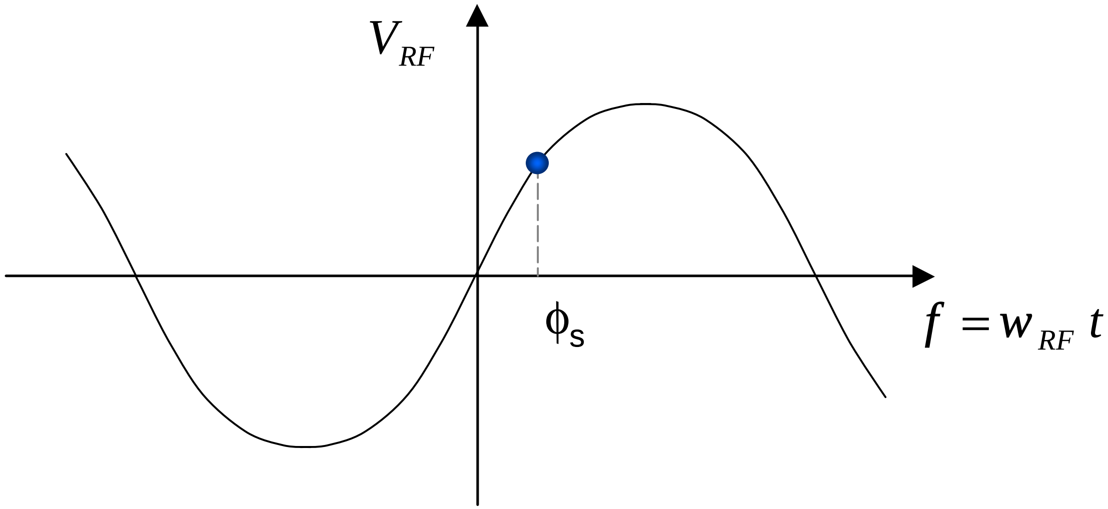


Phase space picture



Synchronous particle

Case with acceleration B increasing $\gamma < \gamma_{tr}$



$$\Delta E = e\hat{V}_{RF} \sin \phi$$

The phase of the synchronous particle is now $\phi_s > 0$ (circular machines convention)

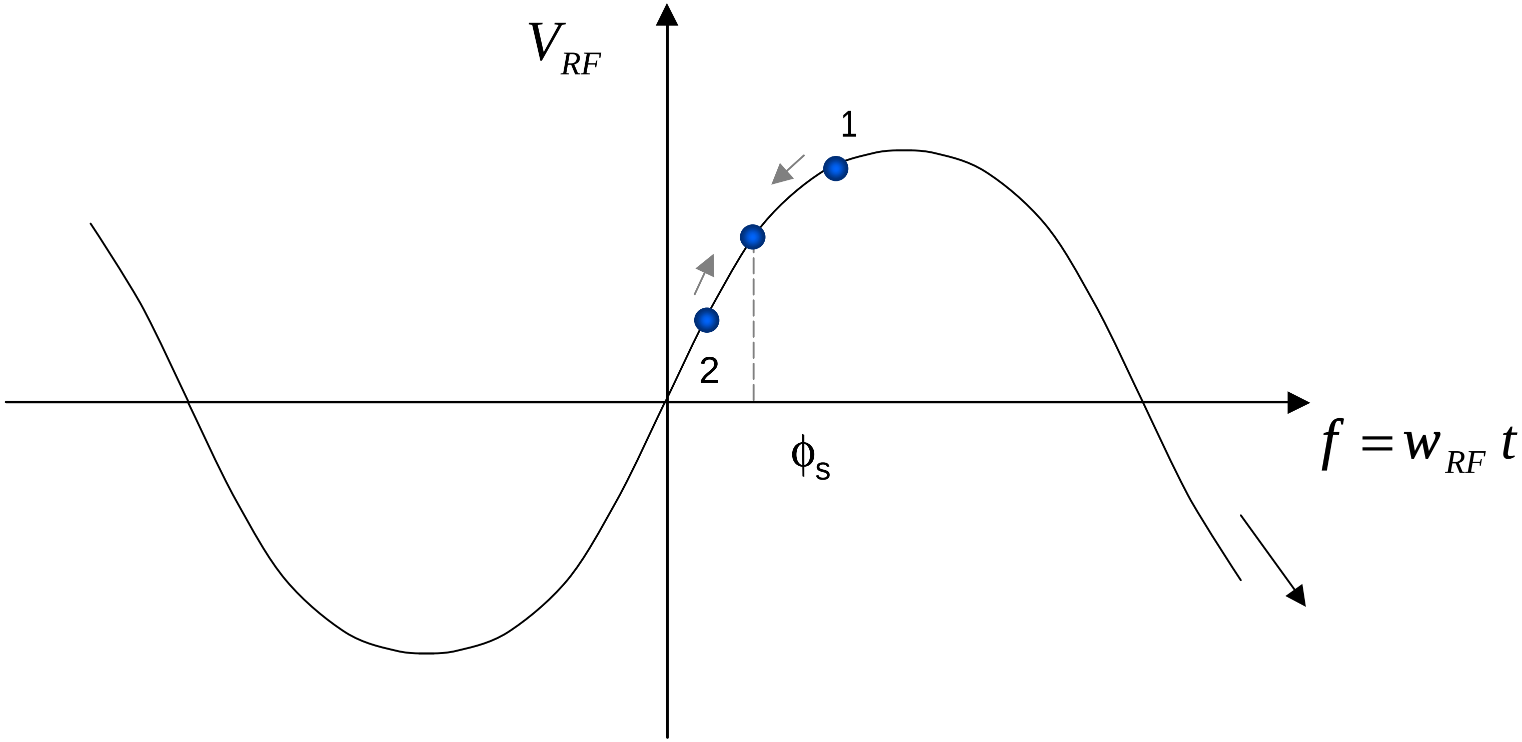
The synchronous particle accelerates, and the magnetic field is increased accordingly to keep the constant radius R.

$$R = \frac{\gamma\, v\, m_0}{eB}$$

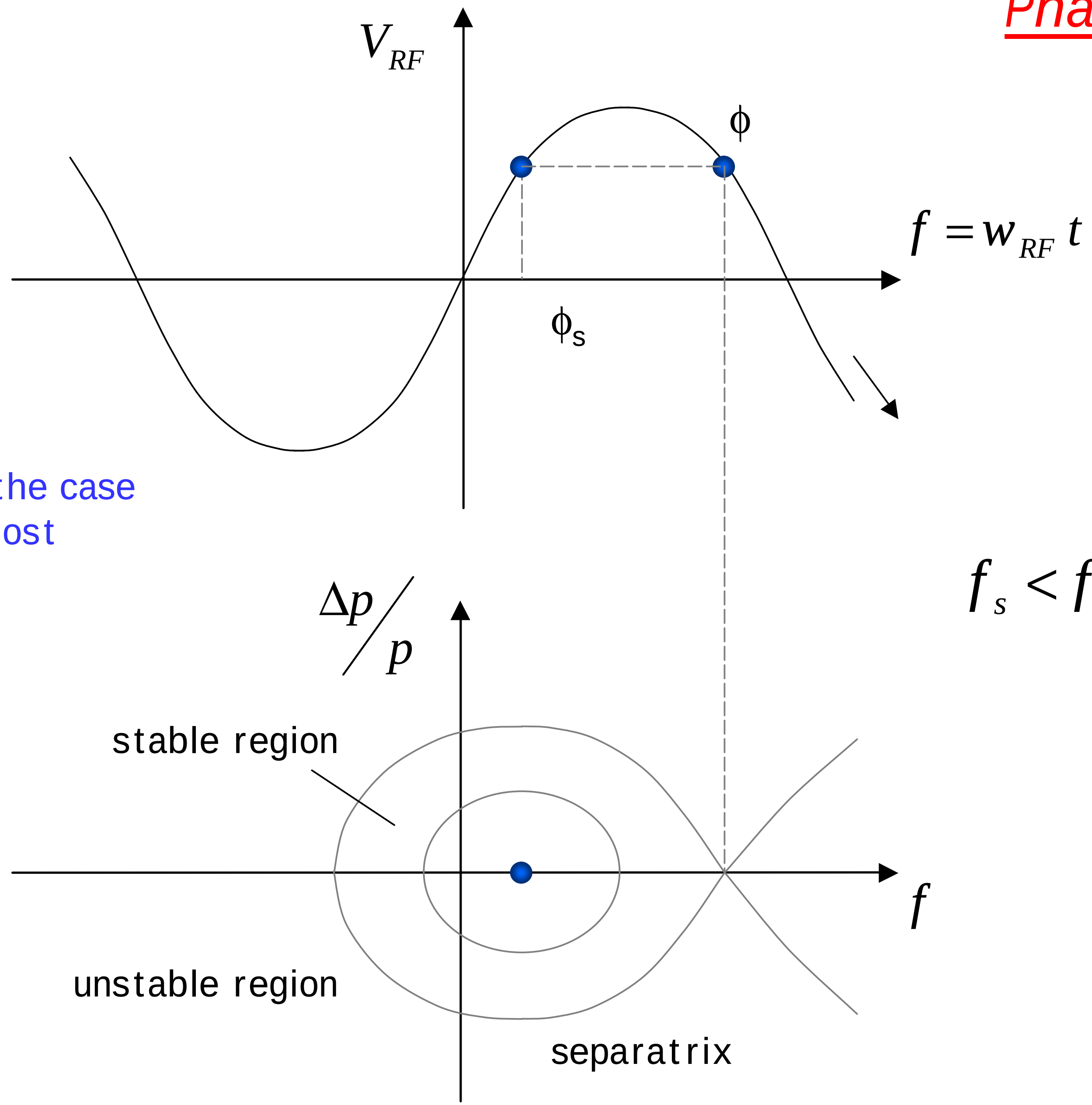
The RF frequency is increased as well is increased accordingly in order to keep the resonant condition

$$\omega = \frac{e\,B}{\gamma\, m_0} = \frac{\omega_{RF}}{h}$$

Phase stability



Phase stability



The simmetry of the case with B = const. is lost

$$f_s < f < p - f_s$$

LESSON IV

RF acceleration for synchronous particle

RF acceleration for non-synchronous particle

Small amplitude oscillations

Large amplitude oscillations – the RF bucket

RF acceleration for synchronous particle - energy gain

Let's assume a synchronous particle with a given $f_s > 0$

We want to calculate its rate of acceleration, and the related rate of increase of B, f.

$$p = e\,B\,r$$

Want to keep r = const

$$\frac{\mathrm{d}p}{\mathrm{d}t} = e\,r\,\frac{\mathrm{d}B}{\mathrm{d}t} = e\,r\,\dot{B}$$

Over one turn:

$$(\Delta p)_{turn} = e\,r\,\dot{B}\,T_{rev} = e\,r\,\dot{B}\,\frac{2p\,R}{b\,c}$$

We know that (relativistic equations) :

$$\Delta p = \frac{\Delta E}{b\,c}$$

$$(\Delta E)_{turn} = e\,r\,\dot{B}\;2p\,R$$

RF acceleration for synchronous particle - phase

$$(\Delta E)_{turn} = e\,\rho\,\dot{B}\;2\pi\;R$$

On the other hand, for the synchronous particle:

$$(\Delta E)_{turn} = e\hat{V}_{RF}\sin\phi_s$$

$$e\,\rho\,\dot{B}\;2\pi\;R = e\hat{V}_{RF}\sin\phi_s$$

Therefore:

1. Knowing ϕ_s, one can calculate the increase rate of the magnetic field needed for a given RF voltage:

$$\dot{B} = \frac{\hat{V}_{RF}}{2\pi\,\rho\,R}\sin\phi_s$$

2. Knowing the magnetic field variation and the RF voltage, one can calculate the value of the synchronous phase:

$$\sin\phi_s = 2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}} \quad\Rightarrow\quad \phi_s = \arcsin\left(2\pi\,\rho\,R\,\frac{\dot{B}}{\hat{V}_{RF}}\right)$$

RF acceleration for synchronous particle - frequency

$$w_{RF} = h w_s = h \frac{e}{m} < B >$$

$$\left(v = \frac{e}{m} B r \right)$$

$$w_{RF} = h \frac{e}{m} \frac{r}{R} B$$

From relativistic equations:

$$w_{RF} = \frac{hc}{R} \sqrt{\frac{B^2}{B^2 + \left(E_0 / ecr\right)^2}}$$

Let

$$B_0 \equiv \frac{E_0}{ecr}$$

$$f_{RF} = \frac{hc}{2p\ R} \left(\frac{B}{B_0} \right) \frac{1}{\sqrt{1 + \left(B / B_0\right)^2}}$$

Example: PS

At the CERN Proton Synchrotron machine, one has:

$$R = 100 \text{ m}$$

$$\dot{B} = 2.4 \text{ T/m}$$

100 dipoles with l_{eff} = 4.398 m. The harmonic number is 20

Calculate:

1. The energy gain per turn
2. The minimum RF voltage needed
3. The RF frequency when B = 1.23 (at extraction)

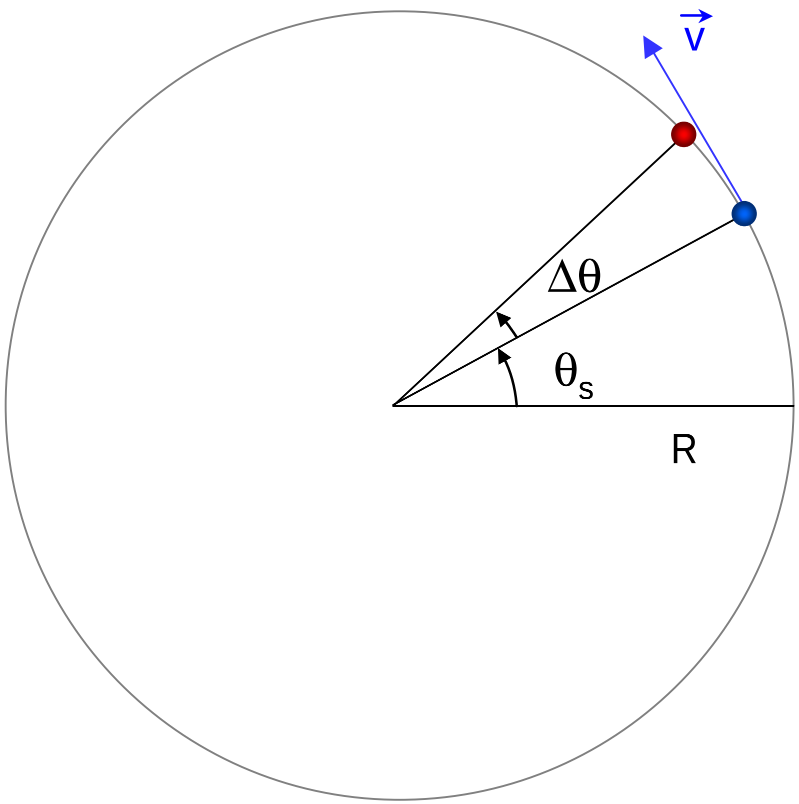
RF acceleration for non synchronous particle

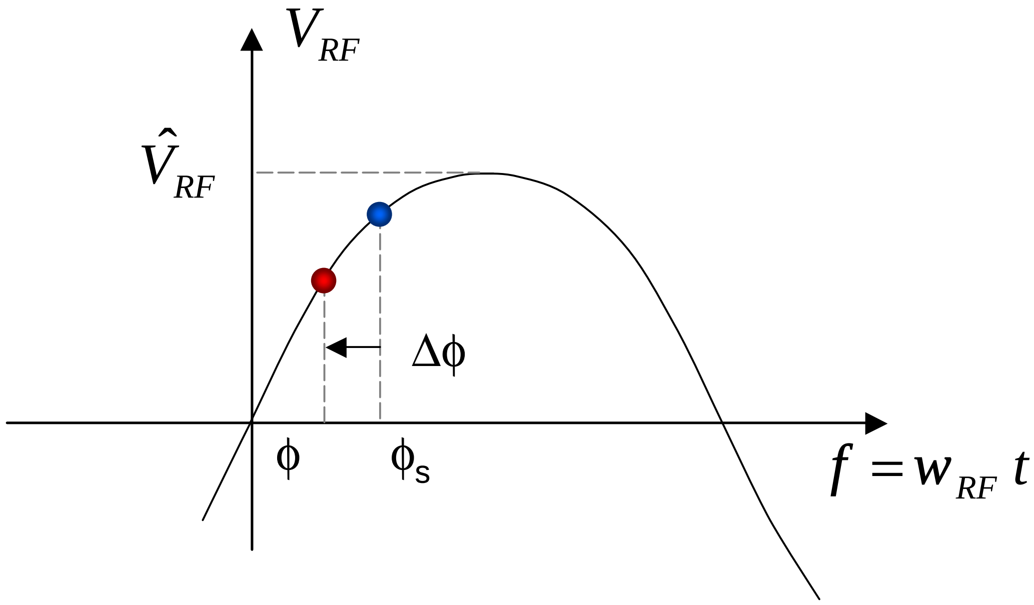
Parameter definition (subscript “s” stands for synchronous particle):

$f = f_s + \Delta f$	revolution frequency
$f = f_s + \Delta f$	RF phase
$p = p_s + \Delta p$	Momentum
$E = E_s + \Delta E$	Energy
$q = q_s + \Delta q$	Azimuth angle

$$\mathrm{d}s = R\,\mathrm{d}q$$

$$q(t) = \int_{t}^{t_0} w(t)\,\mathrm{d}t$$





$$\Delta q > 0 \quad \Rightarrow \quad \Delta f < 0$$

Since $\quad f_{RF} = h\ f_{rev}$

$$\Delta f = -h\,\Delta q$$

Over one turn $\quad q$ varies by 2 p
f varies by 2 p h

Parameters versus $\dot{f}$

1. Angular frequency

$$q(t) = \int_t^{t_0} w(t)\,\mathrm{d}t$$

$$\Delta w = \frac{\mathrm{d}}{\mathrm{d}t}(\Delta q)$$

$$= -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}(\Delta f)$$

$$= -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}(f - f_s)$$

$$\frac{\mathrm{d}f_s}{\mathrm{d}t} = 0$$ by definition

$$= -\frac{1}{h}\frac{\mathrm{d}f}{\mathrm{d}t}$$

$$\Rightarrow \quad \boxed{\Delta w = -\frac{1}{h}\frac{\mathrm{d}f}{\mathrm{d}t}}$$

Parameters versus $\dot{f}$

2. Momentum

$$\eta = \frac{\mathrm{d}\omega / \omega}{\mathrm{d}p / p} = \frac{\Delta\omega / \omega}{\Delta p / p}$$

$$\Delta p = \frac{p_s}{\omega_s}\frac{\Delta\omega}{\eta} = \frac{p_s}{\omega_s \eta}\left(-\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)$$

$$\Rightarrow \quad \boxed{\Delta p = \frac{-p_s}{\omega_s h \eta}\frac{\mathrm{d}\phi}{\mathrm{d}t}}$$

3. Energy

$$\frac{\mathrm{d}E}{\mathrm{d}p} = v$$

$$\frac{\Delta E}{\Delta p} = v = \omega R$$

$$\Rightarrow \quad \boxed{\Delta E = -\frac{R\, p_s}{h\, \eta}\frac{\mathrm{d}\phi}{\mathrm{d}t}}$$

Derivation of equations of motion

Energy gain after the RF cavity

$$(\Delta E)_{turn} = e\hat{V}_{RF}\sin f$$

$$(\Delta p)_{turn} = \frac{e}{w\,R}\,\hat{V}_{RF}\sin f$$

Average increase per time unit

$$\frac{(\Delta p)_{turn}}{T_{rev}} = \frac{e}{2p\;R}\,\hat{V}_{RF}\sin f \qquad 2p\;R\,\dot{p} = e\,\hat{V}_{RF}\sin f$$

valid for any particle !

$$2p\,\left(R\,\dot{p} - R_s\,\dot{p}_s\right) = e\,\hat{V}_{RF}\left(\sin f - \sin f_s\right)$$

Derivation of equations of motion

After some development (see J. Le Duff, in Proceedings CAS 1992, CERN 94-01)

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_s}\right) = e\hat{V}_{RF}\left(\sin\phi - \sin\phi_s\right)$$

An approximated version of the above is

$$\frac{\mathrm{d}(\Delta p)}{\mathrm{d}t} = \frac{e\hat{V}_{RF}}{2\pi R_s}\left(\sin\phi - \sin\phi_s\right)$$

Which, together with the previously found equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\omega_s \eta h}{p_s}\Delta p$$

Describes the motion of the non-synchronous particle in the longitudinal phase space ($\Delta p, \phi$)

Equations of motion I

$$\begin{cases} \dfrac{\mathrm{d}(\Delta p)}{\mathrm{d}t} = A\left(\sin\varphi - \sin\varphi_s\right) \\[2ex] \dfrac{\mathrm{d}\varphi}{\mathrm{d}t} = B\,\Delta p \end{cases}$$

with

$$A = \frac{e\hat{V}_{RF}}{2\pi R_s}$$

$$B = -\frac{\eta\, h}{p_s}\frac{\beta_s c}{R_s}$$

Equations of motion II

1. First approximation – combining the two equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{B}\frac{\mathrm{d}f}{\mathrm{d}t}\right) - A\left(\sin f - \sin f_s\right) = 0$$

We assume that A and B <u>change very slowly</u> compared to the variable $\mathrm{D}f = f - f_s$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + \frac{\Omega_s^{\ 2}}{\cos f_s}\left(\sin f - \sin f_s\right) = 0$$

with $$\frac{\Omega_s^{\ 2}}{\cos f_s} = -AB$$

We can also define: $$\Omega_0^{\ 2} = \frac{\Omega_s^{\ 2}}{\cos f_s} = \frac{e\hat{V}_{RF}\, h\, h\, c^2}{2p\, R_s^{\ 2} E_s}$$

Small amplitude oscillations

2. Second approximation

$$\sin f = \sin\left(f_s + \Delta f\right)$$
$$= \sin f_s \cos \Delta f + \cos f_s \sin \Delta f$$

Δf small $\Rightarrow$ $\sin f \cong \sin f_s + \cos f_s \, \Delta f$

$\dfrac{\mathrm{d} f_s}{\mathrm{d}t} = 0$ $\Rightarrow$ $\dfrac{\mathrm{d}^2 f}{\mathrm{d}t^2} = \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\left(f_s + \Delta f\right) = \dfrac{\mathrm{d}^2 \Delta f}{\mathrm{d}t^2}$

by definition

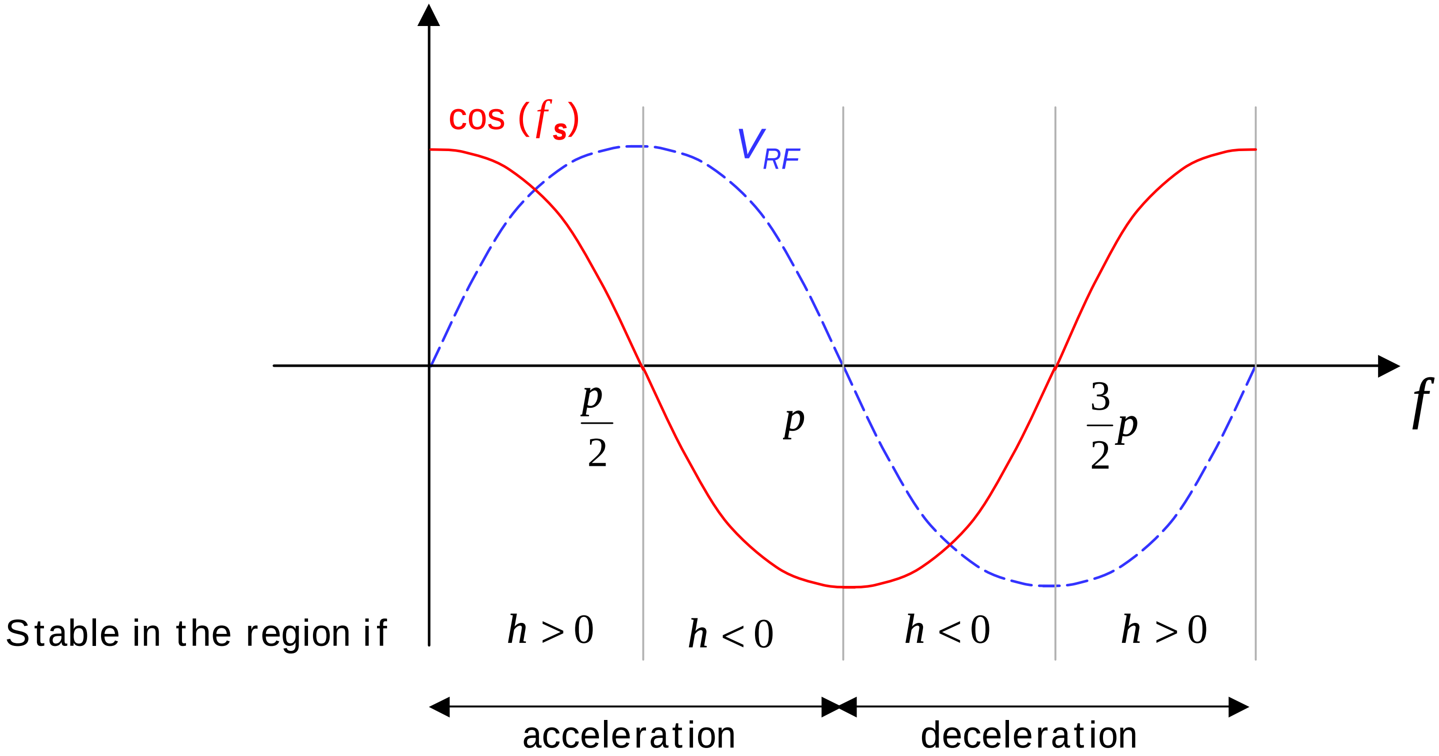
$$\frac{\mathrm{d}^2 \Delta f}{\mathrm{d}t^2} + \Omega_s^{\,2} \, \Delta f = 0$$

Harmonic oscillator !

Stability condition for f_s

Stability is obtained when the angular frequency of the oscillator, $\Omega_s^{\,2}$ is real positive:

$$\Omega_s^{\,2} = \frac{e\hat{V}_{RF}\, h\, h\, c^2}{2p\, R_s^{\,2} E_s}\cos f_s \quad\Rightarrow\quad \Omega_s^{\,2} > 0 \quad\Leftrightarrow\quad h\cos f_s > 0$$



Small amplitude oscillations - orbits

For $h\cos\phi_s > 0$ the motion around the synchronous particle is a stable oscillation:

$$\begin{cases} \Delta\phi = \Delta\phi_{\max}\sin(\Omega_s t + \phi_0) \\ \Delta p = \Delta p_{\max}\cos(\Omega_s t + \phi_0) \end{cases}$$

with $\Delta p_{\max} = \dfrac{\Omega_s}{B}\Delta\phi_{\max}$

Lepton machines

e+, e-

$$b \cong 1 \quad , \quad g \text{ large} \quad , \quad h \cong -a_p$$

$$w_s \cong \frac{c}{R_s} \quad , \quad p_s \cong \frac{E_s}{c}$$

$$\Omega_s = \frac{c}{R_s}\left\{-\frac{e\hat{V}_{RF}\, a_p\, h}{2p\, E_s}\cos f_s\right\}^{1/2}$$

Number of synchrotron oscillations per turn:

$$Q_s = \frac{\Omega_s}{w_s} = \left\{-\frac{e\hat{V}_{RF}\, a_p\, h}{2p\, E_s}\cos f_s\right\}^{1/2}$$ "synchrotron tune"

N.B: in these machines, the RF frequency does not change

Large amplitude oscillations

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

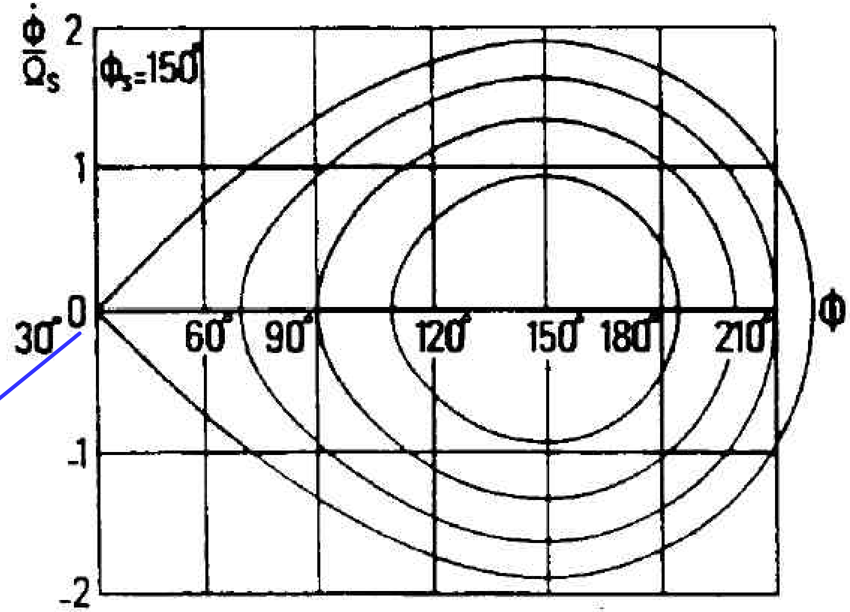
Multiplying by $\dot{\phi}$ and integrating

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = cte$$

Constant of motion

here $\dot{f} = 0$

$f = p - f_s$



Synchronous phase 150°

Equation of the separatrix

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\ \sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left[\cos\left(\pi - \phi_s\right) + \left(\pi - \phi_s\right)\sin\phi_s\right]$$

Energy diagram

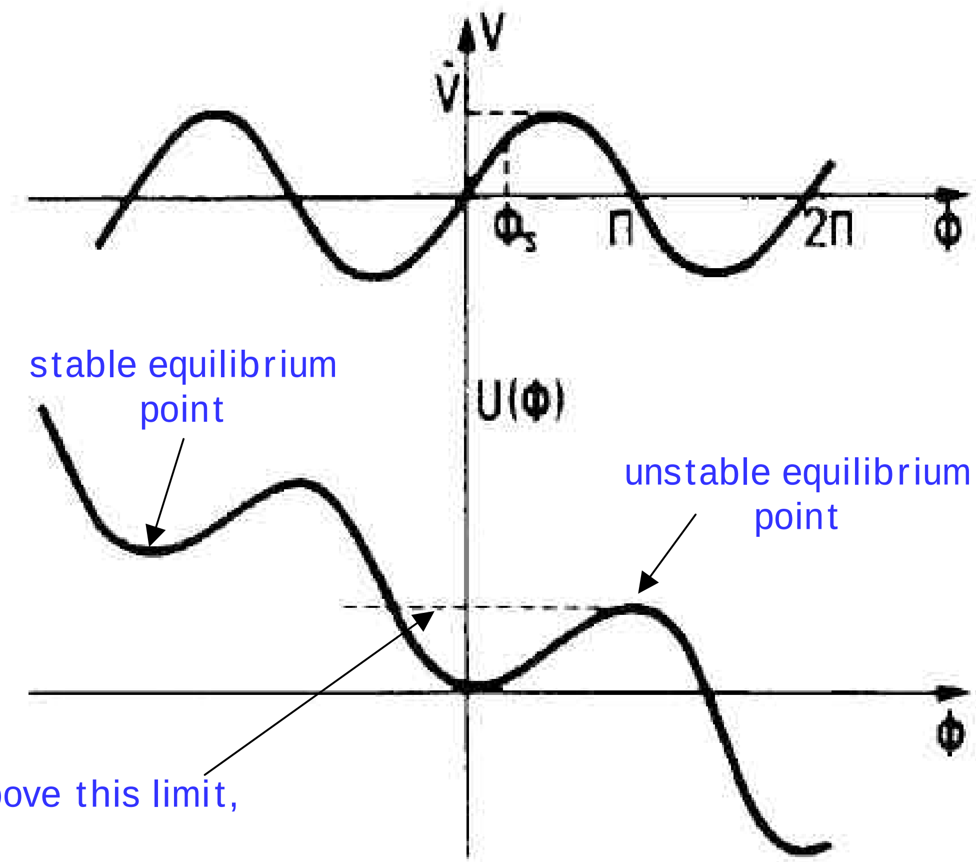
"total energy"

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = cte$$

"kinetic energy" "potential energy *U* "

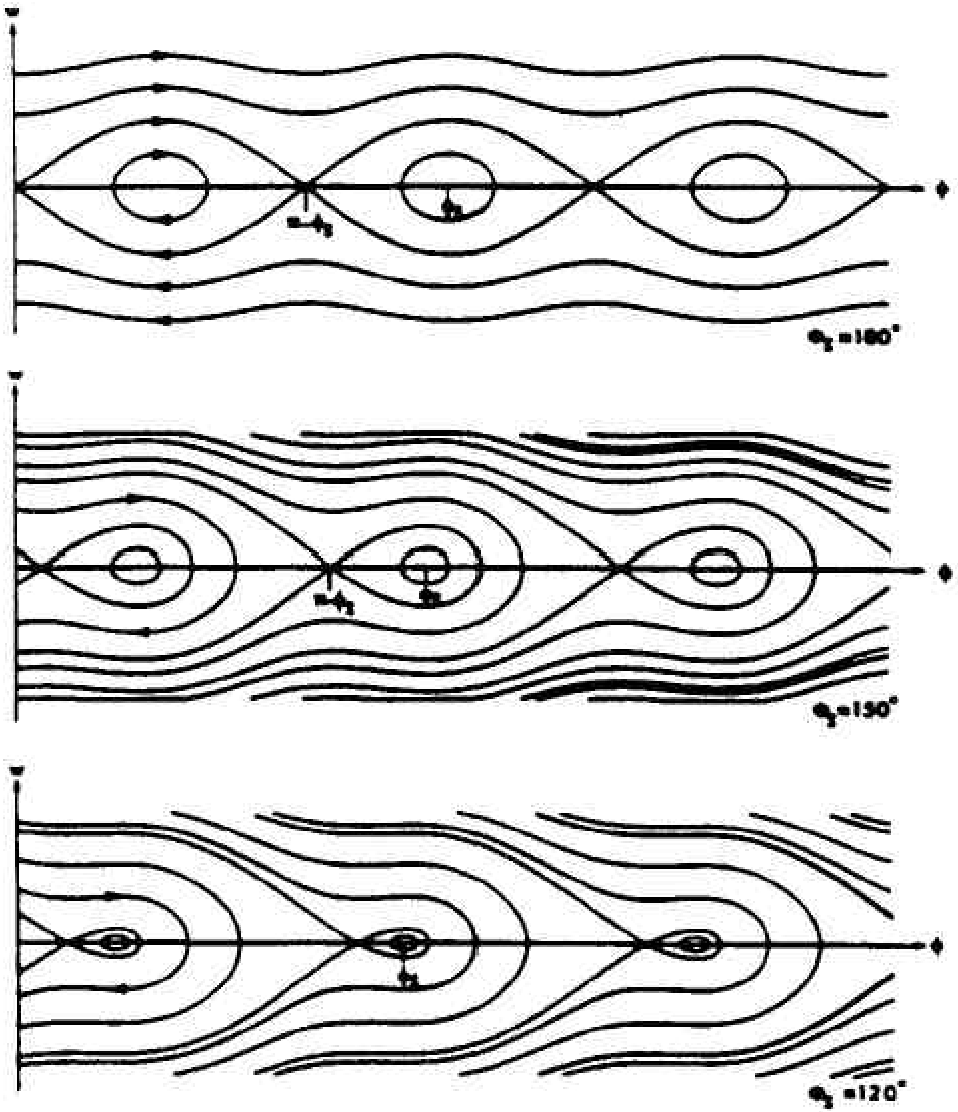
$$\frac{d^2\phi}{dt^2} = F(\phi)$$

$$F(\phi) = -\frac{\partial U}{\partial \phi}$$



If the total energy is above this limit, the motion is unbounded

Phase space trajectories



Phase space trajectories for different synchronous phases