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AN INTRODUCTION TO TRANSVERSE BEAM DYNAMICS IN ACCELERATORS

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Abstract

This text represents an attempt to give a comprehensive introduction to the dynamics of charged particles in accelerator and beam transport lattices. The first part treats the basic principles of linear single particle dynamics in the transverse phase plane. The general equations of motion of a charged particle in magnetic fields are derived. Next, the linearized equations of motion in the presence of bending and focusing fields only are solved. This yields the optical parameters of the lattice, useful in strong focusing machines, from which the oscillatory motion — the betatron oscillation — and also the transfer matrices of the lattice elements are expressed. The second part deals with some of the ideas connected with nonlinearities and resonances in accelerator physics. The equations of motion are revisited, considering the influence of nonlinear magnetic fields as pure multipole magnet fields. A presentation of basic transverse resonance phenomena caused by multipole perturbation terms is given, followed by a semi-quantitative discussion of the third-integer resonance. Finally, chromatic effects in circular accelerators are described and a simple chromaticity correction scheme is given. The beam dynamics is presented on an introductory level, within the framework of differential equations and using only straightforward perturbation methods, discarding the formulation in terms of Hamiltonian formalism.

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1 PARTICLE MOTION IN MAGNETIC FIELDS

1.1 Coordinate system

The motion of a charged particle in a beam transport channel or in a circular accelerator is governed by the Lorentz force equation

$$\boldsymbol{F} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\,, \tag{1.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, $\boldsymbol{v}$ is the particle velocity, and e is the electric charge of the particle. The Lorentz forces are applied as bending forces to guide the particles along a predefined ideal path, the design orbit, on which—ideally—all particles should move, and as focusing forces to confine the particles in the vicinity of the ideal path, from which most particles will unavoidably deviate. The motion of particle beams under the influence of these Lorentz forces is called beam optics.

The design orbit can be described using a fixed, right-handed Cartesian reference system. However, using such a reference system it is difficult to express deviations of individual particle trajectories from the design orbit. Instead, we will use a right-handed orthogonal system $(\boldsymbol{n},\ \boldsymbol{b},\ \boldsymbol{t})$ that follows an ideal particle traveling along the design orbit. The variables s and σ describe the ideal beam path and an individual particle trajectory. We have chosen the convention that $\boldsymbol{n}$ is directed outward if the motion lies in the horizontal plane, and upwards if it lies in the vertical plane.

Let $\delta\boldsymbol{r}(s)$ be the deviation of the particle trajectory $\boldsymbol{r}(s)$ from the design orbit $\boldsymbol{r}_0(s)$. Assume that the design orbit is made of piecewise flat curves, which can be either in the horizontal or vertical plane so that it has no torsion. Hence, from the Frenet–Serret formulae we find

$$\frac{d\boldsymbol{r}_0}{ds} = \boldsymbol{t} \qquad \frac{d\boldsymbol{t}}{ds} = -k(s)\boldsymbol{n} \qquad \frac{d\boldsymbol{b}}{ds} = 0 \qquad \frac{d\boldsymbol{n}}{ds} = k(s)\boldsymbol{t}\,, \tag{1.2}$$

where $k(s)$ is the curvature, $\boldsymbol{t}$ is the target unit vector, $\boldsymbol{n}$ the normal unit vector and $\boldsymbol{b}$ the binomial vector: $\boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n}$.

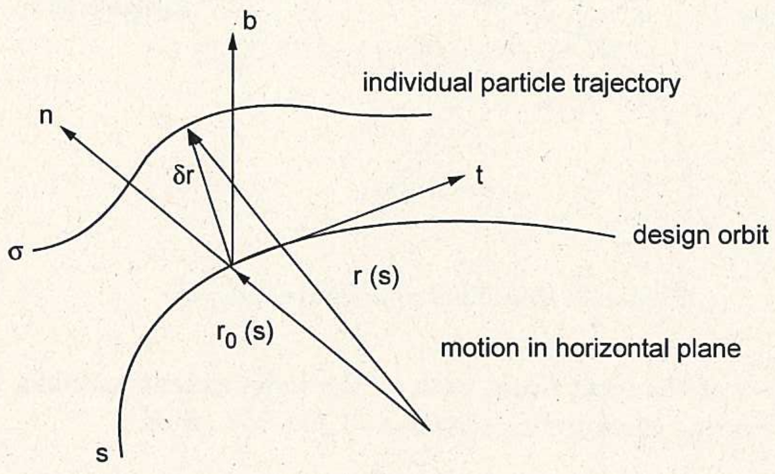

Figure 1: Coordinate system $(\boldsymbol{n},\ \boldsymbol{b},\ \boldsymbol{t})$.

Unfortunately, $\boldsymbol{n}$ charges discontinuously if the design orbit jumps from the horizontal to the vertical plane, and vice versa. Therefore, we instead introduce the new right-handed coordinate system $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{t})$:

$$\boldsymbol{e}_x = \begin{cases} \boldsymbol{n} & \text{if the orbit lies in the horizontal plane} \\ -\boldsymbol{b} & \text{if the orbit lies in the vertical plane} \end{cases}$$

$$\boldsymbol{e}_y = \begin{cases} \boldsymbol{b} & \text{if the orbit lies in the horizontal plane} \\ \boldsymbol{n} & \text{if the orbit lies in the vertical plane.} \end{cases}$$

Thus:

$$\frac{d\boldsymbol{e}_x}{ds} = k_x \boldsymbol{t} \qquad \frac{d\boldsymbol{e}_y}{ds} = k_y \boldsymbol{t} \qquad \frac{d\boldsymbol{t}}{ds} = -k_x \boldsymbol{e}_x - k_y \boldsymbol{e}_y \,. \tag{1.3}$$

The last equation stands, provided we assume

$$k_x(s) k_y(s) = 0 \,, \tag{1.4}$$

where k_x, k_y are the curvatures in the x-direction and the y-direction. Hence the individual particle trajectory reads:

$$\boldsymbol{r}(x, y, s) = \boldsymbol{r}_0(s) + x\, \boldsymbol{e}_x(s) + y\, \boldsymbol{e}_y(s) \,, \tag{1.5}$$

where (x, y, s) are the particle coordinates in the new reference system.

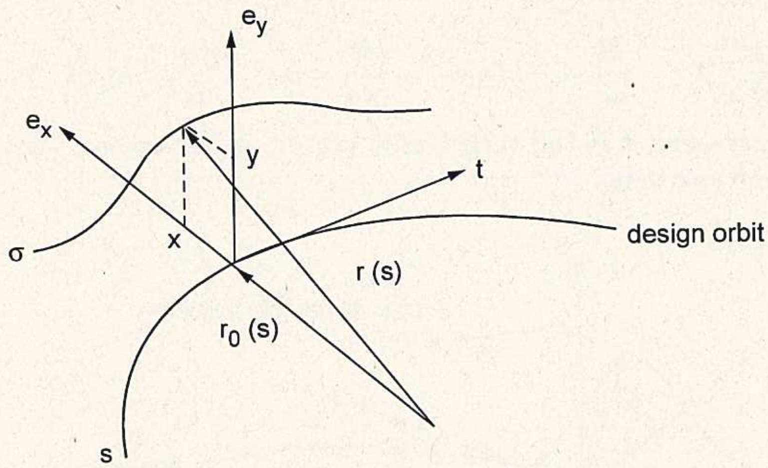

Figure 2: Coordinate system $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{t})$.

Consider the length s of the ideal beam path as the independent variable, instead of the time variable t, to express the Lorentz equation (1.1). Now, from

$$\frac{d}{dt} = \frac{d\sigma}{dt}\frac{d}{d\sigma} = v\frac{ds}{d\sigma}\frac{d}{ds} = \frac{v}{\sigma'}\frac{d}{ds} \,,$$

where $v = d\sigma/dt$ is the particle velocity and a prime denotes the derivative with respect to s, it follows that

$$\frac{d\boldsymbol{r}}{dt} = \frac{v}{\sigma'}\frac{d\boldsymbol{r}}{ds} = \frac{v}{\sigma'}\boldsymbol{r}' \equiv \boldsymbol{v}$$

$$\begin{aligned}
\frac{d^2\boldsymbol{r}}{dt^2} &= \frac{d}{dt}\left(\frac{d\boldsymbol{r}}{dt}\right) = \frac{v}{\sigma'}\frac{d}{ds}\left(v\frac{\boldsymbol{r}'}{\sigma'}\right) = \frac{v^2}{\sigma'^2}\boldsymbol{r}'' - \frac{v^2}{\sigma'}\boldsymbol{r}'\frac{\sigma''}{\sigma'^2} \\
&= \frac{v^2}{\sigma'^2}\left(\boldsymbol{r}'' - \frac{1}{2}\frac{\boldsymbol{r}'}{\sigma'^2}\frac{d}{ds}(\sigma'^2)\right) = \frac{v^2}{\sigma'^2}\left(\boldsymbol{r}'' - \frac{\sigma''}{\sigma'}\boldsymbol{r}'\right) .
\end{aligned}$$

Now, from (1.3) and (1.5),

$$\begin{aligned}
\boldsymbol{r}'(s) &= \boldsymbol{r}_0'(s) + x'\boldsymbol{e}_x + y'\boldsymbol{e}_y + x\,\boldsymbol{e}_x' + y\,\boldsymbol{e}_y' \\
&= \boldsymbol{t} + x'\boldsymbol{e}_x + y'\boldsymbol{e}_y + x\,k_x\boldsymbol{t} + y\,k_y\boldsymbol{t} \\
&= (1 + k_x x + k_y y)\boldsymbol{t} + x'\boldsymbol{e}_x + y'\boldsymbol{e}_y .
\end{aligned}$$

Similarly, we compute the second derivative:

$$\begin{aligned}
\boldsymbol{r}''(s) &= (k_x x' + k_x' x + k_y Y' + k_y' y)\boldsymbol{t} + x''\boldsymbol{e}_x + y''\boldsymbol{e}_y + x'\boldsymbol{e}_x' + y'\boldsymbol{e}_y' \\
&\quad + (1 + k_x x + k_y y)\boldsymbol{t}' \\
&= (k_x x' + k_x' x + k_y y' + k_y' y)\boldsymbol{t} + x''\boldsymbol{e}_x + y''\boldsymbol{e}_y + x'k_x\boldsymbol{t} + y'k_y\boldsymbol{t} \\
&\quad + (1 + k_x x + k_y y)(-k_x\boldsymbol{e}_x - k_y\boldsymbol{e}_y)
\end{aligned}$$

Then,

$$\begin{aligned}
\boldsymbol{r}''(s) &= (k_x' x + k_y' y + 2k_x x' + 2k_y y')\boldsymbol{t} \\
&\quad + [x'' - k_x(1 + k_x x)]\boldsymbol{e}_x + [y'' - k_y(1 + k_y y)]\boldsymbol{e}_y .
\end{aligned}$$

The Lorentz force $\boldsymbol{F}$ may be expressed by the change in the particle momentum $\boldsymbol{p}$ as

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) , \tag{1.6}$$

with

$$\boldsymbol{p} = m\gamma\boldsymbol{v} , \tag{1.7}$$

where m is the particle rest mass and γ the ratio of the particle energy to its rest energy.

Assuming that γ and v are constants (no particle acceleration), the left-hand side of (1.6) can be written as

$$\begin{aligned}
\boldsymbol{F} &= m\gamma\frac{d\boldsymbol{v}}{dt} = m\gamma\frac{d^2\boldsymbol{r}}{dt^2} = m\gamma\frac{v^2}{\sigma'^2}\left(\boldsymbol{r}'' - \frac{\sigma''}{\sigma'}\boldsymbol{r}'\right) \\
&= m\gamma\frac{v^2}{\sigma'^2}\Big\{(k_x' x + k_y' y + 2k_x x' + 2k_y y')\boldsymbol{t} + [x'' - k_x(1 + k_x x)]\boldsymbol{e}_x \\
&\quad + [y'' - k_y(1 + k_y y)]\boldsymbol{e}_y - \frac{\sigma''}{\sigma'}[(1 + k_x x + k_y y)\boldsymbol{t} + x'\boldsymbol{e}_x + y'\boldsymbol{e}_y]\Big\} \\
&= m\gamma\frac{v^2}{\sigma'^2}\left(k_x' x + k_y' y + 2k_x x' + 2k_y y' - \frac{\sigma''}{\sigma'}(1 + k_x x + k_y y)\right)\boldsymbol{t} \\
&\quad + \left(x'' - k_x(1 + k_x x) - \frac{\sigma''}{\sigma'}x'\right)\boldsymbol{e}_x + \left(y'' - k_y(1 + k_y y) - \frac{\sigma''}{\sigma'}y'\right)\boldsymbol{e}_y .
\end{aligned}$$

The magnetic field may be expressed in the $(x,\, y,\, s)$ reference system

$$\boldsymbol{B}(x,y,s) = B_t(x,y,s)\boldsymbol{t} + B_x(x,y,s)\boldsymbol{e}_x + B_y(x,y,s)\boldsymbol{e}_y \,. \tag{1.8}$$

In the absence of an electric field the Lorentz force equation (1.1) becomes

$$\boldsymbol{F} = e(\boldsymbol{v} \times \boldsymbol{B}) = \frac{ev}{\sigma'}(\boldsymbol{r}' \times \boldsymbol{B}) \,. \tag{1.9}$$

Using the above results and defining the variable

$$h = 1 + k_x x + k_y y \,, \tag{1.10}$$

the vector product may be written as

$$\begin{aligned} \boldsymbol{r}' \times \boldsymbol{B} &= (h\boldsymbol{t} + x'\boldsymbol{e}_x + y'\boldsymbol{e}_y) \times (B_t\boldsymbol{t} + B_x\boldsymbol{e}_x + B_y\boldsymbol{e}_y) \\ &= (y'B_t - hB_y)\boldsymbol{e}_x - (xB_t - hB_x)\boldsymbol{e}_y + (x'B_y - y'B_x)\boldsymbol{t} \,. \end{aligned}$$

Finally, equating the expression for the rate of change in the particle momentum, the left-hand side of (1.6) with (1.9) and with (1.4) gives

$$\begin{aligned} &(k_x'x + k_y'y + 2k_xx' + 2k_yy' - \frac{\sigma''}{\sigma'}h)\boldsymbol{t} + \left(x'' - k_xh - \frac{\sigma''}{\sigma'}x'\right)\boldsymbol{e}_x \\ &\quad + \left(y'' - k_yh - \frac{\sigma''}{\sigma'}y'\right)\boldsymbol{e}_y \\ &= \frac{e}{p}\sigma'[(x'B_y - y'B_x)\boldsymbol{t} - (hB_y - y'B_t)\boldsymbol{e}_x + (hB_x - x'B_t)\boldsymbol{e}_y] \,, \end{aligned}$$

since

$$k_x(1 + k_xx) = k_xh \;\text{ and }\; k_y(1 + k_yy) = k_yh \,.$$

Identifying the terms in $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{t}$ leads to the equations of motion

$$\begin{aligned} x'' - \frac{\sigma''}{\sigma'}x' &= k_xh - \frac{e}{p}\sigma'(hB_y - y'B_t), \\ y'' - \frac{\sigma''}{\sigma'}y' &= k_yh + \frac{e}{p}\sigma'(hB_x - x'B_t), \\ \frac{\sigma''}{\sigma'} &= \frac{1}{h}(k_x'x + k_y'y + 2k_xx' + 2k_yy') - \frac{e}{hp}\sigma'(x'B_y - y'B_x) \,. \end{aligned} \tag{1.11}$$

The last equation may be used to eliminate the term σ''/σ' in the other two.

Expanding the particle momentum in the vicinity of the ideal momentum p_0, corresponding to a particle travelling on the design orbit, yields

$$\frac{1}{p} = \frac{1}{p_0(1+\delta)} \,,$$

where

$$\delta = \frac{p - p_0}{p_0} \equiv \frac{\Delta p}{p_0} \tag{1.12}$$

is the relative momentum deviation. Hence, the above general equations of motion for a charged particle in a magnetic field $\boldsymbol{B}$ may be rewritten as:

$$x'' - \frac{\sigma''}{\sigma'}x' = k_x h - (1+\delta)^{-1}\,\frac{e}{p_0}\sigma'(hB_y - y'B_t)\,,$$

$$y'' - \frac{\sigma''}{\sigma'}y' = k_y h + (1+\delta)^{-1}\,\frac{e}{p_0}\sigma'(hB_x - x'B_t)$$

$$\frac{\sigma''}{\sigma'} = \frac{1}{h}(k_x' x + k_y' y + 2k_x x' + 2k_y y') - (1+\delta)^{-1}\,\frac{e}{hp_0}\sigma'(x'B_y - y'B_x)\,. \tag{1.13}$$

Using

$$\boldsymbol{v} = \frac{v}{\sigma'}\boldsymbol{r}'\,,$$

it follows that

$$|\boldsymbol{v}| = v = \frac{v}{\sigma'}|\boldsymbol{r}'| = \frac{v}{\sigma'}\sqrt{\boldsymbol{r}'\cdot\boldsymbol{r}'}\,,$$

then

$$\sigma' = |\boldsymbol{r}'| = \sqrt{h^2 + x'^2 + y'^2}\,. \tag{1.14}$$

On the design orbit (equilibrium orbit) we get

$$\begin{aligned} x &= x' = 0 \\ y &= y' = 0 \\ \delta &= 0\,. \end{aligned}$$

Consequently $h = 1$, $\sigma' = 1$, $\sigma''/\sigma' = 0$, and using (1.13) we find

$$\begin{aligned} k_x &= \frac{e}{p_0}B_y(0,0,s) \equiv \frac{e}{p_0}B_{y0}\,, \\ k_y &= -\frac{e}{p_0}B_x(0,0,s) \equiv -\frac{e}{p_0}B_{x0}\,. \end{aligned} \tag{1.15}$$

Equivalently, introducing the local bending radii ρ_x and ρ_y,

$$\rho_{x,y}(s) = \frac{1}{k_{x,y}(s)}\,, \tag{1.16}$$

we get the bending field for the design momentum p_0

$$\frac{1}{\rho_x} = \frac{e}{p_0}B_{y0} \qquad \frac{1}{\rho_y} = -\frac{e}{p_0}B_{x0}\,. \tag{1.17}$$

We adopt the following sign convention: an observer looking in the positive s-direction sees a positive charge travelling along the positive s-direction deflected to the right (resp.

upwards) by a positive vertical magnetic field $B_y > 0$ (resp. by a positive horizontal magnetic field $B_x > 0$).

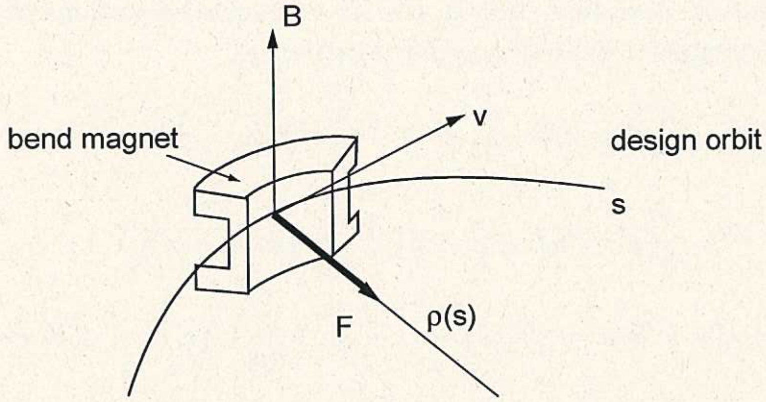

Figure 3: Lorentz force (for a positive charged particle).

The term $|B_0\rho|$ is the beam rigidity (B_0 and ρ stand for B_{x0} and ρ_x or B_{y0} and ρ_y). In more practical units the beam rigidity reads:

$$B_0\rho(\text{tesla}\cdot m) = 3.3356\ p\ (\text{GeV}/c) = 3.3356\ \beta E\ (\text{GeV})\ ,$$

where E is the particle total energy, p the particle momentum, and $\beta = v/c$.

1.2 Linearized equations of motion

From the general equations of motion (1.13) we retain only linear terms in x, x', y, y' and δ. Starting with (1.14) we get

$$\sigma' = h\sqrt{1+\frac{x'^2}{h^2}+\frac{y'^2}{h^2}} \approx h = 1 + k_x x + k_y y$$

$$\sigma'' \approx h' = k_x' x + k_x x' + k_y' y + k_y y'\ .$$

Hence:

$$\frac{\sigma''}{\sigma'} \approx h'$$

With this approximation and with the first-order series expansion $(1+\delta)^{-1} \approx 1-\delta$, the equations of motion (1.13) in the horizontal and vertical planes become, in the absence of a tangential magnetic field (no solenoid field),

$$\begin{aligned} x'' &= k_x h - (1-\delta)h^2\frac{e}{p_0}B_y(x,y,s)\ , \\ y'' &= k_y h + (1-\delta)h^2\frac{e}{p_0}B_x(x,y,s)\ . \end{aligned} \tag{1.18}$$

For x and y, small deviations from the design orbit, the field components may be expanded in series to the first order:

$$\begin{aligned} B_x(x,y,s) &= B_{x0} + \frac{\partial B_x}{\partial x}x + \frac{\partial B_x}{\partial y}y \;, \\ B_y(x,y,s) &= B_{y0} + \frac{\partial B_y}{\partial x}x + \frac{\partial B_y}{\partial_y}y \;. \end{aligned} \tag{1.19}$$

The magnetic field satisfies the Maxwell equations

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = 0 \;,$$

from which we get, using (1.8),

$$\frac{\partial B_x}{\partial x} = -\frac{\partial B_y}{\partial y} \qquad \frac{\partial B_y}{\partial x} = \frac{\partial B_x}{\partial y} \;. \tag{1.20}$$

Thus, introducing the normalized gradient K_0, and the skew normalized gradient $\underline{K}_0$ defined as

$$K_0 = \frac{e}{p_0}\left(\frac{\partial B_y}{\partial x}\right)_{x=y=0} \qquad \underline{K}_0 = \frac{e}{p_0}\left(\frac{\partial B_x}{\partial x}\right)_{x=y=0} \;, \tag{1.21}$$

the field components may be written with (1.15)

$$\begin{aligned} \frac{e}{p_0}B_x(x,y,s) &= -k_y + \underline{K}_0 x + K_0 y \;, \\ \frac{e}{p_0}B_y(x,y,s) &= k_x + K_0 x - \underline{K}_0 y \;. \end{aligned} \tag{1.22}$$

Hence, the transverse equations of motion become

$$\begin{aligned} x'' &= k_x h - (1-\delta)h^2(k_x + K_0 x - \underline{K}_0 y) \;, \\ y'' &= k_y h - (1-\delta)h^2(k_y - \underline{K}_0 x - K_0 y) \;. \end{aligned} \tag{1.23}$$

From (1.4) and (1.10) we compute to the first order in x, y, δ,

$$h^2 \approx 1 + 2k_x x + 2k_y y$$

$$k_u h = k_u + k_u^2 u$$

$$(1-\delta)h^2 k_u \approx k_u + 2k_u^2 u - k_u \delta$$

$$(1-\delta)h^2 K_0 u \approx K_0 u \text{ and } (1-\delta)h^2 \underline{K}_0 u \approx \underline{K}_0 u \;,$$

where u stands for x or y. Substituting these approximations into (1.23) and using the radius $\rho_{x,y}$ instead of the curvature $k_{x,y}$ we obtain the linearized equations of motion

$$x'' + \left(K_0 + \frac{1}{\rho_x^2}\right)x - \underline{K}_0 y = \frac{\delta}{\rho_x} \;,$$

$$y'' - \left(K_0 - \frac{1}{\rho_y^2}\right) y - \underline{K}_0 x = \frac{\delta}{\rho_y} \; . \tag{1.24}$$

The term $\underline{K}_0$ introduces a linear coupling into the equations of motion. If we restrict this to magnetic fields which does not introduce any coupling, we have to set $\underline{K}_0 = 0$ (no skew linear magnets). Then the equations of motion read:

$$x'' + \left(K_0 + \frac{1}{\rho_x^2}\right) x = \frac{\delta}{\rho_x} \; ,$$
$$y'' - \left(K_0 - \frac{1}{\rho_y^2}\right) y = \frac{\delta}{\rho_y} \; . \tag{1.25}$$

The terms K_0 and $\rho_{x,y}^{-2}$ in the above expressions represent the gradient focusing and weak sector magnet focusing, respectively. When the deflection occurs only in the horizontal plane, which is the usual case for synchrotrons, the equation of motion in the vertical plane simplifies to

$$y'' - K_0 y = 0 \; . \tag{1.26}$$

The magnet parameters $\rho_{x,y}$, K_0 and $\underline{K}_0$ are functions of the s-coordinate. In practice, they have zero values in magnet-free sections and assume constant values within the magnets. The arrangement of magnets in a beam transport channel or in a circular accelerator is called a lattice.

1.3 Weak and strong focusing

Consider the simple case at a circular accelerator with a bending magnet which deflects only in the horizontal plane and whose field does not change azimuthally

$$\frac{\partial B_x}{\partial s} = \frac{\partial B_y}{\partial s} = 0 \; .$$

The design orbit is then a circle of constant radius ρ_x, with the constant magnetic field B_{y0} on this circle evaluated for the design momentum

$$p_0 = e\, B_{y0}\, \rho_x \; . \tag{1.27}$$

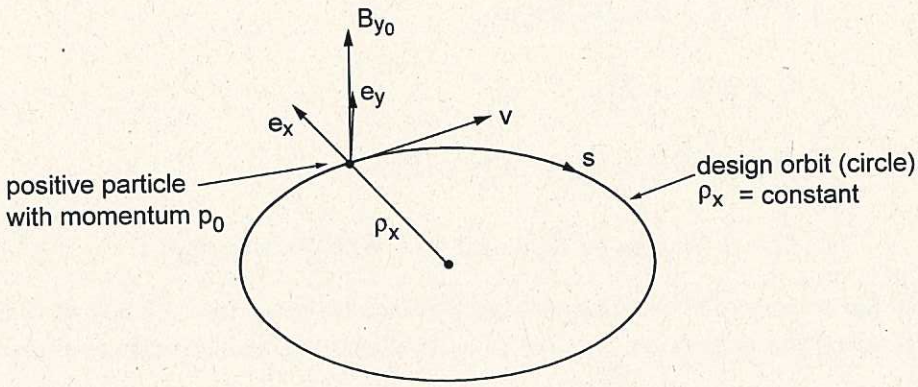

Figure 4: Circular design orbit in the horizontal plane.

For a particle with the design momentum the equations of motion read:

$$x'' + \left(K_0 + \frac{1}{\rho_x^2}\right) x = 0 \ ,$$

$$y'' - K_0 y = 0 \ , \tag{1.28}$$

where K_0 is the constant normalized gradient (1.21). The stability in the horizontal motion is achieved if the solution is oscillatory, that is:

$$K_0 + \frac{1}{\rho_x^2} > 0 \ .$$

Vertically, the stability is achieved if

$$K_0 < 0 \ .$$

Hence, the stability is achieved in both planes provided

$$0 < -K_0 < \frac{1}{\rho_x^2} \ . \tag{1.29}$$

Using (1.17) and (1.21) the latter inequalities write:

$$0 < -\frac{1}{B_{y0}} \left(\frac{\partial B_y}{\partial x}\right)_{x=y=0} < \frac{1}{\rho_x} \ . \tag{1.30}$$

This condition is called weak- or constant-gradient focusing. It allows stable motion in both planes.

The oscillatory solution of (1.28), called betatron oscillations, is

$$u(s) = a \ \cos\left(\sqrt{K} s - \varphi\right) \ , \tag{1.31}$$

where u stands for x or y, $K = \rho_x^{-2} + K_0$ or $K = -K_0$, and where a and φ are integration constants. The wavelength λ of the betatron oscillation is

$$\lambda = \frac{2\pi}{\sqrt{K}} \ . \tag{1.32}$$

The number of betatron oscillations performed by particles around a machine circumference C is called the tune Q of the circular accelerator (Q stands for Q_x or Q_y):

$$Q = \frac{C}{\lambda} = \rho_x \sqrt{K} \ . \tag{1.33}$$

The weak focusing condition (1.29) show that $K < \rho_x^{-2}$, that is:

$$Q < 1 \ .$$

The betatron oscillation wavelength is larger than the machine circumference. This means that the amplitude of the oscillations may become very large as the size ρ_x of the machine increases and hence the magnet aperture may also be very big. This yields a practical limit on the size of weak focusing accelerators. Historically, the field index n was introduced instead of the normalized gradient K_0 for the weak focusing accelerators

$$n = -\frac{\rho_x}{B_{y0}} \left(\frac{\partial B_y}{\partial x} \right)_{x=y=0} . \tag{1.34}$$

The weak focusing condition (1.30) is then expressed as

$$0 < n < 1 . \tag{1.35}$$

Using (1.33) and (1.34) the machine tune may be written as

$$Q_x = \sqrt{1-n} \qquad Q_y = \sqrt{n} . \tag{1.36}$$

The combination of bending and focusing forces required for weak focusing accelerators may be obtained by the magnetic field shape called synchrotron magnet or combined dipole-quadrupole magnet.

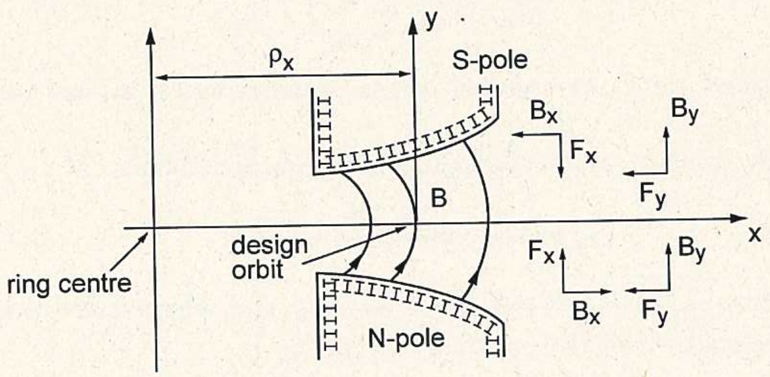

Figure 5: Synchrotron magnet (positive particles approach the reader).

The limitations at weak focusing accelerators have been overcome by the invention at the strong- or alternating-gradient focusing principle. This method amounts to splitting up the machine into an alternate sequence of strongly horizontally focusing ($n \ll -1$ or $K_0 \gg \rho_x^{-2}$) and strongly horizontally defocusing ($n \gg 1$ or $K_0 \ll -\rho_x^{-2}$) magnets. By (1.36) this implies that the tunes Q_x and Q_y may become arbitrarily large. This means that the betatron amplitudes can be kept small for a given angular deflection as the size ρ_x increases, and the magnet aperture may be reduced.

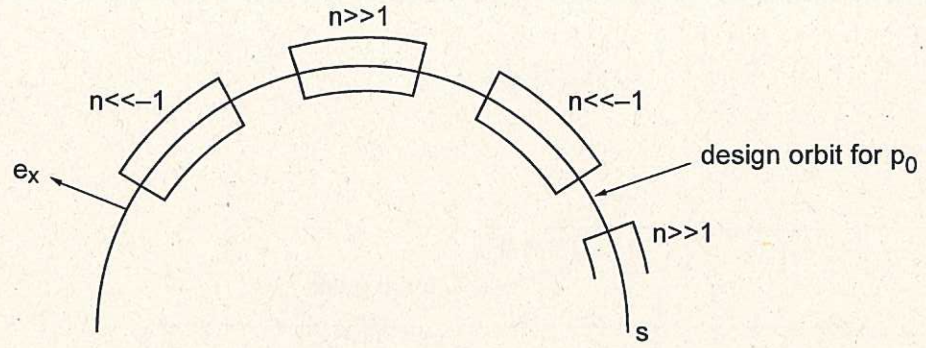

Figure 6: Strong focusing machine.

The bending and focusing forces for strong focusing accelerators may be achieved either within synchrotron magnets or in a separate bending magnet, called a dipole, and a focusing magnet, called a quadrupole.

The quadrupole magnet provides focusing forces through its normalized gradient, given by (1.21) and (1.22):

$$\frac{e}{p_0}B_x = K_0 y \qquad \frac{e}{p_0}B_y = K_0 x \; . \tag{1.37}$$

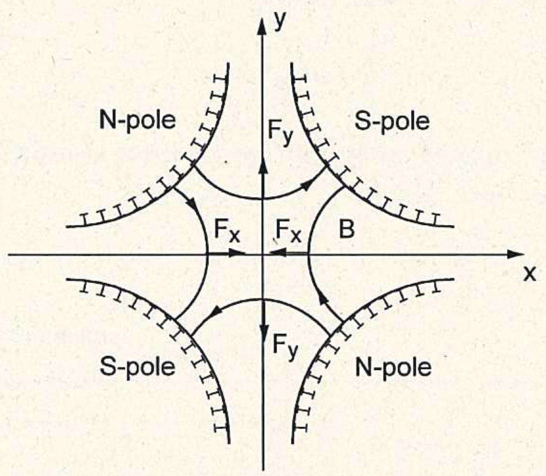

Figure 7: Horizontally focusing, *F*-type quadrupole magnet (positive particles approach the reader).

A quadrupole that focuses in one plane defocuses in the other plane. The horizontally defocusing, *D*-type quadrupole is obtained by permuting the *N*- and *S*-poles of an *F*-quadrupole.

In geometrical optics the focal length f of a lens is related to the angle of deflection θ of the lens by

$$\tan\,\theta = -\frac{x}{f}\,.$$

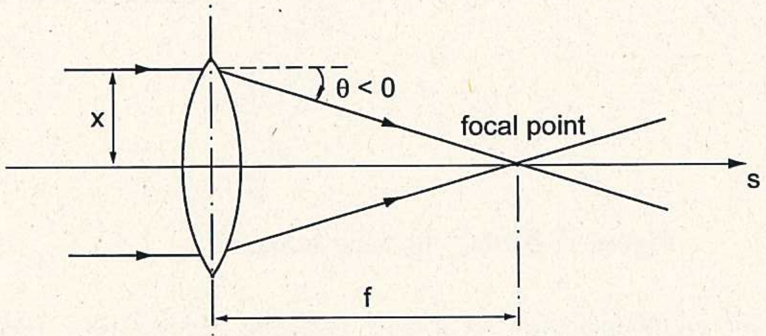

Figure 8: Principle of focusing for light.

It is known that a pair of glass lenses, one focusing with focal length $f_F > 0$, and the other defocusing with focal length $f_D < 0$, separated by a distance d, yields a total focal length f given by

$$\frac{1}{f} = \frac{1}{f_F} + \frac{1}{f_D} - \frac{d}{f_F f_D}\,. \tag{1.38}$$

This lens doublet is focusing if $f_D = -f_F$. Hence,

$$\frac{1}{f} = \frac{d}{f_F^2}\,. \tag{1.39}$$

The strong focusing scheme is based upon the quadrupole doublet arrangement, which becomes a system focusing in both planes.

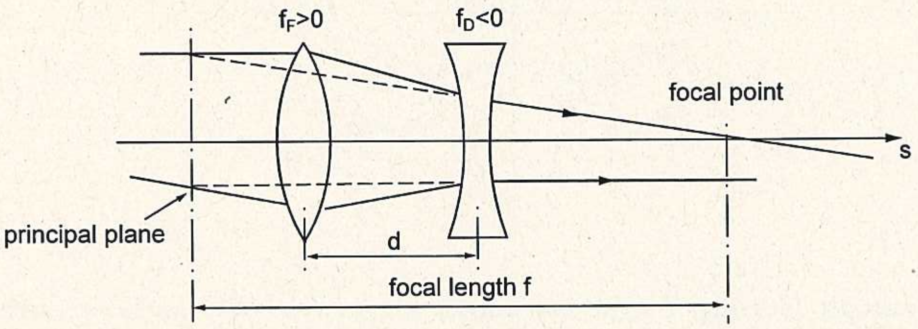

Figure 9: Principle of strong focusing for light.

The linear equations of motion (1.25) for strong focusing lattices (where the bending and focusing forces depend on s) may be written as

$$u'' + K(s)u = 0\,, \tag{1.40}$$

where u stands for either the horizontal x or the vertical y coordinates, and where the bending and focusing functions are combined in one function (plus sign for x, minus sign for y):

$$K(s) = \pm K_0(s) + \frac{1}{\rho^2_{x,y}(s)} . \tag{1.41}$$

For synchrotron magnets both terms in $K(s)$ are non-zero, while for separated function magnets (separate dipoles and quadrupoles) either $K_0(s)$ or $\rho_{x,y}(s)$ is set to zero.

Integrating (1.40) over a short distance, ℓ, where $K(s) \approx$ constant) we find in the paraxial approximation, where the defection angle θ is equal to the slope of the particle trajectory,

$$\int_0^\ell u'' ds = u'(\ell) - u'(0) = \tan\theta$$

and

$$\int_0^\ell K(s) u ds \approx K u \ell .$$

Therefore by (1.40)

$$\tan\theta \approx -K\ell u .$$

By analogy with the expression at the focal length of a glass lens, we defined the focal length of a quadrupole as

$$\tan\theta = -\frac{u}{f} . \tag{1.42}$$

Hence we get for a thin quadrupole of length ℓ [for $K(s) = K_0 > 0$]

$$\frac{1}{f} = \pm K_0 \ell . \tag{1.43}$$

with f positive in the focusing plane and negative in the defocusing plane.

2 LINEAR BEAM OPTICS

2.1 Betatron functions for periodic closed lattices

In the case where the bending magnets of a circular accelerator deflect only in the horizontal plane, the linear unperturbed equations of motion for a particle having the design momentum p_0 (i.e. $\delta = 0$) are, according to (1.25),

$$x'' + \left(K_0 + \frac{1}{\rho_x^2}\right) x = 0$$
$$y'' - K_0 y = 0 , \tag{2.1}$$

where $K_0(s)$ and $\rho_x(s)$ are periodic functions of the s-coordinate due to the orbit being a closed curve. The period L may be the accelerator circumference C or the length of a "cell" repeated N times around the circumference: $C = NL$. Both the above equations may be cast in the form

$$u'' + K(s)u = 0 \ , \tag{2.2}$$

with

$$K(s+L) = K(s) \ , \tag{2.3}$$

where u stands for x or y and where $K(s) = K_0(s) + \rho_x^{-2}(s)$ or $K(s) = -K_0(s)$.

Equation (2.2) with periodic coefficient $K(s)$ is called Hill's equation. It has a pair of independent stable solutions of the form (see appendix A1)

$$\begin{aligned} u_1(s) &= w_1(s)e^{i\mu(s)} \ , \\ u_2(s) &= w_2(s)e^{-i\mu(s)} \ , \end{aligned} \tag{2.4}$$

such that $w_1(s)$ and $w_2(s)$ are periodic with period L,

$$w_i(s+L) = w_i(s) \qquad i = 1, 2 \ , \tag{2.5}$$

and $\mu(s)$ is a function such that

$$\mu(s+L) - \mu(s) = \mu \ , \tag{2.6}$$

where μ is called the characteristic exponent of Hill's equation defined by

$$\cos\mu = \frac{1}{2}\mathrm{Tr}\ [M(s+L/s)] \ , \tag{2.7}$$

and is independent of the length s. The matrix $M(s+L/s)$ is called the transfer matrix over one period L [shortly written as $M(s)$ defined by (see A.9)]

$$M(s) \equiv M(s+L/s) = M(s+L/s_0)M(s/s_0)^{-1} \ ,$$

in which $M(s/s_0)$ is called the transfer matrix between the reference point s_0 and s, given by (see A.6)

$$M(s/s_0) = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \ ,$$

where $C(s)$ and $S(s)$ are two independent solutions of Hill's equation called cosine-like and sine-line solutions, which satisfy the particular initial conditions (see A.4)

$$C(s_0) = 1 \quad C'(s_0) = 0 \quad S(s_0) = 0 \quad S'(s_0) = 1 \ .$$

As the functions $e^{i\mu(s)}$ and $e^{-i\mu(s)}$ are already linearly independent, we can arbitrarily make $w_1(s)$ identical to $w_2(s)$ so that the pair at independent stable solutions (2.4) now read:

$$u_{1,2}(s) = w(s)e^{\pm i\mu(s)} \ , \tag{2.8}$$

where $w(s)$ is periodic with period L. Differentiating the latter equation twice,

$$\begin{aligned} u'_{1,2}(s) &= w'(s)e^{i\mu(s)} \pm i\mu'(s)w(s)e^{i\mu(s)} \\ u''_{1,2}(s) &= [w''(s) \pm i\mu'(s)w'(s) \pm i\mu''(s)w(s) - \mu'(s)^2 w(s) \pm i\mu'(s)w'(s)]e^{i\mu(s)} \\ &= [w''(s) \pm 2i\mu'(s)w'(s) - \mu'(s)^2 w(s) \pm i\mu''(s)w(s)]e^{i\mu(s)} , \end{aligned}$$

we obtain by substitution into Hill's equation and cancelling $e^{i\mu(s)}$

$$w'' - \mu'^2 w + K(s)w \pm i(2\mu'w' + \mu''w) = 0 .$$

Equating real and imaginary parts to zero yields

$$w'' - \mu'^2 w + K(s)w = 0 \tag{2.9}$$

and

$$2\mu'w' + \mu''w = 0 ,$$

or equivalently

$$\frac{2w'}{w} + \frac{\mu''}{\mu'} = 0 .. \tag{2.10}$$

The last equation may be rewritten as

$$[\ln w(s)^2]' + [\ln \mu'(s)]' = 0 ,$$

which can be integrated to give

$$\ln \mu'(s) = \ln \left(\frac{1}{w(s)^2}\right) + \ln c .$$

Hence, choosing the integration constant c equal to unity

$$\mu'(s) = \frac{1}{w(s)^2} , \tag{2.11}$$

which yields on integration

$$\mu(s) = \int_{s_0}^{s} \frac{dt}{w(t)^2} . \tag{2.12}$$

Substituting (2.11) into (2.9) gives a new differential equation for $w(s)$:

$$w'' - \frac{1}{w^3} + K(s)w = 0 . \tag{2.13}$$

Defining the so-called betatron function $\beta(s)$ as

$$\beta(s) = w^2(s) , \tag{2.14}$$

equations (2.12) and (2.13) transform into

$$\mu(s) = \int_{s_0}^{s} \frac{dt}{\beta(t)} \tag{2.15}$$

and

$$\beta^{1/2''} + K(s)\beta^{1/2} - \beta^{-3/2} = 0 \, . \tag{2.16}$$

The latter expression may be further transformed to give

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K(s)\beta^2 = 1 \, , \tag{2.17}$$

since

$$\beta^{3/2}\beta^{1/2''} = \frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 \, .$$

The function $\mu(s)$ given by (2.15) is called the phase function. Then, the two independent solutions of Hill's equations become

$$u_{1,2}(s) = \sqrt{\beta(s)}e^{\pm i\mu(s)} \, . \tag{2.18}$$

Every solution of Hill's equation is a linear combination of these two solutions:

$$u(s) = c_1\sqrt{\beta(s)}e^{i\mu(s)} + c_2\sqrt{\beta(s)}e^{-i\mu(s)} \, ,$$

where c_1 and c_2 are constants, or equivalently

$$u(s) = a\sqrt{\beta(s)} \, \cos\left[\mu(s) - \varphi\right] \, , \tag{2.19}$$

with

$$a = 2\sqrt{c_1 c_2} \qquad \tan\varphi = i\left(\frac{c_1 - c_2}{c_1 + c_2}\right) \, .$$

Any solution $\beta(s)$ that satisfies (2.17) together with the phase function $\mu(s)$, whose derivative is $\beta(s)^{-1}$, can be used to make (2.19) a real solution of Hill's equation. Such a solution is a pseudo-harmonic oscillation with varying amplitude and frequency, called betatron oscillation.

From (2.15) we compute:

$$\mu(s+L) - \mu(s) = \int_{s}^{s+L} \frac{dt}{\beta(t)} \, ;$$

and using (2.6) we find:

$$\mu = \int_{s}^{s+L} \frac{dt}{\beta(t)} \, . \tag{2.20}$$

Thus the characteristic exponent μ may be identified with the phase advance per period or cell (of length L).

The oscillation's local frequency $f(s)$ and wavelength $\lambda(s)$ can be related to the phase function by

$$\mu'(s) = 2\pi f(s) = \frac{2\pi}{\lambda(s)} . \tag{2.21}$$

Thus, the betatron function may be interpreted as the local wavelength divided by 2π since $\mu'(s) = \beta(s)^{-1}$:

$$\lambda(s) = 2\pi\beta(s) . \tag{2.22}$$

Let the tune Q of a circular accelerator be defined as the number of betatron oscillations executed by particles travelling once around the machine circumference C. Since the local frequency $f(s)$ denotes the number of oscillations per unit of length, the tune reads:

$$Q = \int_s^{s+C} f(t)dt . \tag{2.23}$$

If the accelerator has N cells of period L (i.e., $C = NL$), then using (2.20) to (2.23), Q is given by

$$Q = \frac{N\mu}{2\pi} = \frac{1}{2\pi}\int_s^{s+C} \frac{dt}{\beta(t)} , \tag{2.24}$$

since $\beta(s)$ is a periodic function of period L.

Any solution of Hill's equation can be described in terms of the betatron oscillation (2.19). In particular, the cosine-like solution $C(s)$ and the sine-like solution $S(s)$ may be represented by (2.19);

$$C(s) = a\sqrt{\beta(s)}\ \cos\left[\mu(s) - \varphi\right] , \tag{2.25}$$

where a and φ are constants which depend upon the initial conditions at the reference point s_0. Setting $\beta(s_0) = \beta_0$ and since $\mu(s_0) = 0$, $C(s_0) = 1$, $C'(s_0) = 0$ and

$$C'(s) = \frac{a}{\sqrt{\beta(s)}}\left(\frac{\beta'(s)}{2}\ \cos\left[\mu(s) - \varphi\right] - \sin\left[\mu(s) - \varphi\right]\right) , \tag{2.26}$$

we get

$$a\sqrt{\beta_0}\ \cos\varphi = 1 ,$$

$$\frac{a}{\sqrt{\beta_)}}\left(\frac{\beta_0'}{2}\ \cos\varphi + \sin\varphi\right) = 0 ,$$

from which we obtain

$$a\ \cos\varphi = \frac{1}{\sqrt{\beta_0}} ,$$

$$a\ \sin\varphi = -\frac{\beta_0'}{2\sqrt{\beta_0}} . \tag{2.27}$$

Substituting these two expressions into $C(s)$ and $C'(s)$ we find:

$$C(s) = \sqrt{\frac{\beta(s)}{\beta_0}} \left(\cos\mu(s) - \frac{\beta_0'}{2}\sin\mu(s)\right)$$

$$\begin{aligned} C'(s) &= \frac{1}{\sqrt{\beta(s)\beta_0}} \left\{ \frac{\beta'(s)}{2} \left(\cos\mu(s) - \frac{\beta_0'}{2}\sin\mu(s)\right) - \left(\sin\mu(s) + \frac{\beta_0'}{2}\cos\mu(s)\right)\right\} \\ &= \frac{\beta'(s) - \beta_0'}{2\sqrt{\beta(s)\beta_0}} \cos\mu(s) - \frac{1 + [\beta'(s)\beta_0/4]}{\sqrt{\beta(s)\beta_0}} \sin\mu(s) \, . \end{aligned}$$

Defining the new variables

$$\alpha(s) = -\frac{\beta'(s)}{2} \tag{2.28}$$

and $\Delta\mu(s) = \mu(s) - \mu(s_0) \equiv \mu(s)$, we get

$$C(s) = \sqrt{\frac{\beta(s)}{\beta_0}} (\cos\ \Delta\mu(s) + \alpha_0\ \sin\ \Delta\mu(s)) \ , \tag{2.29}$$

$$C'(s) = \frac{1}{\sqrt{\beta(s)\beta_0}} \{[\alpha(s) - \alpha_0]\ \cos\ \Delta\mu(s) - [1 + \alpha(s)\alpha_0]\ \sin\ \Delta\mu(s)\} \ . \tag{2.30}$$

The sine-like solutions $S(s)$ and $S'(s)$ can be computed similarly. Hence, the transfer matrix (A.6) between s_0 and s reads:

$$M(s/s_0) = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \tag{2.31}$$

$$= \begin{pmatrix} \sqrt{\frac{\beta(s)}{\beta_0}} (\cos\Delta\mu(s) + \alpha_0 \sin\Delta\mu(s)) & \sqrt{\beta(s)\beta_0} \sin\Delta\mu(s) \\ \frac{\alpha_0 - \alpha(s)}{\sqrt{\beta(s)\beta_0}} \cos\Delta\mu(s) - \frac{1 + \alpha(s)\alpha_0}{\sqrt{\beta(s)\beta_0}} \sin\ \Delta\mu(s) & \sqrt{\frac{\beta_0}{\beta(s)}} (\cos\Delta\mu(s) - \alpha(s)\sin\Delta\mu(s)) \end{pmatrix} .$$

From this and using (A.19) the transfer matrix $M(s)$ over one period L may be written as

$$M(s) = \begin{pmatrix} C(s+L) & S(s+L) \\ C'(s+L) & S'(s+L) \end{pmatrix} \begin{pmatrix} S'(s) & -S(s) \\ -C'(s) & C(s) \end{pmatrix} ,$$

because the determinant of any transfer matrix is equal to unity. Performing the above matrix multiplication yields

$$M(s) = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\frac{1+\alpha(s)^2}{\beta(s)} \sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix} , \tag{2.32}$$

since $\alpha(s+L) = \alpha(s)$, $\beta(s+L) = \beta(s)$ and $\mu(s+L) = \mu(s) + \mu$. Introducing the new variable

$$\gamma(s) = \frac{1+\alpha(s)^2}{\beta(s)}, \tag{2.33}$$

the transfer matrix reads:

$$M(s) = \begin{pmatrix} \cos\mu + \alpha(s)\sin\mu & \beta(s)\sin\mu \\ -\gamma(s)\sin\mu & \cos\mu - \alpha(s)\sin\mu \end{pmatrix}. \tag{2.34}$$

We check immediately that the determinant of the transfer matrix for one period is equal to unity and its trace is equal to 2 $\cos\mu$ as expected. The transfer matrix $M(s)$ is called the Twiss matrix and the periodic functions $\alpha(s)$, $\beta(s)$, $\gamma(s)$ are called Twiss parameters. In summary, the solution of Hill's equation and the transfer matrices may be expressed in terms of the single function $\beta(s)$, determined by searching the periodic solutions of

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K(s)\beta^2 = 1 .$$

2.2 Hill's equation with piecewise periodic constant coefficients

The nonlinear differential equation

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K(s)\beta^2 = 1 \tag{2.35}$$

is completely specified by the linear optical properties of the lattice (focusing magnets) and the condition of periodicity. Unfortunately it is not any easier to solve than the original Hill's equation

$$u'' + K(s)u = 0 . \tag{2.36}$$

However, when $K(s)$ is a piecewise constant periodic function, explicit determination of the betatron function $\beta(s)$ may be found. Assume $K(s)$ to be a constant K over a distance ℓ between the azimuthal locations s_0 and s_1. There are three cases: K is positive, K is negative, and K is equal to zero. For the first two cases, Hill's equation reduces to the simple harmonic oscillator equation. The solutions may be expressed in terms of the functions $C(s)$ and $S(s)$—(see A.6)—satisfying the initial conditions (A.4) at s_0. In terms of transfer matrices we get:

1) For $K > 0$

$$M(s_1/s_0) = \begin{pmatrix} \cos\sqrt{K}\ell & \frac{1}{\sqrt{K}}\sin\sqrt{K}\ell \\ -\sqrt{K}\sin\sqrt{K}\ell & \cos\sqrt{K}\ell \end{pmatrix}, \tag{2.37}$$

since between s_0 and s_1

$$C(s) = \cos\sqrt{K}(s-s_0) \text{ and } S(s) = \frac{1}{\sqrt{K}}\sin\sqrt{K}(s-s_0) ;$$

2) For $K < 0$

$$M(s_1/s_0) = \begin{pmatrix} \cosh\sqrt{|K|}\ell & \frac{1}{\sqrt{|K|}}\sinh\sqrt{|K|}\ell \\ \sqrt{|K|}\sinh\sqrt{|K|}\ell & \cosh\sqrt{|K|}\ell \end{pmatrix} ; \tag{2.38}$$

3) For $K = 0$

$$M(s_1/s_2) = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} , \tag{2.39}$$

since $C(s) = 1$ and $S(s) = s - s_0$ between s_0 and s_1.

The transfer matrix $M(s)$ over one cell (of length L) is then the product of the individual matrices composing the cell. If the cell is made of N elements having transfer matrices M_1, M_2, ... M_N [with $M_k = M(s_k/s_{k-1})$ for short], we get

$$M(s_0) \equiv M(s_0 + L/s_0) = M_N \,...\, M_2 M_1 . \tag{2.40}$$

Let $m_{ij}(s_0)$ the components of $M(s_0)$ obtained by (2.40). Equating the two versions (2.34) and (2.40) of $M(s_0)$ gives the Twiss parameters at the reference point s_0. We find:

$$\begin{aligned} \beta(s_0) &= \frac{m_{12}(s_0)}{\sin\mu} , \\ \gamma(s_0) &= -\frac{m_{21}(s_0)}{\sin\mu} , \\ \alpha(s_0) &= \frac{m_{11}(s_0) - m_{22}(s_0)}{2\sin\mu} , \\ \cos\mu &= \frac{1}{2}[m_{11}(s_0) + m_{22}(s_0)] . \end{aligned} \tag{2.41}$$

Consequently, once the Twiss matrix has been derived by multiplication of the individual transfer matrices in the cell, the Twiss parameters are obtained by (2.41) at any reference point s.

The principle of strong focusing is based on the quadrupole doublet system composed of one focusing quadrupole and one defocusing quadrupole separated by a straight section of length d.

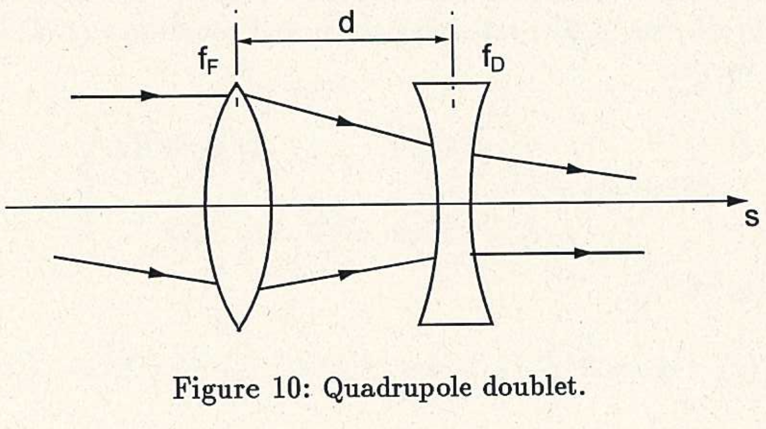

Figure 10: Quadrupole doublet.

Consider the thin lens approximation which assumes that $\sqrt{K_0}\ell \ll 1$ with $K_0 > 0$ and $\ell \to 0$ as $K_0\ell$ remains constant, where ℓ is quadrupole length and K_0 the normalized gradient. In that approximation, the transfer matrix (2.37) or (2.38) of a thin quadrupole reduces to

$$M(\ell/0) = \begin{pmatrix} 1 & 0 \\ \mp K_0\ell & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mp\frac{1}{f} & 1 \end{pmatrix}, \tag{2.42}$$

with $f^{-1} = K_0\ell$ and where the minus sign corresponds to a horizontally focusing quadrupole and the plus sign to a horizontally defocusing quadrupole.

The transfer matrix $M_{\rm db}(d/0)$ of the doublet with focal lengths $f_F > 0$ and $f_D < 0$ is thus

$$M_{\rm db}(d/0) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_D} & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_F} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{d}{f_F} & d \\ -\frac{1}{f} & 1-\frac{d}{f_D} \end{pmatrix}, \tag{2.43}$$

with

$$\frac{1}{f} = \frac{1}{f_F} + \frac{1}{f_D} - \frac{d}{f_F f_D} = \frac{d}{f_F^2} > 0\,, \tag{2.44}$$

when $f_D = -f_F$, with $f_F^{-1} = K_0\ell$.

As another example, consider a symmetric thin-lens FODO cell composed of a horizontally focusing quadrupole (F) of focal length $f_F = f > 0$, followed by a drift space (O) of length L, then a horizontally defocusing quadrupole (D) of focal length $f_D = -|f| < 0$ and a drift space (O) of length L. The transfer matrix through the FODO cell of total length $2L$ is then, by (2.39) and (2.42)

$$M_{\rm FODO}(2L/0) = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} = \begin{pmatrix} 1-\frac{L}{f}-\frac{L^2}{f^2} & 2L+\frac{L^2}{f} \\ -\frac{L}{f^2} & 1+\frac{L}{f} \end{pmatrix}$$

where $f^{-1} = K_0\ell, K_0$ and ℓ being the quadrupole strength and length.

2.3 Emittance and beam envelope

An invariant can be found from the solution of Hill's equation:

$$u(s) = a\sqrt{\beta(s)}\,\cos\left[\mu(s) - \varphi\right]. \tag{2.45}$$

Computing the derivative

$$\begin{aligned} u'(s) &= -\frac{a}{\sqrt{\beta(s)}}\Big(\alpha(s)\,\cos\left[\mu(s)-\varphi\right] + \sin\left[\mu(s)-\varphi\right]\Big) = \\ & -\frac{\alpha(s)}{\beta(s)}u(s) - \frac{a}{\sqrt{\beta(s)}}\,\sin\left[\mu(s)-\varphi\right], \end{aligned}$$

or equivalently,

$$\alpha(s)u(s) + \beta(s)u'(s) = a\sqrt{\beta(s)}\sin\left[\mu(s)-\varphi\right]$$

Squaring and summing the above equations using (2.33) gives

$$\gamma(s)u(s)^2 + 2\alpha(s)u(s)u'(s) + \beta(s)u'(s)^2 = a^2 \,. \tag{2.46}$$

This is a constant of motion, called the Courant–Snyder invariant. It represents the equation of an ellipse in the plane (u, u'), with the area πa^2.

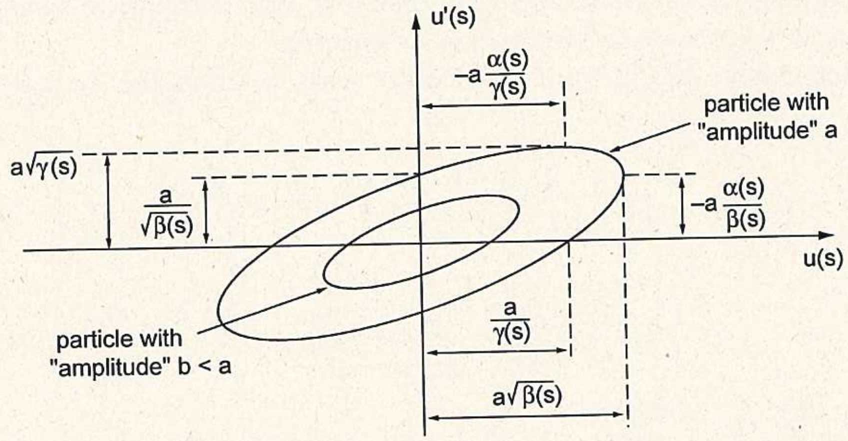

Figure 11: Phase plane ellipses for particles with different amplitudes.

It is customary to surround a given fraction at the beam in the (u, u') plane, say 95%, at a certain point s (for example, the injection point) by a phase ellipse described by

$$\gamma(s)u(s)^2 + 2\alpha(s)u(s)u'(s) + \beta(s)u'(s)^2 = \epsilon \,, \tag{2.47}$$

where the parameter ϵ is called the beam emittance:

$$\iint_{\text{ellipse}} du du' = \pi\epsilon \,. \tag{2.48}$$

All particles inside the phase ellipse will evolve a homothetic invariant ellipse with parameters $a \leq \epsilon$ dictated by the optical properties of the lattice (Courant–Snyder invariant). Thus the phase ellipse will contain the same fraction at the beam on successive machine turns: the beam emittance is conserved (in the absence of acceleration, radiation, and some collective effects).

The betatron oscillation for a particle on the phase ellipse reads:

$$u(s) = \sqrt{\beta(s)\epsilon}\cos\left[\mu(s) - \varphi\right], \tag{2.49}$$

where φ is an arbitrary phase constant. The envelope of the beam containing the specified fraction of particles is defined by

$$E(s) = \sqrt{\beta(s)\epsilon} \tag{2.50}$$

and the beam divergence is defined by

$$A(s) = \sqrt{\gamma(s)\epsilon}\,. \tag{2.51}$$

The Twiss parameters $\alpha(s), \beta(s), \gamma(s)$ determine the shape and orientation of the ellipse at azimuthal location s along the lattice. As the particle trajectory $u(s)$ evolves along the ring, the ellipse continuously changes its form and orientation but not its area. Every time the trajectory covers one cell of length L along s the ellipse is the same, since the Twiss parameters are periodic with period L. Consequently, on every machine revolution the particle coordinates (u, u') will appear on the same ellipse;

$$u(s+kC) = a\sqrt{\beta(s)}\{\cos\,[\mu(s)-\varphi]\,\cos\,2\pi kQ - \sin\,[\mu(s)-\varphi]\sin\,2\pi kQ\}\,,$$

since $\mu(s+kC) - \mu(s) = 2\pi kQ$. Thus the particle will appear cyclically at only n points on the ellipse if the tune is a rational number $Q = m/n$ and the particle trajectory will cover the ellipse densely if the tune is an irrational number.

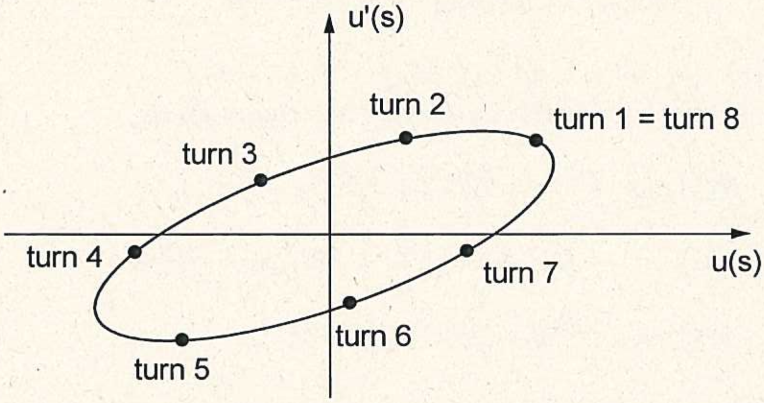

Figure 12: Phase plane motion for one particle after many turns.

This representation of the motion where the trajectory coordinates u and u' are picked up at fixed azimuthal location s on successive tunes is called 'stroboscopic' representation in phase plane (u, u') or Poincaré mapping: the series of u_i, $u_i'(i = 1, 2...)$ dots is a mapping of the (u, u') plane onto itself.

The Courant–Snyder invariant enables us to determine how the Twiss parameters transform through the lattice. Consider the reference point s_0 where the initial conditions of the cosine- and sine-like solutions are given. Setting $\alpha(s_0) = \alpha_0$, $\beta(s_0) = \beta_0$, $\gamma(s_0) = \gamma_0$, and $u(s_0) = u_0$, $u'(s_0) = u_0'$ for short, we get

$$\gamma_0 u_0^2 + 2\alpha_0 u_0 u_0' + \beta_0 u_0'^2 = a^2\,.$$

Furthermore, by definition of a transfer matrix from s_0 to s,

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} u_0 \\ u_0' \end{pmatrix}\,,$$

or equivalently by matrix inversion,

$$\begin{pmatrix} u_0 \\ u_0' \end{pmatrix} = \begin{pmatrix} S'(s) & -S(s) \\ -C'(s) & C(s) \end{pmatrix} \begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} ,$$

the Courant–Snyder invariant reads as

$$\begin{aligned} &\gamma_0(S'u - Su')^2 + 2\alpha_0(S'u - Su')(-C'u + Cu') + \beta_0(-C'u + Cu')^2 \\ &= (S'^2\gamma_0 - 2S'C'\alpha_0 + C'^2\beta_0)u^2 + \\ &+2(-SS'\gamma_0 + (S'C + SC')\alpha_0 - C'C\beta_0)uu' + \\ &+(S^2\gamma_0 - 2SC\alpha_0 + C^2\beta_0)u'^2 = a^2 , \end{aligned}$$

which is the expression of the Courant–Snyder invariant at s;

$$\gamma(s)u^2 + 2\alpha(s)uu' + \beta(s)u'^2 = a^2 ,$$

where we have set $u(s) = u$, $u'(s) = u'$ and $S(s) = S$, $C(s) = C$.

Identifying the coefficients of the latter two invariants gives

$$\begin{aligned} \beta(s) &= C^2\beta_0 - 2SC\alpha_0 + S^2\gamma_0 \\ \alpha(s) &= -CC'\beta_0 + (S'C + SC')\alpha_0 - SS'\gamma_0 \\ \gamma(s) &= C'^2\beta_0 - 2S'C'\alpha_0 + S'^2\gamma_0 , \end{aligned}$$

or in matrix formulation,

$$\begin{pmatrix} \beta(s) \\ \alpha(s) \\ \gamma(s) \end{pmatrix} = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & S'C + SC' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \alpha_0 \\ \gamma_0 \end{pmatrix} . \tag{2.52}$$

This expression is the transformation rule for phase ellipses through the lattice.

2.4 Betatron functions for beam transport lattices

The linear unperturbed equation of motion for a particle through an arbitrary beam transport lattice (non-periodic) has a form similar to equation (2.2):

$$u'' + K(s)u = 0 , \tag{2.53}$$

where $K(s)$ is an arbitrary function of s. By analogy with the solution (2.19) of Hill's equation (2.2) we try a solution of the form

$$u(s) = a\sqrt{\beta^*(s)} \cos\left[\mu^*(s) - \varphi\right] , \tag{2.54}$$

in which a and φ are integration constants, $\beta^*(s)$ and $\mu^*(s)$ are functions of s to be determined. Differentiating this expression twice—writing for short $\beta^* = \beta^*(s)$ and $\mu^* = \mu^*(s)$—

$$u' = a\frac{\beta^{*\prime}}{2\sqrt{\beta}} \cos\,[\mu^* - \varphi] - a\sqrt{\beta^*}\mu^{*\prime} \sin\,[\mu^* - \varphi] \ ,$$

$$\begin{aligned} u'' &= \frac{a}{4}\left(\frac{2\beta^*\beta^{*\prime\prime} - \beta^{*\prime 2}}{\beta^{*3/2}}\right) \cos\,[\mu^* - \varphi] - a\frac{\beta^{*\prime}}{\sqrt{\beta^*}}\mu^{*\prime} \sin\,[\mu^* - \varphi] \\ &\quad -a\sqrt{\beta^*}\mu^{*\prime\prime} \sin\,(\mu^* - \varphi) - a\sqrt{\beta^*}\mu^{*\prime 2} \cos\,[\mu^* - \varphi] \ , \end{aligned}$$

and inserting into (2.53) yields

$$\begin{aligned} &\frac{a}{\beta^{*3/2}}\left(\frac{\beta^*\beta^{*\prime\prime}}{2} - \frac{\beta^{*\prime 2}}{4} - \beta^{*2}\mu^{*\prime 2} + \beta^{*2}K(s)\right) \cos\,[\mu^* - \varphi] \\ &\qquad - \frac{a}{\sqrt{\beta^*}}(\beta^{*\prime}\mu^{*\prime} + \beta^*\mu^{*\prime\prime}) \sin\,(\mu^* - \varphi) = 0 \ . \end{aligned}$$

Since the cosine and sine terms must cancel separately to make (2.54) true for all $\mu^*(s)$, we obtain

$$\frac{\beta^*\beta^{*\prime\prime}}{2} - \frac{\beta^{*\prime 2}}{4} - \beta^{*2}\mu^{*\prime 2} + \beta^{*2}K(s) = 0 \ , \tag{2.55}$$

and

$$\beta^{*\prime}\mu^{*\prime} + \beta * \mu^{*\prime\prime} = (\beta^*\mu^{*\prime})' = 0 \ .$$

The latter equation gives on integration

$$\beta^*(s)\mu^{*\prime}(s) = c \ .$$

Then, choosing the integration constant c equal to unity, we get

$$\mu^{*\prime}(s) = \frac{1}{\beta^*(s)} \ , \tag{2.56}$$

which after integration gives

$$\mu^*(s) = \int_{s_0}^{s} \frac{dt}{\beta^*(t)} \ . \tag{2.57}$$

Hence, inserting (2.56) into (2.55) we obtain

$$\frac{1}{2}\beta^*\beta^{*\prime\prime} - \frac{1}{4}\beta^{*\prime 2} + K(s)\beta^{*2} = 1 \ . \tag{2.58}$$

The last two equations are identical to (2.15) and (2.17), derived from Hill's equation. Therefore, $\mu^*(s)$ and $\beta^*(s)$ are also called phase and betatron functions. However, unlike that of a periodic lattice, the betatron function of a non-periodic lattice is not uniquely determined by the periodicity condition on the cells, but depends on the initial conditions at the entrance of the lattice. Similarly to the periodic lattice case, we can define the parameters

$$\begin{aligned} \alpha^*(s) &= -\frac{\beta^{*\prime}(s)}{2}, \\ \gamma^*(s) &= \frac{1+\alpha^*(s)^2}{\beta^*(s)}, \end{aligned} \tag{2.59}$$

and the Courant–Snyder invariant may be derived in the same way:

$$\gamma^*(s)u(s)^2 + 2\alpha^*(s)u(s)u'(s) + \beta^*(s)u'(s)^2 = a^2 \tag{2.60}$$

Since the parameters $\alpha^*(s)$, $\beta^*(s)$, $\gamma^*(s)$ of the non-periodic case, and $\alpha(s)$, $\beta(s)$, $\gamma(s)$ of the periodic case are of similar nature, the star above the letters α^*, β^*, γ^* will be removed. The derivation of the formulae (2.31) for the general transformation matrix and (2.52) for the transformation of Twiss parameters proceed in the same way as in the periodic lattice case.

Designing a beam transport lattice, the Twiss parameters $\alpha(s_0)$, $\beta(s_0)$, $\gamma(s_0)$ at its entrance s_0 may be chosen such that the phase ellipse

$$\gamma(s_0)u(s_0)^2 + 2\alpha(s_0)u(s_0)u'(s_0) + \beta(s_0)u'(s_0)^2 = \epsilon$$

closely surrounds a given fraction at the incoming particle beam distribution in the (u, u') phase plane. The parameter ϵ is the beam emittance. The description of the distribution of particles within the beam is thus reduced to that of a particle travelling along the phase space ellipse

$$u(s) = \sqrt{\beta(s)\epsilon}\ \cos\left[\mu(s) - \varphi\right].$$

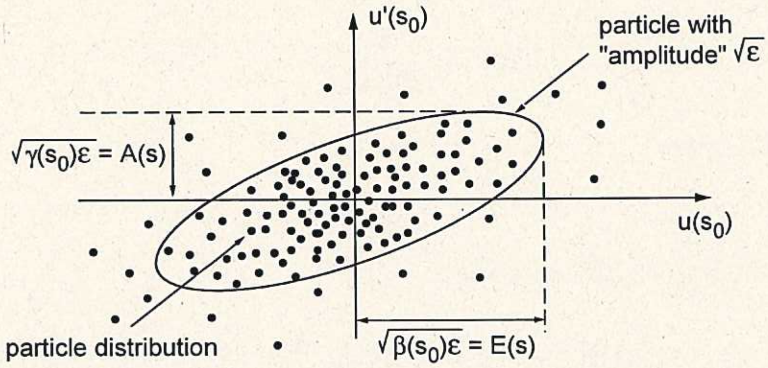

Figure 13: Particle distribution in phase plane, beam envelope and divergence.

The Twiss parameters at the entrance of a transport lattice may also be chosen as those coming from a join circular machine (at ejection point).

Another beam emittance definition frequently used is

$$\epsilon = \frac{\sigma_u(s)^2}{\beta(s)} ,$$

where $\sigma_u(s)$ is the standard deviation of the projected beam distribution onto the u-axis (beam profile).

2.5 Dispersive periodic closed lattices

Particle beams have a finite dispersion of momenta about the ideal momentum p_0. A particle with $\Delta p = p - p_0$ will perform betatron oscillations about a different closed orbit from that of the reference particle. For small deviations in momentum the equation of motion is, according to (1.25)

$$u'' + K(s)u = \frac{\delta}{\rho(s)} , \tag{2.61}$$

with

$$\delta = \frac{p - p_0}{p_0} = \frac{\Delta p}{p_0} , \tag{2.62}$$

where $\rho(s)$ stands for the local bend radii $\rho_x(s)$ or $\rho_y(s)$.

The individual particle deviation from the design orbit can be regarded as being the sum of two parts:

$$u(s) = u_\beta(s) + u_\delta(s) , \tag{2.63}$$

where $u_\delta(s)$ is the displacement at the closed orbit for the off-momentum particle from that of the reference particle (with $\delta = 0$), and $u_\beta(s)$ is the betatron oscillation around this off-momentum orbit. The particular solution $u_\delta(s)$ of the inhomogeneous equation (2.61) is generally re-expressed as

$$u_\delta(s) = D(s)\delta , \tag{2.64}$$

where $D(s)$ is called the dispersion function, which evidently satisfies the equation

$$D'' + K(s)D = \frac{1}{\rho(s)} . \tag{2.65}$$

A particular solution of this equation is

$$D_p(s) = S(s) \int_{s_0}^{s} \frac{C(t)}{\rho(t)} dt - C(s) \int_{s_0}^{s} \frac{S(t)}{\rho(t)} dt , \tag{2.66}$$

where $C(s)$ and $S(s)$ are the cosine- and sine-like solutions of the homogeneous Hill's equation (2.61) (with $\delta = 0$). The substitution of (2.66) into (2.65) verifies that this is a valid solution. We compute first

$$
\begin{aligned}
D'_p(s) &= S'(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt + S(s)\frac{C(s)}{\rho(s)} - C'(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt - C(s)\frac{S(s)}{\rho(s)} \\
&= S'(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C'(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt
\end{aligned}
$$

and

$$
\begin{aligned}
D''_p(s) &= S''(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt + S'(s)\frac{C(s)}{\rho(s)} - C''(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt - C'(s)\frac{S(s)}{\rho(s)} \\
&= \frac{1}{\rho(s)} + S''(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C''(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt\ ,
\end{aligned}
$$

since the Wronskian is equal to unity (A.7). Hence (2.65) for D_p reads

$$
[S'' + K(s)S]\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - [C'' + K(s)C]\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt = 0\ ,
$$

where $C(s)$ and $S(s)$ satisfy the homogeneous Hill's equation

$$
S'' + K(s)S = 0 \quad \text{and} \quad C'' + K(s)C = 0\ .
$$

For a closed lattice we must find a dispersion function which leads to an off-momentum closed orbit, one for which

$$
u_\delta(s + C) = u_\delta(s) \qquad u'_\delta(s + C) = u'_\delta(s)\ , \tag{2.67}
$$

where we have considered the machine circumference $C = NL$ as the period, instead of the cell length L. Other possible solutions $u_\delta(s)$ are $D_p(s)\delta$ plus a linear combination of $C(s)$ and $S(s)$:

$$
\begin{aligned}
u_\delta(s) &= c_1 C(s) + c_2 S(s) + D_p(s)\delta \\
u'_\delta(s) &= c_1 C'(s) + c_2 S'(s) + D'_p(s)\delta\ ,
\end{aligned} \tag{2.68}
$$

where c_1 and c_2 are constants. The application of the boundary condition (2.67) to the reference point $s = s_0$, from which $C(s)$ and $S(s)$ are derived, yields

$$
\begin{aligned}
&c_1 C(s_0 + C) + c_2 S(s_0 + C) + D_p(s_0 + C)\delta = c_1\ , \\
&c_1 C'(s_0 + C) + c_2 S'(s_0 + C) + D'_p(s_0 + C)\delta = c_2\ ,
\end{aligned} \tag{2.69}
$$

since

$$
C(s_0) = 1 \qquad C'(s_0) = 0 \qquad S(s_0) = 0 \qquad S'(s_0) = 1
$$

and

$$
D_p(s_0) = 0 \qquad D'_p(s_0) = 0\ .
$$

In matrix form the system (2.69) reads:

$$\begin{pmatrix} C(s_0+C)-1 & S(s_0+C) \\ C'(s_0+C) & S'(s_0+C)-1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\begin{pmatrix} D_p(s_0+C) \\ D'_p(s_0+C) \end{pmatrix} \delta ;$$

its solution is

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -\frac{\begin{pmatrix} S'(s_0+C)-1 & -S(s_0+C) \\ -C'(s_0+C) & C(s_0+C)-1 \end{pmatrix} \begin{pmatrix} D_p(s_0+C) \\ D'_p(s_0+C) \end{pmatrix}}{[C(s_0+C)-1][S'(s_0+C)-1]-C'(s_0+C)S(s_0+C)} \delta . \tag{2.70}$$

The denominator of this expression may be written as

$$\begin{aligned} & C(s_0+C)S'(s_0+C)-C'(s_0+C)S(s_0+C)-[C(s_0+C)+S'(s_0+C)]+1 \\ & = |M(s_0)| - \text{Tr}\,[M(s_0)]+1 = 2(1-\cos 2\pi Q) = 4\,\sin^2 \pi Q\ , \end{aligned}$$

since

$$M(s_0) = \begin{pmatrix} C(s_0+C) & S(s_0+C) \\ C'(s_0+C) & S'(s_0+C) \end{pmatrix}$$

is the transfer matrix for one machine revolution, whose determinant is equal to unity, and the trace is

$$\text{Tr}\,[M(s_0)] = 2\,\cos\mu(s_0+C) = 2\,\cos 2\pi Q\ ,$$

with

$$\mu(s_0+C) = \int_{s_0}^{s_0+C} \frac{dt}{\beta(t)} = 2\pi Q\ .$$

The solution for the constant c_1 is then

$$c_1 = \frac{S(s_0+C)D'_p(s_0+C)-[S'(s_0+C)-1]D_p(s_0+C)}{4\,\sin^2 \pi Q}\delta\ . \tag{2.71}$$

The numerator may be expressed as

$$\begin{aligned} & S(s_0+C)D'_p(s_0+C)-S'(s_0+C)D_p(s_0+C)+D_p(s_0+C) \\ = \ & -[S(s_0+C)C'(s_0+C)-S'(s_0+C)C(s_0+C)]\int_{s_0}^{s_0+C}\frac{S(t)}{\rho(t)}dt \\ & +S(s_0+C)\int_{s_0}^{s_0+C}\frac{C(t)}{\rho(t)}dt - C(s_0+C)\int_{s_0}^{s_0+C}\frac{S(t)}{\rho(t)}dt \\ = \ & [1-C(s_0+C)]\int_{s_0}^{s_0+C}\frac{S(t)}{\rho(t)}dt + S(s_0+C)\int_{s_0}^{s_0+C}\frac{C(t)}{\rho(t)}dt \end{aligned}$$

Furthermore the transfer matrix over a full turn also reads:

$$M(s_0) = \begin{pmatrix} \cos 2\pi Q + \alpha(s_0)\sin 2\pi Q & \beta(s_0)\sin 2\pi Q \\ -\gamma(s_0)\sin 2\pi Q & \cos 2\pi Q - \alpha(s_0)\sin 2\pi Q \end{pmatrix} ,$$

so that

$$C(s_0+C) = \cos 2\pi Q + \alpha(s_0)\sin 2\pi Q\ ,$$

$$S(s_0+C) = \beta(s_0)\sin 2\pi Q\ .$$

The expression (2.31) for the transfer matrix $M(t/s_0)$ yields

$$\begin{aligned} C(t) &= \sqrt{\frac{\beta(t)}{\beta(s_0)}}[\cos\,\Delta\mu(t) + \alpha(s_0)\sin\,\Delta\mu(t)] \\ S(t) &= \sqrt{\beta(t)\beta(s_0)}\sin\,\Delta\mu(t) \end{aligned}$$

where $\Delta\mu(t) = \mu(t) - \mu(s_0)$. Hence the constant c_1 becomes

$$\begin{aligned} c_1 &= \frac{\delta}{4\sin\,^2\pi Q}\Bigg([1-\cos\,2\pi Q - \alpha(s_0)\sin\,2\pi Q]\int_{s_0}^{s_0+C}\frac{1}{\rho(t)}\sqrt{\beta(s_0)\beta(t)}\sin\,\Delta\mu(t)dt \\ &\quad +\beta(s_0)\sin\,2\pi Q\int_{s_0}^{s_0+C}\frac{1}{\rho(t)}\sqrt{\frac{\beta(t)}{\beta(s_0)}}[\cos\,\Delta\mu(t)+\alpha(s_0)\sin\,\Delta\mu(t)]dt\Bigg) \\ &= \frac{\sqrt{\beta(s_0)}\delta}{4\sin\,^2\pi Q}\int_{s_0}^{s_0+C}\frac{1}{\rho(t)}\sqrt{\beta(t)}\,[(1-\cos\,2\pi Q)\sin\,\Delta\mu(t)+\sin\,2\pi Q\cos\,\Delta\mu(t)]\,dt \\ &= \frac{\sqrt{\beta(s_0)}\delta}{2\sin\,\pi Q}\int_{s_0}^{s_0+C}\frac{\sqrt{\beta(t)}}{\rho(t)}[\sin\,\pi Q\sin\,\Delta\mu(t)+\cos\,\pi Q\cos\,\Delta\mu(t)]\,dt \\ &= \frac{\sqrt{\beta(s_0)}\delta}{2\sin\,\pi Q}\int_{s_0}^{s_0+C}\frac{\sqrt{\beta(t)}}{\rho(t)}\cos\,[\Delta\mu(t)-\pi Q]dt \end{aligned}$$

At the reference point s_0 from (2.64), (2.66) and (2.68) we get

$$c_1 = u_\delta(s_0) = D(s_0)\delta\ .$$

Since the origin s_0 was chosen arbitrarily, it may be replaced by the general coordinate s, and the dispersion function $D(s)$ is found to be

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin\,\pi Q}\int_{s}^{s+C}\frac{\sqrt{\beta(t)}}{\rho(t)}\cos\,[\mu(t)-\mu(s)-\pi Q]dt\ . \qquad (2.72)$$

The periodic dispersion function $D(s)$—also written as $\eta(s)$—depends on all bending magnets in the accelerator. Furthermore, to get stable off-momentum orbits, the machine tune Q must not be an integer, otherwise resonance effect will occur: $u_\delta(s)$ becomes infinite.

2.6 Dispersive beam transport lattices

The dispersion function $D(s)$ in a beam transport line is the general solution of the equation

$$D'' + K(s)D = \frac{1}{\rho(s)}\,, \tag{2.73}$$

where $K(s)$ and $\rho(s)$ are arbitrary functions of s.

Already derived is the solution

$$D(s) = c_1 C(s) + c_2 S(s) + S(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt \tag{2.74}$$

where c_1 and c_2 are constants and remembering that $C(s)$ and $S(s)$ are solutions of the equation

$$D'' + K(s)D = 0\,.$$

From the initial conditions at s_0 of the cosine- and sine-like solutions, we obtain

$$D(s) = C(s)D(s_0) + S(s)D'(s_0) + S(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt\,, \tag{2.75}$$

or, in matrix formulation, differentiating (2.75),

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} C(s) & S(s) & S(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt \\ C'(s) & S'(s) & S'(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C'(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s_0) \\ D'(s_0) \\ 1 \end{pmatrix}. \tag{2.76}$$

This 3×3 matrix is the extended transfer matrix $M(s/s_0)$:

$$M(s/s_0) = \begin{pmatrix} C(s) & S(s) & S(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt \\ C'(s) & S'(s) & S'(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C'(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.77}$$

The solution of the equation of motion with a momentum deviation may then be expressed as

$$\begin{pmatrix} u(s) \\ u'(s) \\ \delta \end{pmatrix} = M(s/s_0) \begin{pmatrix} u(s_0) \\ u'(s_0) \\ \delta \end{pmatrix}. \tag{2.78}$$

Indeed, using (2.63), (2.64), (2.76) and (A.5) we get

$$\begin{aligned}
u(s) &= C(s)[u_\beta(s_0)+u_\delta(s_0)] + S(s)[u'_\beta(s_0)+u'_\delta(s)] \\
&\quad + S(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt\ \delta - C(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt\ \delta \\
&= C(s)u_\beta(s_0) + S(s)u'_\beta(s_0) + C(s)D(s_0)\delta + S(s)D'(s_0)\delta \\
&\quad + S(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt\delta - C(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt\delta = u_\beta(s) + D(s)\delta \\
&= u_\beta(s) + u_\delta(s)\ ,
\end{aligned}$$

and, similarly, we verify that

$$u'(s) = u'_\beta(s) + u'_\delta(s)\ .$$

The 3×3 extended transfer matrix (2.77) is easily computed when the strength $K(s)$ and the bending radius $\rho(s)$ are constant functions through the magnet.

Consider a dipole sector magnet, which is a bending magnet with entry and exit pole faces normal to the incoming and outgoing design orbit, respectively. For a sector magnet: $K(s) = 1/\rho_0^2$ (between s_0 and s_1 over a distance ℓ). The cosine- and sine-like solutions are

$$C(s) = \cos\left(\frac{s-s_0}{\rho_0}\right) \qquad \text{and} \qquad S(s) = \rho_0 \sin\left(\frac{s-s_0}{\rho_0}\right)\ .$$

Hence

$$\begin{aligned}
S(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt &= \sin\left(\frac{s-s_0}{\rho_0}\right)\int_{s_0}^{s}\cos\left(\frac{t-s_0}{\rho_0}\right)dt = \rho_0 \sin^2\left(\frac{s-s_0}{\rho_0}\right)^2 \\
C(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt &= \cos\left(\frac{s-s_0}{\rho_0}\right)\int_{s_0}^{s}\sin\left(\frac{t-s_0}{\rho_0}\right)dt \\
&= -\rho_0\cos\left(\frac{s-s_0}{\rho_0}\right)\left(\cos\left(\frac{s-s_0}{\rho_0}\right)-1\right)
\end{aligned}$$

and then

$$S(s)\int_{s_0}^{s}\frac{C(t)}{\rho(t)}dt - C(s)\int_{s_0}^{s}\frac{S(t)}{\rho(t)}dt = \rho_0\left(1-\cos\left(\frac{s-s_0}{\rho_0}\right)\right)$$

By (2.77) the transfer matrix of a sector magnet in the deflecting plane is

$$M(s_1/s_0) = \begin{pmatrix} \cos\theta & \rho_0\sin\theta & \rho_0(1-\cos\theta) \\ -\frac{1}{\rho_0}\sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}, \qquad (2.79)$$

where $\theta = \ell/\rho_0$ is the bending angle and ℓ the arc length of the magnet.

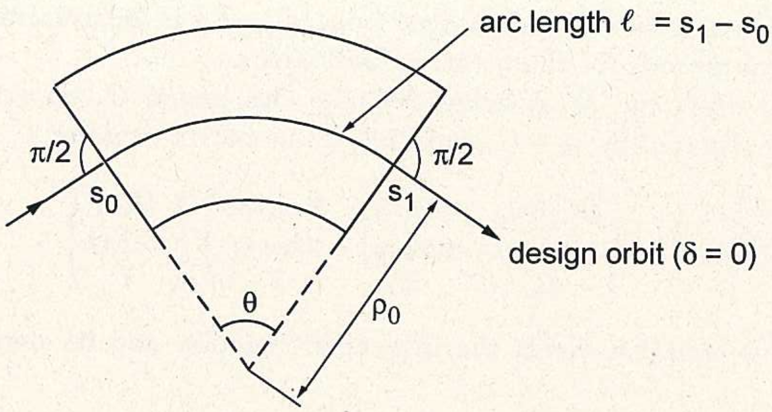

Figure 14: Dipole sector magnet.

The 3×3 transfer matrices for synchrotron magnets (combined dipole-quadrupole magnets) may be derived similarly. For a strong focusing synchrotron sector magnet: $K(s) = K_0 + 1/\rho_0^2 > 0$ (between s_0 and s_1 over a distance ℓ). In analogy to the dipole sector magnet case, replacing $1/\rho_0$ by $\sqrt{K}$ in the above solutions $C(s)$ and $S(s)$, the transfer matrix (2.77) reads, after computation of the matrix components m_{13} and m_{23}

$$M(s_1/s_0) = \begin{pmatrix} \cos\theta & \frac{\sin\theta}{\sqrt{K}} & \frac{1-\cos\theta}{\rho_0 K} \\ -\sqrt{K}\sin\theta & \cos\theta & \frac{\sin\theta}{\rho_0\sqrt{K}} \\ 0 & 0 & 1 \end{pmatrix} \tag{2.80}$$

with $\theta = \sqrt{K}\ell$, where K_0 is the normalized gradient and ℓ the magnet length.

For a strong defocusing synchrotron sector magnet: $K(s) = K_0 + 1/\rho_0^2 < 0$. The transfer matrix may be written as

$$M(s_1/s_0) = \begin{pmatrix} \cosh\theta & \frac{\sinh\theta}{\sqrt{|K|}} & \frac{\cosh\theta - 1}{\rho_0 |K|} \\ -\sqrt{|K|}\sinh\theta & \cosh\theta & \frac{\sinh\theta}{\rho_0\sqrt{|K|}} \\ 0 & 0 & 1 \end{pmatrix} \tag{2.81}$$

with $\theta = \sqrt{|K|}\ell$.

Consider again closed lattices. Like the method used for the betatron oscillations, the periodic dispersion function may also be derived applying the matrix formalism rather than using (2.72). For a ring of circumference C (or one periodic cell of length L) composed of N elements having extended 3×3 transfer matrices $M_1, M_2, \cdots M_N$ [with $M_k = M(s_k/s_{k-1})$], the total transfer matrix over the whole machine (or one machine period) is then obtained by multiplying the various matrices, yielding

$$M(s_0) \equiv M(s_0 + C/s_0) = M_N \cdots M_2 M_1 = \begin{pmatrix} m_{11}(s_0) & m_{12}(s_0) & m_{13}(s_0) \\ m_{21}(s_0) & m_{22}(s_0) & m_{23}(s_0) \\ 0 & 0 & 1 \end{pmatrix} \tag{2.82}$$

in which the 2×2 sub-matrix with components $(m_{ij}(s_0))_{i,j=1,2}$ is the usual transfer matrix over one turn (or one period) for the betatron oscillations.

The functions $D(s)$ and $D'(s)$ being periodic with period C, we get from (2.64), (2.78) in which s is replaced by $s_0 + C$, and (2.82), the matrix equation

$$\begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} = \begin{pmatrix} m_{11}(s_0) & m_{12}(s_0) & m_{13}(s_0) \\ m_{21}(s_0) & m_{22}(s_0) & m_{23}(s) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} D(s) \\ D'(s) \\ 1 \end{pmatrix} . \tag{2.83}$$

The solution of this equation yields the dispersion function and its derivative at the starting point s_0

$$\begin{aligned} D(s_0) &= \frac{m_{13}(s_0)(1 - m_{22}(s_0)) + m_{12}(s_0)m_{23}(s_0)}{2 - m_{11}(s_0) - m_{22}(s_0)} \\ D'(s_0) &= \frac{m_{23}(s_0)(1 - m_{11}(s_0)) + m_{21}(s_0)m_{13}(s_0)}{2 - m_{11}(s_0) - m_{22}(s_0)} \end{aligned} \tag{2.84}$$

since

$$m_{11}(s_0)m_{22}(s_0) - m_{12}(s_0)m_{21}(s_0) = 1 .$$

The same calculations can be performed at any azimuthal location s along the machine circumference, so that the dispersion function (2.84) can be obtained everywhere in the lattice. The matrix approach is useful when the strength $K(s)$ and the radius of curvature $\rho(s)$ are piecewise constant functions over the magnets, allowing an explicit determination of the periodic dispersion.

As an example, consider a thin-lens FODO lattice with cell length $2L$, in which the drift spaces are replaced by dipole sector magnets of length L and bending radius ρ_0 (assuming that the bending angle of the dipole is small: $L \ll \rho_0$). In accordance with (2.42), (2.77) and (2.79) the 3×3 transfer matrices of the cell elements read

$$M_{\mathrm{QF,D}} = \begin{pmatrix} 1 & 0 & 0 \\ \mp\frac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{\mathrm{O}} = \begin{pmatrix} 1 & L & \frac{L^2}{2\rho_0} \\ 0 & 1 & \frac{L}{\rho_0} \\ 0 & 0 & 1 \end{pmatrix}$$

where the dipole transfer matrix M_{O} in the deflecting plane has been simplified by expanding (2.79) to the first order in L/ρ_0.

The extended 3×3 transfer matrix through a FODO cell is obtained from the latter formulae with $M_{\mathrm{FODO}} = M_{\mathrm{O}} M_{\mathrm{QD}} M_{\mathrm{O}} M_{\mathrm{QF}}$

$$M_{\mathrm{FODO}} = \begin{pmatrix} 1 - \frac{L}{f} - \frac{L^2}{f^2} & 2L + \frac{L^2}{f} & \frac{L^2}{2\rho_0}\left(4 + \frac{L}{f}\right) \\ -\frac{L}{f^2} & 1 + \frac{L}{f} & \frac{L}{2\rho_0}\left(4 + \frac{L}{f}\right) \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.85}$$

Hence, by (2.84) the periodic dispersion in the focusing or defocusing thin quadrupole is

$$D_{\mathrm{QF,D}} = \frac{4f^2}{\rho_0}\left(1 \pm \frac{L}{4f}\right) . \tag{2.86}$$

The dispersion in the defocusing quadrupole (minus sign in the formula) is derived by replacing the focal length f by $-f$ in (2.85) (i.e. considering a DOFO cell instead, with $M_{\mathrm{DOFO}} = M_{\mathrm{O}} M_{\mathrm{QF}} M_{\mathrm{O}} M_{\mathrm{QD}}$).

3 MULTIPOLE FIELD EXPANSION

3.1 General multipole field components

Nonlinear magnetic fields will be considered as pure multipole magnet components to be used in the equations of motion of a charged particle in a transport channel or in a circular accelerator. To go beyond the linear expression of the magnetic field $\boldsymbol{B}$, we shall use a multipole expansion of $\boldsymbol{B}$ in a fixed, right-handed, Cartesian coordinate system (x, y, z), where the z-axis coincides with the s-axis of the coordinate system moving on the design orbit. The effects of curvature will thus be neglected in the derivation of the multipole field components in the magnets forming a lattice.

In a vacuum environment in the vicinity of the design orbit, the following Maxwell equations hold:

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \times \boldsymbol{B} = 0 \tag{3.1}$$

because the current density is zero in the region of interest.

The latter equation permits the expression of the magnetic field as the gradient of a magnetic scalar potential $U(x, y, z)$

$$\boldsymbol{B} = -\nabla U , \tag{3.2}$$

since for any scalar function U

$$\nabla \times \nabla U = 0 .$$

This leads to the Laplace equation by the first equation (3.1)

$$\nabla \cdot \nabla U \equiv \nabla^2 U = 0 . \tag{3.3}$$

We assume that the field does not vary along the z-axis, as is the case for long magnets far from the ends, and that there are only transverse field components (no solenoid field). Then the Laplace equation reads in polar coordinates (r, φ)

$$r\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{\partial^2 U}{\partial \varphi^2} = 0 . \tag{3.4}$$

Writing the potential $U(r, \varphi)$ as the product of two functions $f(r)$ and $g(\varphi)$

$$U(r, \varphi) = f(r) g(\varphi) \tag{3.5}$$

the Laplace equation transforms into

$$g(\varphi)r\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)+f(r)\frac{\partial^2 g}{\partial \varphi^2}=0 \tag{3.6}$$

or equivalently

$$\frac{r}{f(r)}\frac{d}{dr}\left(r\frac{df}{dr}\right)=-\frac{1}{g(\varphi)}\frac{d^2 g}{d\varphi^2}\equiv n^2\,. \tag{3.7}$$

Each side of this equation must be equal to a constant n^2 since the left-hand side depends only on r and the right-hand side on φ. Hence, we get

$$\begin{gathered} r^2\frac{d^2 f}{dr^2}+r\frac{df}{dr}-n^2 f=0 \\ \frac{d^2 g}{d\varphi^2}+n^2 g=0\,. \end{gathered} \tag{3.8}$$

The solutions of these equations may be written as

$$f(r)=-\frac{p}{e}\frac{r^n}{n!} \qquad g(\varphi)=A_n\,e^{in\varphi}\,. \tag{3.9}$$

The terms $(-p/e)$ and $n!$ have been added for convenience. The constant n is an integer since the potential $U(r,\varphi)$ is a periodic function of φ with period 2π (and also the function $g(\varphi)$). The constant A_n is assumed to be real. Hence, by (3.5) the magnetic scalar potential is written

$$U(r,\varphi)=-\frac{p}{e}\sum_{n=1}^{\infty}\frac{1}{n!}A_n r^n e^{in\varphi}\,. \tag{3.10}$$

Values of $n\leq 0$ have been discarded to avoid field singularities for $r\to 0$, and because $n=0$ gives a constant potential term which does not contribute to the magnetic field. In Cartesian coordinates the magnetic potential becomes

$$U(x,y)=-\frac{p}{e}\sum_{n=1}^{\infty}\frac{1}{n!}A_n(x+iy)^n\,. \tag{3.11}$$

Thus, the magnetic potential may be decomposed into independent multipole terms

$$U_n(x,y)=-\frac{p}{e}\frac{A_n}{n!}(x+iy)^n\,. \tag{3.12}$$

The real and imaginary parts of equation (3.12) are two independent solutions of the Laplace equation (3.3). From

$$(x+iy)^n=\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}x^{n-k}(iy)^k$$

and with for, $k \geq 0$,

$$i^{2k} = (-1)^k \qquad i^{2k+1} = (-1)^k i$$

it follows that

$$\mathrm{Re}\,((x+iy)^n) = \sum_{m=0}^{n/2} \frac{n!}{(2m)!(n-2m)!} x^{n-2m} y^{2m} (-1)^m$$

$$\mathrm{Im}\,((x+iy)^n) = \sum_{m=0}^{(n-1)/2} (-1)^m \frac{n!}{(2m+1)!(n-2m-1)!} x^{n-2m-1} y^{2m+1}$$

where the sums extend over the greatest integer less than or equal to $n/2$ and $(n-1)/2$, respectively.

Hence, assigning different coefficients $\underline{A}_n$ and A_n to the real and imaginary parts of (3.12), the potential for the nth order multipole becomes

$$\mathrm{Re}\,[U_n(x,y)] = -\frac{p}{e} \sum_{m=0}^{n/2} (-1)^m \underline{A}_n \frac{x^{n-2m}}{(n-2m)!} \frac{y^{2m}}{(2m)!} \tag{3.13}$$

$$\mathrm{Im}\,[U_n(x,y)] = -\frac{p}{e} \sum_{m=0}^{(n-1)/2} (-1)^m A_n \frac{x^{n-2m-1}}{(n-2m-1)!} \frac{y^{2m+1}}{(2m+1)!}\,. \tag{3.14}$$

The real and the imaginary part differentiate between two classes of magnet orientation. The imaginary part has the so-called midplane symmetry, that is

$$\mathrm{Im}\,[U_n(x,y)] = -\mathrm{Im}\,[U_n(x,-y)]\,. \tag{3.15}$$

The magnetic field components for the nth order multipoles derived from the imaginary solution (3.14) are, according to (3.2),

$$\begin{aligned} B_{nx}(x,y) &= -\frac{\partial}{\partial x} \mathrm{Im}\,[U_n(x,y)] = \\ &= \frac{p}{e} \sum_{m=0}^{(n-2)/2} (-1)^m A_n \frac{x^{n-2m-2}}{(n-2m-2)!} \frac{y^{2m+1}}{(2m+1)!} \end{aligned} \tag{3.16}$$

$$\begin{aligned} B_{ny}(x,y) &= -\frac{\partial}{\partial y} \mathrm{Im}\,[U_n(x,y)] = \\ &= \frac{p}{e} \sum_{m=0}^{(n-1)/2} (-1)^m A_n \frac{x^{n-2m-1}}{(n-2m-1)!} \frac{y^{2m}}{(2m)!} \end{aligned} \tag{3.17}$$

since $(\partial/\partial x) x^{n-2m-1} = 0$ if $m = (n-1)/2$. The midplane symmetry (3.15) yields

$$B_{nx}(x,y) = -B_{nx}(x,-y)$$

and

$$B_{ny}(x,y) = B_{ny}(x,-y)\,.$$

Thus, in this symmetry there is no horizontal field in the midplane, $B_{nx}(x,0) = 0$, and a particle travelling in the horizontal midplane will remain in this plane.

The magnets derived from the imaginary solution of the potential are called upright or regular magnets. Rewriting Eq. (3.17) as

$$B_{ny}(x,y) = \frac{p}{e}A_n\frac{x^{n-1}}{(n-1)!} + \frac{p}{e}A_n\sum_{m=1}^{(n-1)/2}(-1)^m\frac{x^{n-2m-1}}{(n-2m-1)!}\frac{y^{2m}}{(2m)!}$$

and differentiating this expression $n-1$ times leads to

$$A_n = \frac{e}{p}\frac{\partial^{n-1}B_{ny}}{\partial x^{n-1}} = -\frac{e}{p}(-1)^{n/2}\frac{\partial^{n-1}B_{nx}}{\partial y^{n-1}}\,. \tag{3.18}$$

The coefficient A_n is called the multipole strength parameter.

The magnets derived from the real solution of the potential are called rotated or skew magnets. The magnetic field components for the nth order skew multipole derived from the real solution are

$$\begin{aligned} B_{nx}(x,y) &= -\frac{\partial}{\partial x}\mathrm{Re}\,[U_n(x,y)] = \\ &= \frac{p}{e}\underline{A}_n\sum_{m=0}^{(n-1)/2}(-1)^m\frac{x^{n-2m-1}}{(n-2m-1)!}\frac{y^{2m}}{(2m)!} \end{aligned} \tag{3.19}$$

$$\begin{aligned} B_{ny}(x,y) &= -\frac{\partial}{\partial y}\mathrm{Re}\,[U_n(x,y)] = \\ &= \frac{p}{e}\underline{A}_n\sum_{m=0}^{n/2}(-1)^m\frac{x^{n-2m}}{(n-2m)!}\frac{y^{2m-1}}{(2m-1)!} \end{aligned} \tag{3.20}$$

since $(\partial/\partial x)x^{n-2m} = 0$ if $m = n/2$.

The skew multipole strength parameters $\underline{A}_n$ are derived from (3.19)

$$\underline{A}_n = \frac{e}{p}\frac{\partial^{n-1}B_{nx}}{\partial x^{n-1}} = \frac{e}{p}(-1)^{n/2}\frac{\partial^{n-1}B_{ny}}{\partial y^{n-1}}\,. \tag{3.21}$$

The skew magnets differ from the regular magnets only by a rotation about the z-axis by an angle $\phi_n = \pi/2n$, where n is the order at the multipole.

Magnetic multipole potentials

Dipole	$-\frac{e}{p}U_1(x,y) = -\frac{1}{\rho_y}x + i\frac{1}{\rho_x}y$
Quadrupole	$-\frac{e}{p}U_2(x,y) = \frac{1}{2}\underline{K}(x^2 - y^2) + iKxy$
Sextupole	$-\frac{e}{p}U_3(x,y) = \frac{1}{6}\underline{S}(x^3 - 3xy^2) + i\frac{1}{6}S(3x^2y - y^3)$
Octupole	$-\frac{e}{p}U_4(x,y) = \frac{1}{24}\underline{O}(x^4 - 6x^2y^2 + y^4) + i\frac{1}{6}O(x^3y - xy^3)$

Here we have used the notation: $A_2 = K$, $A_3 = S$, $A_4 = O$, and $\underline{A}_2 = \underline{K}$, $\underline{A}_3 = \underline{S}$, $\underline{A}_4 = O$. In particular the dipole strength parameters A_1 and $\underline{A}_1$ are given by

$$A_1 = \frac{e}{p}B_{1y} = k_x = \frac{1}{\rho_x} \qquad \underline{A}_1 = \frac{e}{p}B_{1x} = -k_y = -\frac{1}{\rho_y} \, .$$

Regular multipole fields

Dipole	$\frac{e}{p}B_{1x} = 0$	$\frac{e}{p}B_{1y} = \frac{1}{\rho_x}$
Quadrupole	$\frac{e}{p}B_{2x} = Ky$	$\frac{e}{p}B_{2y} = Kx$
Sextupole	$\frac{e}{p}B_{3x} = Sxy$	$\frac{e}{p} = B_{3y} = \frac{1}{2}S(x^2 - y^2)$
Octupole	$\frac{e}{p}B_{4x} = \frac{1}{6}O(3x^2y - y^3)$	$\frac{e}{p}B_{4y} = \frac{1}{6}O(x^3 - 3xy^2)$

Skew multipole fields

Dipole ($\phi = 90°$)	$\frac{e}{p}B_{1x} = -\frac{1}{\rho_y}$	$\frac{e}{p}B_{1y} = 0$
Quadrupole ($\phi = 45°$)	$\frac{e}{p}B_{2x} = \underline{K}x$	$\frac{e}{p}B_{2y} = -\underline{K}y$
Sextupole ($\phi = 30°$)	$\frac{e}{p}B_{3x} = \frac{1}{2}\underline{S}(x^2 - y^2)$	$\frac{e}{p}B_{3y} = -\underline{S}xy$
Octupole ($\phi = 22.5°$)	$\frac{e}{p}B_{4x} = \frac{1}{6}\underline{O}(x^3 - 3xy^2)$	$\frac{e}{p}B_{4y} = -\frac{1}{6}\underline{O}(3x^2y - y^3)$

The general magnetic field expansion including the most commonly used regular multipole elements reads

$$\begin{aligned}
\frac{e}{p}B_x(x,y) &= Ky + Sxy + \frac{1}{6}O(3x^2y - y^3) + \ldots \\
\frac{e}{p}B_y(x,y) &= \frac{1}{\rho_x} + Kx + \frac{1}{2}S(x^2 - y^2) + \frac{1}{6}O(x^3 - 3xy^2) + \ldots
\end{aligned}$$

3.2 Pole profile

Multipole fields may be generated by iron magnets in which the metallic surfaces, in the limit of infinite magnetic permeability, are equipotential surfaces for magnetic fields. Ignoring the end field effects, the potential of a horizontally deflecting dipole magnet is

$$\frac{e}{p}U_1(x,y) = -\frac{y}{\rho_x}\,. \tag{3.22}$$

An equipotential iron surface is determined by

$$\frac{y}{\rho_x} = \text{constant} \tag{3.23}$$

or $y = \text{constant}$ since ρ_x is constant inside the magnet.

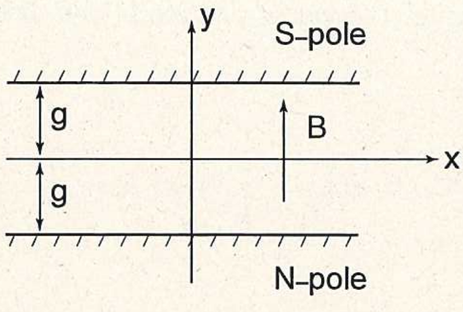

Figure 15: Dipole magnet pole shape.

For the midplane magnet to be at $y = 0$, two pole profiles are needed at $y = \pm g$, where $2g$ is the gap width.

Similarly, the equipotential iron-surface for a regular quadrupole magnet is

$$Kxy = \text{constant} . \tag{3.24}$$

Let R be the inscribed radius at the iron-free region limited by the hyperbola. For $x = y$: $x = \pm R/\sqrt{2}$. Then

$$K\frac{1}{2}R^2 = \text{constant} .$$

Identifying this expression with (3.24) gives the pole profile equation

$$xy = \pm\frac{1}{2}R^2 . \tag{3.25}$$

Similarly, the equipotential for a skew quadrupole magnet is

$$\frac{1}{2}\underline{K}(x^2 - y^2) = \text{constant} . \tag{3.26}$$

For $y = 0$: $x = \pm R$. Then

$$\frac{1}{2}\underline{K}R^2 = \text{constant}$$

The pole profile equation is therefore

$$x^2 - y^2 = \pm R^2 . \tag{3.27}$$

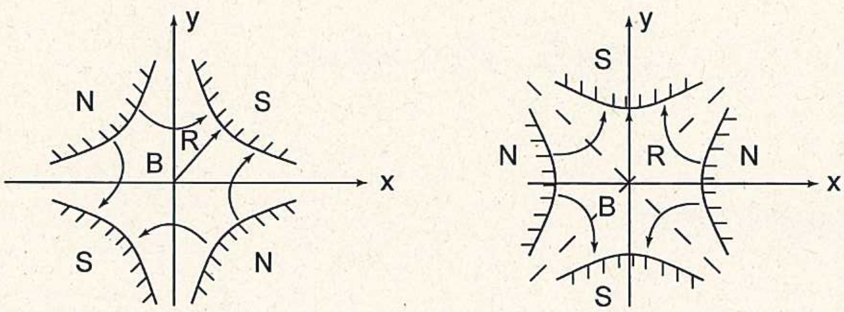

Figure 16: Regular and skew quadrupole magnet pole shapes.

A regular quadrupole magnet focuses in one plane and defocuses in the other. The field pattern is

$$\frac{e}{p}B_{2x} = Ky \qquad \frac{e}{p}B_{2y} = Kx \ .$$

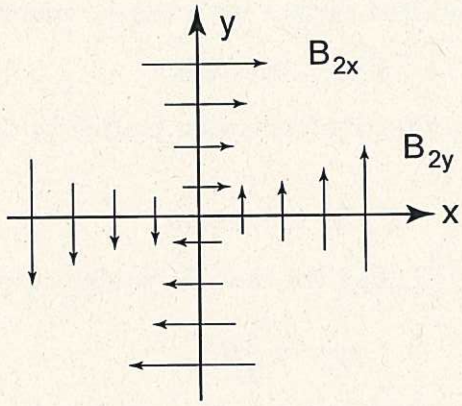

Figure 17: Magnetic filed for an horizontally focusing quadrupole (positive particles approach the reader).

The equipotential iron surface for a regular sextupole magnet is

$$S(3x^2y - y^3) = \text{constant} \tag{3.28}$$

and for a skew sextupole magnet

$$\underline{S}(x^3 - 3xy^2) = \text{constant} \ . \tag{3.29}$$

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Figure 18: Regular and skew sextupole magnet pole shapes.

4 TRANSVERSE RESONANCES

4.1 Nonlinear equations of motion

In the absence of a tangential magnetic field (no solenoid field) the transverse equations of a motion (3.13) for a charged particle in a magnetic field read

$$x'' - \frac{\sigma''}{\sigma'} x' = k_x h - (1+\delta)^{-1} \frac{e}{p_0} \sigma' h B_y$$
$$y'' - \frac{\sigma''}{\sigma'} y' = k_y h + (1+\delta)^{-1} \frac{e}{p_0} \sigma' h B_x \qquad (4.1)$$

where

$$h = 1 + k_x x + k_y y$$

and

$$\sigma' = \sqrt{h^2 + x'^2 + y'^2}\ . \qquad (4.2)$$

The local curvatures of the design orbit are $k_{x,y}$, the design momentum is p_0, and δ is the relative momentum deviation of a particle: $\delta = (p - p_0)/p_0$.

The equations of motion will be expanded to the second order in δ, x, y, and their derivatives. Differentiating Eq. (4.2) gives, dividing the result by σ',

$$\frac{\sigma''}{\sigma'} = \frac{hh' + x'x'' + y'y''}{\sigma'^2}$$

which, after some sorting may be expressed as

$$\frac{\sigma''}{\sigma'} \approx h' - \frac{1}{2}\frac{d}{ds}(k_x^2x^2 + k_y^2y^2 - x'^2 - y'^2) \tag{4.3}$$

where we have assumed a piecewise flat design orbit, so that

$$k_x(s)k_y(s) = 0 \; .$$

Then

$$\frac{\sigma''}{\sigma'}x' \approx k_x x'^2 + k_y x'y' + k_x' x x' + k_y' y x' \; .$$

Furthermore it will be assumed that

$$u' \ll 1 \qquad u \ll \rho_u = \frac{1}{k_u} \; , \tag{4.4}$$

where u stands for x or y, so that u'^2 and uu'/ρ_u^2 may be neglected since

$$k_u' = -\frac{\rho_u'}{\rho_u^2}$$

where $\rho_u(s)$ is the local bending radius. Therefore, to the second order we find

$$\frac{\sigma''}{\sigma'}x' = \frac{\sigma''}{\sigma'}y' \approx 0 \; .$$

Expressing the general magnetic field in terms of multipole components, the equations of motion (4.1) become to the second-order expansion

$$x'' = k_x h - (1 - \delta + \delta^2)\sigma' h\left(k_x + K_0 x + \frac{1}{2}S_0(x^2 - y^2) - \underline{K}_0 y - \underline{S}_0 xy \right)$$

$$y'' = k_y h + (1 - \delta + \delta^2)\sigma' h\left(- k_y + K_0 y + S_0 xy + \underline{K}_0 x + \frac{1}{2}\underline{S}_0(x^2 - y^2) \right) \tag{4.5}$$

The index zero in K_0, $\underline{K}_0$, S_0 and $\underline{S}_0$ means that the quadrupole and sextupole strengths are evaluated at the design momentum p_0. We compute the following quantities to the second order:

$$h^2 \;=\; 1 + 2k_x x + 2k_y y + k_x^2 x^2 + k_y^2 y^2$$

$$\sigma' h = h^2\sqrt{1+\frac{x'^2}{h^2}+\frac{y'^2}{h^2}} \approx h^2\left(1+\frac{x'^2}{2}+\frac{y'^2}{2}\right) \approx h^2 ,$$

since x'^2 and y'^2 are neglected due to (4.4). Furthermore

$$(1-\delta+\delta^2)\sigma' h k_x \approx k_x(1-\delta+\delta^2)+(2k_x^2-2k_x^2\delta+k_x^3 x)x$$

Similarly

$$\begin{aligned}
&(1-\delta+\delta^2)\sigma' h K_0 x \approx (1+2k_x x+2k_y y-\delta)K_0 x\\
&(1-\delta+\delta^2)\sigma' h \underline{K}_0 y \approx (1+2k_x x+2k_y y-\delta)\underline{K}_0 y\\
&(1-\delta+\delta^2)\sigma' h \frac{1}{2}S_0(x^2-y^2) \approx \frac{1}{2}S_0(x^2-y^2)\\
&(1-\delta+\delta^2)\underline{S}_0 xy \approx \underline{S}_0 xy
\end{aligned}$$

Thus the equations of motion (4.5) become

$$\begin{aligned}
x'' &+ (K_0+k_x^2)x-\underline{K}_0 y = k_x\delta-k_x\delta^2+(K_0+2k_x^2)x\delta-\\
&- (2K_0+k_x^2)k_x x^2-\frac{1}{2}S_0(x^2-y^2)-2K_0k_y xy-\\
&- \underline{K}_0 y\delta+2\underline{K}_0 k_y y^2+(\underline{S}_0+2\underline{K}_0 k_x)xy \qquad (4.6)
\end{aligned}$$

and

$$\begin{aligned}
y'' &- (K_0-k_y^2)y-\underline{K}_0 x = k_y\delta-k_y\delta^2-(K_0-2k_y^2)y\delta+\\
&+ (2K_0-k_y^2)k_y y^2+(S_0+2K_0k_x)xy-\\
&- \underline{K}_0\delta+2\underline{K}_0 k_x x^2+\underline{K}_0 k_y xy+\frac{1}{2}\underline{S}_0(x^2-y^2)\,. \qquad (4.7)
\end{aligned}$$

We shall further assume that

$$K_0 \ll \rho_{x,y}S_0 \qquad \underline{K}_0 \ll \rho_{x,y}\underline{S}_0\,, \qquad (4.8)$$

which is generally valid in large synchrotrons. Hence, some terms in (4.6) and (4.7) may be neglected since

$$\begin{aligned}
&K_0k_y xy \ll S_0(x^2-y^2) && \underline{K}_0 k_y xy \ll \underline{S}_0(x^2-y^2)\\
&K_0k_y xy \ll S_0 xy && \underline{K}_0 k_x xy \ll \underline{S}_0 xy\\
&K_0k_u u^2 \ll K_0 u && \underline{K}_0 k_u u^2 \ll \underline{K}_0 u\\
&k_u^3 u^2 \ll k_u^2 u
\end{aligned}$$

where u denotes x or y. Then the equations of motion reduce to the form

$$x'' + \left(K_0+\frac{1}{\rho_x^2}\right)x = \underline{K}_0 y+\frac{\delta}{\rho_x}-\frac{\delta^2}{\rho_x}+\left(K_0+\frac{2}{\rho_x^2}\right)x\delta-$$

$$- \quad \frac{1}{2}S_0(x^2 - y^2) - \underline{K}_0 y\delta + \underline{S}_0 xy \tag{4.9}$$

$$y'' \quad - \quad \left(K_0 - \frac{1}{\rho_y^2}\right) y = \underline{K}_0 x + \frac{\delta}{\rho_y} - \frac{\delta^2}{\rho_y} - \left(K_0 - \frac{2}{\rho_y^2}\right) y\delta +$$
$$+ \quad S_0 xy - \underline{K}_0 x\delta + \frac{1}{2}\underline{S}_0(x^2 - y^2)\,. \tag{4.10}$$

If we consider particles with the design momentum p_0, the equations of motion expanded up to sextupole terms read

$$x'' + \left(K_0 + \frac{1}{\rho_x^2}\right) x = \underline{K}_0 y - \frac{1}{2}S_0(x^2 - y^2) + \underline{S}_0 xy \tag{4.11}$$

$$y'' - \left(K_0 - \frac{1}{\rho_y^2}\right) y = \underline{K}_0 x + S_0 xy + \frac{1}{2}\underline{S}_0(x^2 - y^2)\,. \tag{4.12}$$

When the design orbit lies only in the horizontal plane, the vertical equation of motion simplifies to

$$y'' - K_0 y = \underline{K}_0 x + S_0 xy + \frac{1}{2}\underline{S}_0(x^2 - y^2)\,. \tag{4.13}$$

Remember that the magnet parameters $\rho_{x,y}$, K_0, $\underline{K}_0$, S_0 and $\underline{S}_0$ are generally functions of the s-coordinate. In practice they are piecewise constant functions along the lattice.

4.2 Description of motion in normalized coordinates

All terms on the right-hand sides of equations (4.9) and (4.10) will be treated as small perturbations. The left-hand side of these equations yields the linear unperturbed equations of motion

$$x'' + \left(K_0 + \frac{1}{\rho_x^2}\right) x = 0$$

$$y'' - \left(K_0 - \frac{1}{\rho_y^2}\right) y = 0 . \tag{4.14}$$

For closed lattices these equations are Hill's equations with periodic coefficients. They can be written in the compact form, writing u for either x or y,

$$u'' + K(s)u = 0 \tag{4.15}$$

where $K(s)$ is a periodic function with period L equal to the length of a machine cell

$$K(s + L) = K(s) \tag{4.16}$$

with

$$K(s) = \pm K_0(s) + \frac{1}{\rho_{x,y}^2(s)} .$$

Perturbation terms in the equations of motion may lead to unstable beam motion, called resonances, when the perturbating field acts in synchronism with the particle oscillations. A multipole of nth order is said to generate resonances of order n. Resonances below the third order (i.e. due to dipole and quadrupole field errors for instance) are called linear resonances. The nonlinear resonances are those of third order and above.

By an appropriate transformation of variables we can express the equations of motions (4.9) and (4.10) in the form of a perturbed harmonic oscillator with constant frequency. To this end we introduce the normalized coordinates (η, ϕ), called Floquet's coordinates, though the transformation $(u, s) \rightarrow (\eta, \phi)$

$$\eta = \frac{u}{\sqrt{\beta(s)}} \qquad \phi = \frac{1}{Q} \int_{s_0}^{s} \frac{dt}{\beta(t)} \tag{4.17}$$

where Q is the machine tune

$$Q = \frac{1}{2\pi} \int_{s}^{s+C} \frac{dt}{\beta(t)} \tag{4.18}$$

C is the ring circumference: $C = NL$, where N is the number of periodic cells.

Since $u = u(\beta, \eta)$ we compute

$$u' = \frac{\partial u}{\partial \beta}\beta' + \frac{\partial u}{\partial \eta}\eta' = \frac{1}{2\beta^{1/2}}\eta\beta' + \beta^{1/2}\eta' .$$

Similarly, since $u' = u'(\beta, \beta', \eta, \eta')$ we find

$$\begin{aligned} u'' &= \frac{\partial u'}{\partial \beta}\beta' + \frac{\partial u'}{\partial \beta'}\beta'' + \frac{\partial u'}{\partial \eta}\eta' + \frac{\partial u'}{\partial \eta'}\eta'' = \\ &= \left(-\frac{\beta'}{4\beta^{3/2}}\eta + \frac{1}{2\beta^{1/2}}\eta'\right)\beta' + \frac{1}{2\beta^{1/2}}\eta\beta'' + \frac{\beta'}{2\beta^{1/2}}\eta' + \beta^{1/2}\eta'' = \\ &= \beta^{1/2}\eta'' + \frac{\beta'}{\beta^{1/2}}\eta' + \frac{\eta}{2}\left(\frac{\beta''}{\beta^{1/2}} - \frac{1}{2}\frac{\beta'^2}{\beta^{3/2}}\right) . \end{aligned}$$

Hence, with

$$\frac{d}{ds} = \frac{d\phi}{ds}\frac{d}{d\phi} = \frac{1}{\beta Q}\frac{d}{d\phi}$$

and

$$\frac{d^2}{ds^2} = \frac{d}{ds}\left(\frac{1}{\beta Q}\frac{d}{d\phi}\right) = -\frac{\beta'}{\beta^2 Q}\frac{d}{d\phi} + \frac{1}{\beta Q}\frac{d}{ds}\left(\frac{d}{d\phi}\right) = -\frac{\beta'}{\beta^2 Q}\frac{d}{d\phi} + \frac{1}{\beta^2 Q^2}\frac{d^2}{d\phi^2} ,$$

it follows that

$$u'' = \beta^{1/2}\left(-\frac{\beta'}{\beta^2 Q}\dot{\eta} + \frac{1}{\beta^2 Q^2}\ddot{\eta}\right) + \beta^{-1/2}\beta'\left(\frac{1}{\beta Q}\dot{\eta}\right) + \frac{\eta}{2}\left(\frac{\beta''}{\beta^{1/2}} - \frac{1}{2}\frac{\beta'^2}{\beta^{3/2}}\right)$$

in which a point denotes the derivative with respect to ϕ. Inserting u'' into the Hill's equation (4.15) yields the unperturbed equation of motion expressed in normalized coordinates

$$\frac{1}{Q^2}\beta^{-3/2}\ddot{\eta} + \frac{1}{2}\eta\left(\beta^{-1/2}\beta'' + 2K(s)\beta^{1/2} - \frac{1}{2}\beta^{-3/2}\beta'^2\right) = 0 .$$

At this stage we introduce a general perturbation term $p(x, y, s)$ in the right-hand side of the latter equation [such a term may be the right-hand side of equation (4.9) or (4.10)]. Multiplying the resulting equation by $Q^2\beta^{3/2}$ yields

$$\ddot{\eta} + Q^2\eta\left(\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K(s)\beta^2\right) = Q^2\beta^{3/2}p(x, y, s) . \tag{4.19}$$

It is known that the betatron function $\beta(s)$ satisfies the differential equation

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + K(s)\beta^2 = 1 . \tag{4.20}$$

Consequently, the perturbed Hill's equation is transformed into

$$\ddot{\eta} + Q^2\eta = Q^2\beta^{3/2}p(\eta, \phi) \tag{4.21}$$

assuming a perturbation of the form $p(u, s)$ (one degree of freedom). When the perturbation vanishes, equation (4.21) reduces to a harmonic oscillator with frequency Q, whose solution is

$$\eta(\phi) = a\cos(\mu(\phi) - \varphi) \tag{4.22}$$

with

$$\mu(\phi) = Q\phi = \int_{s_0}^{s} \frac{dt}{\beta(t)} \, . \tag{4.23}$$

Differentiating (4.22) with respect to the variable μ gives

$$\frac{d\eta}{d\mu} = -a \sin\left[\mu(\phi) - \varphi\right]$$

and then

$$\eta^2 + \left(\frac{d\eta}{d\mu}\right)^2 = a^2 \, . \tag{4.24}$$

The particle trajectory in the phase plane (η, μ) is thus a circle of radius equal to the amplitude a of the oscillation. The phase $\mu(\phi)$ advances by 2π every betatron oscillation (i.e. the trajectory describes one full phase circle) or by $2\pi Q$ every machine revolution.

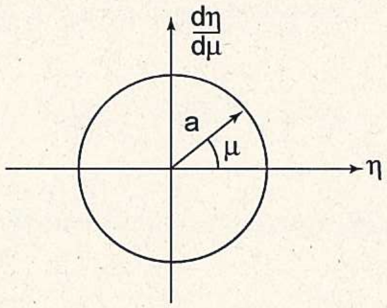

Figure 19: Phase circle in normalized coordinates.

In terms of normalized coordinates, the equations of motion (4.11) and (4.12) for an on-momentum particle (with $\delta = 0$) reads

$$\ddot{\eta} + Q_x^2\eta = -Q_x^2\left(-\beta_x^{3/2}\beta_y^{1/2}\underline{K}_0\xi + \beta_x^{5/2}\frac{S_0}{2}\eta^2 - \beta_x^{3/2}\beta_y\frac{S_0}{2}\xi^2 - \beta_x^2\beta_y^{1/2}\underline{S}_0\eta\xi\right) \tag{4.25}$$

$$\ddot{\xi} + Q_y^2\xi = Q_y^2\left(\beta_x^{1/2}\beta_y^{3/2}\underline{K}_0\eta + \beta_x\beta_y^{3/2}\frac{1}{2}\underline{S}_0\eta^2 - \beta_y^{5/2}\frac{1}{2}\underline{S}_0\xi^2 + \beta_x^{1/2}\beta_y^2 S_0\eta\xi\right) \tag{4.26}$$

in which ξ is the normalized coordinate for the vertical plane

$$\eta = \frac{x}{\sqrt{\beta_x(s)}} \qquad \xi = \frac{y}{\sqrt{\beta_y(s)}} \tag{4.27}$$

where $\beta_{x,y}(s)$ is the horizontal or vertical betatron function, and since ϕ is the independent variable in the differential equations (4.25) and (4.26), it can be defined as

$$\phi = \frac{1}{Q_{x,y}} \int_{s_0}^{s} \frac{dt}{\beta_{x,y}(t)} \tag{4.28}$$

where $Q_{x,y}$ is the horizontal or vertical machine tune. The multipole strengths K_0, $\underline{K}_0$, S_0, $\underline{S}_0$, and the betatron functions $\beta_{x,y}$ are periodic functions of ϕ with period 2π.

Now we consider the case where the general perturbation term $p(x, y, s)$ represents the magnetic field error $\Delta B_{x,y}$, that is the field not considered in setting up the design orbit of the machine. Equivalently, $\Delta B_{x,y}$ is the field error with respect to the ideal lattice, even if such a lattice includes nonlinear magnets like sextupoles. Expanding $\Delta B_{x,y}$ into multipoles up to the third order, in which the strength parameters are replaced by their variations: $\delta B_{x_0,y_0}$ for the dipole field error, δK and $\delta \underline{K}$ for the quadrupole gradient errors, δS and $\delta \underline{S}$ for the sextupole errors, we obtain

$$\begin{aligned}
\frac{e}{p_0}\Delta B_x &= \frac{e}{p_0}\delta B_{x_0} + \delta K\, y + \delta \underline{K}\, x + \delta S\, xy + \frac{1}{2}\delta \underline{S}(x^2 - y^2) \\
\frac{e}{p_0}\Delta B_y &= \frac{e}{p_0}\delta B_{y_0} + \delta K\, x - \delta \underline{K}\, y + \frac{1}{2}\delta S(x^2 - y^2) - \delta \underline{S}\, xy\,.
\end{aligned}$$

Hence, using (4.21), (4.27) and (4.28) we get the equations of motion in normalized coordinates

$$\begin{aligned}
\ddot{\eta} + Q_x^2\eta = Q_x^2\Bigg(&\beta_x^{3/2}\frac{e}{p_0}\delta B_{y_0} + \beta_x^2\delta K\,\eta - \beta_x^{3/2}\beta_y^{1/2}\delta \underline{K}\,\xi + \\
&+ \beta_x^{5/2}\frac{\delta S}{2}\eta^2 - \beta_x^{3/2}\beta_y\frac{\delta S}{2}\xi^2 - \beta_x^2\beta_y^{1/2}\delta \underline{S}\,\eta\xi\Bigg)
\end{aligned}$$

$$\begin{aligned}
\ddot{\xi} + Q_y^2\xi = Q_y^2\Bigg(&\beta_y^{3/2}\frac{e}{p_0}\delta B_{x_0} + \beta_y^2\delta K\,\xi - \beta_x^{1/2}\beta_y^{3/2}\delta \underline{K}\,\eta + \\
&+ \beta_x^{1/2}\beta_y^2\delta S\,\eta\xi + \beta_x\beta_y^{3/2}\frac{1}{2}\delta \underline{S}\,\eta^2 - \beta_y^{5/2}\frac{1}{2}\delta \underline{S}\,\xi^2\Bigg)\,.
\end{aligned} \tag{4.29}$$

We can rewrite the last two equations considering only the nth order multipole term in the horizontal plane and the nth order multipole term in the vertical plane. We get

$$\begin{aligned}
\ddot{\eta} + Q_x^2\eta &= p_{nrx}(\phi)\eta^{n-1}\xi^{r-1} \\
\ddot{\xi} + Q_y^2\xi &= p_{nry}(\phi)\eta^{n-1}\xi^{r-1}\,.
\end{aligned} \tag{4.30}$$

Perturbation terms

Order n	r	$p_{nrx}(\phi)\eta^{n-1}\xi^{r-1}$	$p_{nry}(\phi)\eta^{n-1}\xi^{n-1}$
1	1	$Q_x^2\beta_x^{3/2}\frac{e}{p_0}\delta B_{y_0}$	$Q_y^2\beta_y^{3/2}\frac{e}{p_0}\delta B_{x_0}$
1	2	$-Q_x^2\beta_x^{3/2}\beta_y^{1/2}\delta \underline{K}\xi$	$Q_y^2\beta_y^2\delta K\xi$
2	1	$Q_x^2\beta_x^2\delta K\eta$	$-Q_y^2\beta_x^{1/2}\beta_y^{3/2}\delta \underline{K}\,\eta$
2	2	$-Q_x^2\beta_x^2\beta_y^{1/2}\delta \underline{S}\,\eta\xi$	$Q_y^2\beta_x^{1/2}\beta_y^2\delta S\,\eta\xi$
1	3	$-Q_x^2\beta_x^{3/2}\beta_y\frac{1}{2}\delta S\,\xi^2$	$-Q_y^2\beta_y^{5/2}\frac{1}{2}\delta \underline{S}\,\xi^2$
3	1	$Q_x^2\beta_x^{5/2}\frac{1}{2}\delta S\,\eta^2$	$Q_y^2\beta_x\beta_y^{3/2}\frac{1}{2}\delta \underline{S}\,\eta^2$

4.3 One-dimensional resonances

Considering only the nth order multipole horizontal perturbation term, the equation of motion (4.30) in one dimension may be written [with $r = 1$ and Q stands for Q_x and $p_n(\phi)$ for $p_{n1x}(\phi)$] as

$$\ddot{\eta} + Q^2\eta = p_n(\phi)\eta^{n-1} \,. \tag{4.31}$$

Expanding the perturbation into Fourier series yields

$$p_n(\phi) = \sum_{m=-\infty}^{\infty} \hat{p}_n(m)e^{im\phi} \tag{4.32}$$

where $\hat{p}_n(m)$ is the Fourier coefficient of the expansion

$$\hat{p}_n(m) = \frac{1}{2\pi}\int_0^{2\pi} p_n(\phi)e^{-im\phi}d\phi \,. \tag{4.33}$$

The unperturbed oscillation, solution of the inhomogeneous equation (4.31), may be written as

$$\eta_0(\phi) = a\,e^{iQ\phi} + b\,e^{-iQ\phi} \tag{4.34}$$

where a and b are constants of integration.

Assuming the perturbation is small, we can insert $\eta_0(\phi)$ into the right-hand side of the equation of motion (4.31). We find

$$\ddot{\eta} + Q^2\eta = \sum_{m=-\infty}^{\infty} \hat{p}_n(m)e^{im\phi}(a\,e^{iQ\phi} + b\,e^{-iQ\phi})^{n-1} \,.$$

Using the binomial expansion, we get

$$\begin{aligned}\eta_0(\phi)^{n-1} &= (a\,e^{iQ\phi}+b\,e^{-iQ\phi})^{n-1} = \sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-k-1)!}a^{n-k-1}e^{i(n-k-1)Q\phi}b^k e^{-ikQ\phi} \\ &= \sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-k-1)!}a^{n-k-1}b^k e^{i(n-2k-1)Q\phi}\,.\end{aligned}$$

Performing the change of variable $p = 2k - n + 1$, the above expression may be rewritten as

$$\eta_0(\phi)^{n-1} = \sum_{p=-n+1}^{n-1} c(p)e^{-ipQ\phi}$$

with

$$c(p) = \frac{(n-1)!}{\left(\frac{n+p-1}{2}\right)!\left(\frac{n-p-1}{2}\right)!}\frac{1}{2}\left[1+(-1)^{n+p-1}\right]a^{\frac{n-p-1}{2}}b^{\frac{n+p-1}{2}}\,.$$

Hence (4.31) reads

$$\ddot{\eta} + Q^2\eta = \sum_{m=-\infty}^{\infty}\sum_{p=-(n-1)}^{n-1}\hat{p}_n(m)c(p)e^{i(m-pQ)\phi}\,. \tag{4.35}$$

Whenever a term on the right-hand side of equation (4.35) has the same frequency as the frequency Q of the harmonic oscillator, the motion gets in resonance. To see this, we consider a single Fourier component of the perturbation

$$\ddot{\eta} + Q^2\eta = \hat{p}_n(m)c(p)e^{i(m-pQ)\phi}\,. \tag{4.36}$$

The solution of this equation is the sum of the solution of the homogeneous equation and a particular solution of the form

$$\bar{\eta} = A\,e^{i(m-pQ)\phi}\,. \tag{4.37}$$

The constant A is determined by introducing (4.37) into (4.36)

$$\left[-(m-pQ)^2+Q^2\right]A\,e^{i(m-pQ)\phi} = \hat{p}_n(m)c(p)e^{i(m-pQ)\phi}$$

from which A is found and

$$\bar{\eta}(\phi) = -\frac{\hat{p}_n(m)c(p)}{[Q(p-1)-m][Q(p+1)-m]}e^{i(m-pQ)\phi}\,. \tag{4.38}$$

Thus the motion becomes unstable when the tune satisfies the resonance conditions

$$m = (p\pm 1)Q \quad \text{with} \quad |p| \le n-1 \tag{4.39}$$

or equivalently

$$|m| = (|p|\pm 1)Q \tag{4.40}$$

where $|p|+1$ is called the order of resonance and m the order of the perturbation Fourier harmonic.

The resonance conditions apply only for index p such that the coefficient $c(p)$ is non-zero. Thus for the third order resonance $n=3$ driven by sextupole fields, we compute for $p=-2, -1, 0, 1, 2$:

$$c(-2)=a^2 \quad c(-1)=0 \quad c(0)=2ab \quad c(1)=0 \quad c(2)=b^2 \, .$$

The resonance conditions are then $m = \pm 3Q$: (third-order resonance) and $m = \pm Q$: (integer resonance). Hence the Q values driving third integer resonances are

$$Q = k \pm \frac{1}{3} \tag{4.41}$$

where k is any positive integer.

4.4 Coupling resonances

The two-dimensional equations of motion with only the nth order multipole horizontal and rth order vertical multipole perturbation are given by (4.30)

$$\begin{aligned} \ddot{\eta} + Q_x^2 \eta &= p_{nrx}(\phi)\eta^{n-1}\xi^{r-1} \\ \ddot{\xi} + Q_y^2 \xi &= p_{nry}(\phi)\eta^{n-1}\xi^{r-1} \, . \end{aligned}$$

Again expanding the perturbation in Fourier series yields

$$\begin{aligned} p_{nrx}(\phi) &= \sum_{m=-\infty}^{\infty} \hat{p}_{nrx}(m) e^{im\phi} \\ p_{nry}(\phi) &= \sum_{m=-\infty}^{\infty} \hat{p}_{nry}(m) e^{im\phi} \, . \end{aligned} \tag{4.42}$$

Furthermore, the solutions of the unperturbed equations are

$$\begin{aligned} \eta_0(\phi) &= a_x e^{iQ_x\phi} + b_x e^{-iQ_y\phi} \\ \xi_0(\phi) &= a_y e^{iQ_y\phi} + b_y e^{-iQ_y\phi} \end{aligned} \tag{4.43}$$

in which $a_{x,y}$ and $b_{x,y}$ are constants at integration. Hence

$$\begin{aligned} \eta_0(\phi)^{n-1} &= \sum_{\ell=-n+1}^{n-1} c_x(\ell) e^{-i\ell Q_x\phi} \\ \xi_0(\phi)^{r-1} &= \sum_{p=-r+1}^{r-1} c_y(p) e^{-ipQ_y\phi} \end{aligned}$$

with

$$
\begin{aligned}
c_x(\ell) &= \frac{(n-1)!}{\left(\frac{n+\ell-1}{2}\right)!\left(\frac{n-\ell-1}{2}\right)!}\frac{1}{2}\left[1+(-1)^{n-\ell-1}\right]a_x^{\frac{n-\ell-1}{2}}b_x^{\frac{n+\ell-1}{2}} \\
c_y(p) &= \frac{(r-1)!}{\left(\frac{r+p-1}{2}\right)!\left(\frac{r-p-1}{2}\right)!}\frac{1}{2}\left[1+(-1)^{r-p-1}\right]a_y^{\frac{r-p-1}{2}}b_y^{\frac{r+\ell-1}{2}} .
\end{aligned}
$$

In the same way as in the uncoupled treatment of resonances, we obtain, after substitution of the unperturbed oscillations into the right-hand side of the perturbed differential equations (4.30),

$$
\begin{aligned}
\ddot{\eta}+Q_x^2\eta &= \sum_m\sum_\ell\sum_p \hat{p}_{nrx}(m)c_x(\ell)c_y(p)e^{i(m-\ell Q_x-pQ_y)\phi} \\
\ddot{\xi}+Q_y^2\xi &= \sum_m\sum_\ell\sum_p \hat{p}_{nry}(m)c_x(\ell)c_y(p)e^{i(m-\ell Q_x-pQ_y)\phi} .
\end{aligned} \tag{4.44}
$$

Then considering a single component of the perturbation, particular solutions of these equations may be derived. We obtain

$$
\begin{aligned}
\bar{\eta}(\phi) &= -\frac{\hat{p}_{nrx}(m)c_x(\ell)c_y(p)}{[Q_x(\ell-1)+pQ_y-m]\,[Q_x(\ell+1)+pQ_y-m]}e^{i(m-\ell Q_x-pQ_y)\phi} \\
\bar{\xi}(\phi) &= -\frac{\hat{p}_{nry}(m)c_x(k)c_y(q)}{[Q_y(q-1)+kQ_x-m][Q_y(q+1)+kQ_x-m]}e^{i(m-kQ_x-qQ_y)\phi} .
\end{aligned} \tag{4.45}
$$

The resonance conditions are then

$$
\begin{aligned}
& m=(\ell\pm 1)Q_x+pQ_y \quad \text{with } |\ell|\le n-1 \text{ and } |p|\le r-1 , \\
& m=(q\pm 1)Q_y+kQ_x \quad \text{with } |k|\le n-1 \text{ and } |q|\le r-1 .
\end{aligned} \tag{4.46}
$$

For example, we consider the resonances driven by a regular sextupole for which the equations of motion are

$$
\begin{aligned}
\ddot{\eta}+Q_x^2\eta &= p_{31x}(\phi)\eta^2+p_{13x}(\phi)\xi^2 \\
\ddot{\xi}+Q_y^2\xi &= p_{22y}(\phi)\eta\xi .
\end{aligned} \tag{4.47}
$$

The resonance conditions are given in the following table.

n	Order r	ℓ	p	Horizontal motion $m = (\ell \pm 1)Q_x + pQ_y$	Vertical motion $m = (p \pm 1)Q_y + \ell Q_x$
2	2	−1	−1		$m = \begin{cases} -Q_x \\ -2Q_y - Q_x \end{cases}$
		−1	1		$m = \begin{cases} -Q_x \\ 2Q_y - Q_x \end{cases}$
		1	−1		$m = \begin{cases} -2Q_y + Q_x \\ Q_x \end{cases}$
		1	1		$m = \begin{cases} 2Q_y + Q_x \\ Q_x \end{cases}$
1	3	0	−2	$m = \begin{cases} Q_x - 2Q_y \\ -Q_x - 2Q_y \end{cases}$	
		0	0	$m = \begin{cases} Q_x \\ -Q_x \end{cases}$	
		0	2	$m = \begin{cases} Q_x + 2Q_y \\ -Q_x + 2Q_y \end{cases}$	
3	1	−2	0	$m = \begin{cases} -Q_x \\ -3Q_x \end{cases}$	
		0	0	$m = \begin{cases} Q_x \\ -Q_x \end{cases}$	
		2	0	$m = \begin{cases} 3Q_x \\ Q_x \end{cases}$	

Discarding the redundant conditions, we are left with

$$|m| = Q_x \quad |m| = 2Q_y \pm Q_x \quad |m| = 3Q_x \ .$$

Having considering a skew sextupole instead, the resonance conditions would have been

$$|m| = Q_y \quad |m| = 2Q_x \pm Q_y \quad |m| = 3Q_y \ .$$

Thus, the general resonance conditions in two degrees of freedom may be cast into the form

$$MQ_x + NQ_y = P \ , \tag{4.48}$$

where M, N, and P are integers, P being non-negative, and $|M| + |N|$ is the order of the resonance, and P is the order of the perturbation harmonic. Plotting the resonance lines (4.47) for different values of M, N and P in the (Q_x, Q_y) plane yield the so called resonance or tune diagram.

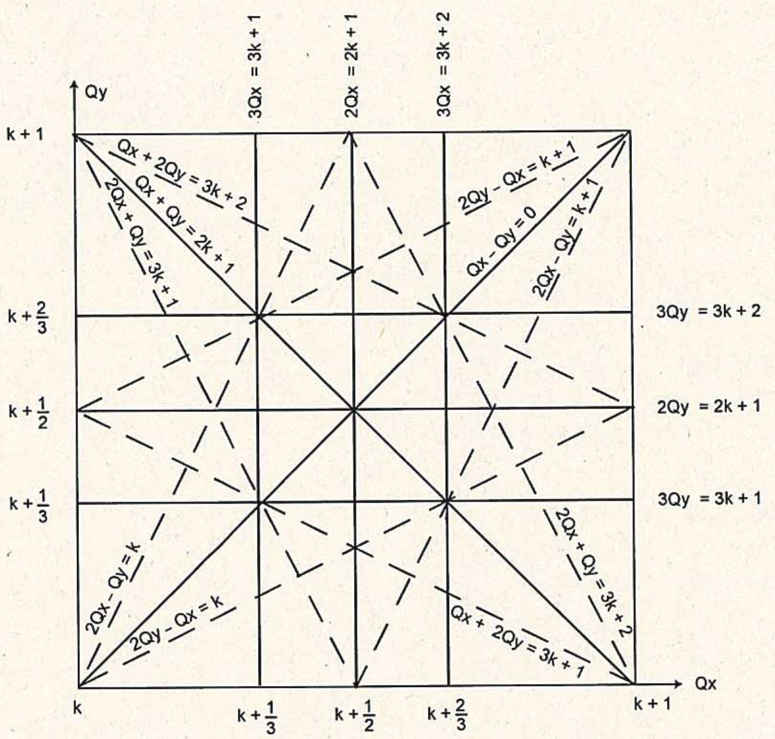

Figure 20: Resonance diagram in a unit square for regular and skew multipole fields up to the 3rd order.

If the integers M and N are positive, the resonance is called a sum resonance and can lead to a loss of beam. If M and N are of opposite sign, the resonance is called a coupling resonance or a difference resonance and does not cause a loss of beam, but rather leads to a beating between the horizontal and the vertical oscillations. It can be shown that the "strength" of the resonances decrease with increasing order, so that only low-order resonances need to be avoided.

Around every resonance line in the resonance diagram, there is a band with some thickness, called the resonance width, in which the motion may be unstable, depending on the oscillation amplitude. When the resonance is linear (below the third order), the resonance width is called a stopband because the entire beam becomes unstable if the operating point Q_x, Q_y reaches this region of tune values. The largest oscillation amplitude which is still stable in the presence of nonlinearities is called the dynamic aperture.

The following figures are tune diagrams in a unit square around the origin showing the resonance lines driven by regular multipole fields up to the 12th order.

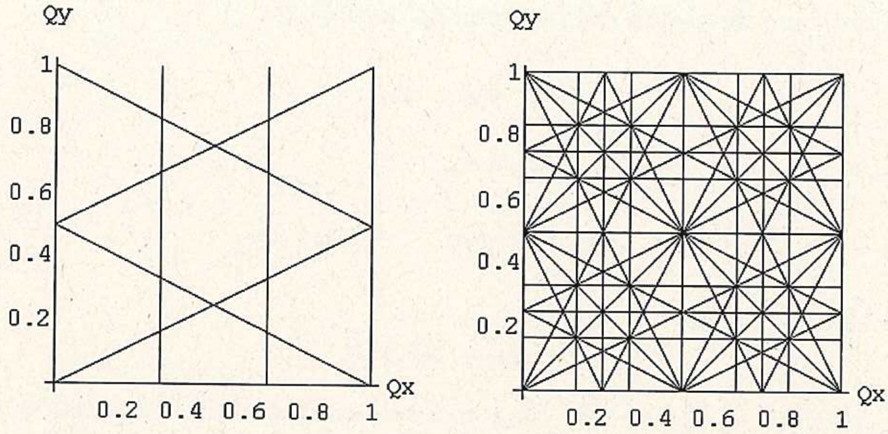

Figure 21: Resonance diagrams for regular multipole fields up to 3rd and 6th orders.

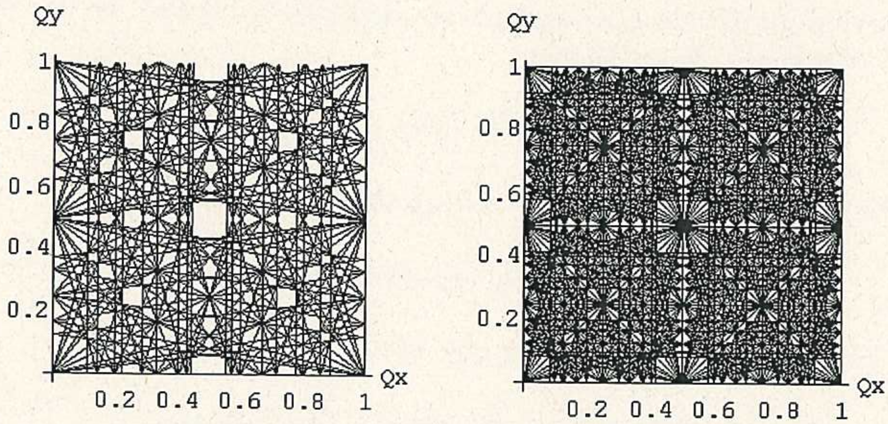

Figure 22: Resonance diagrams for regular multipole fields up to 9th and 12th orders.

5 THE THIRD-INTEGER RESONANCE

5.1 The averaging method

Consider the equation of motion in Floquet coordinates for an nth order multipole perturbation

$$\ddot{\eta} + Q^2\eta = p_n(\phi)\eta^{n-1} \tag{5.1}$$

where the perturbation term $p_n(\phi)$ is periodic in ϕ, with period 2π.

Assume that the tune Q is close to an nth order resonance

$$Q \approx Q_r = \frac{m}{n} \, . \tag{5.2}$$

in which m is a harmonic of the perturbation $p_n(\phi)$.

Defining the tune deviation δQ between Q_r and Q as

$$\delta Q = Q - Q_r$$

the equation of motion can then be written as

$$\ddot{\eta} + Q_r^2 \eta = p_n(\phi)\eta^{n-1} - 2Q_r \delta Q \eta \tag{5.3}$$

since for small tune deviation

$$Q^2 \approx Q_r^2 + 2Q_r\delta Q \ .$$

The averaging method of Krilov and Bogoliubov assumes that the solution of (5.3) is nearly periodic in the variable ϕ. Writing this solution as

$$\eta(\phi) = a(\phi)\cos\left[Q_r\phi + \varphi(\phi)\right] \tag{5.4}$$

with slowly-varying amplitude $a(\phi)$ and phase $\varphi(\phi)$ over a "cycle" 2π, we impose the condition that $\dot{\eta}(\phi)$ should have the form

$$\dot{\eta}(\phi) = -a(\phi)Q_r \sin\left[Q_r\phi + \varphi(\phi)\right] \ . \tag{5.5}$$

By differentiating (5.4), this assumption leads to the additional condition

$$\dot{a}(\phi)\cos\psi(\phi) - a(\phi)\dot{\varphi}(\phi)\sin\psi(\phi) = 0 \ , \tag{5.6}$$

where $\psi(\phi) = Q_r\phi + \varphi(\phi)$. Differentiating now $\dot{\eta}(\phi)$ yields

$$\ddot{\eta} = -Q_r^2 a \cos\psi - \dot{a}Q_r \sin\psi - Q_r a\,\dot{\varphi}\cos\psi \ .$$

Substituting η and $\ddot{\eta}$ into (5.3) gives

$$-Q_r\dot{a}\sin\psi - Q_r a\dot{\varphi}\cos\psi = p_n(\phi)a^{n-1}\cos^{n-1}\psi - 2Q_r\delta Q a\cos\psi \ .$$

Adding this expression, multiplied by $\cos\psi$, to equation (5.6) multiplied by $Q_r \sin\psi$, one obtains

$$\frac{d\varphi}{d\phi} = -\frac{p_n(\phi)}{Q_r}a^{n-2}\cos^n\psi + 2\delta Q\cos^2\psi \tag{5.7}$$

$$\frac{da}{d\phi} = -\frac{p_n(\phi)}{Q_r}a^{n-1}\cos^{n-1}\psi\sin\psi + 2a\delta Q\cos\psi\sin\psi \ , \tag{5.8}$$

where the latter equation is derived from (5.6), using $\dot{\varphi}$ given in (5.7).

The periodic perturbation $p_n(\phi)$ may be expanded in a trigonometric Fourier series

$$p_n(\phi) = \frac{\hat{p}_n(0)}{2} + \sum_{k=1}^{\infty}\left[\hat{p}_n(k)\cos k\phi + \hat{p}_n^*(k)\sin k\phi\right] \tag{5.9}$$

with

$$\hat{p}_n(k) = \frac{1}{\pi}\int_0^{2\pi} p_n(\phi)\cos k\phi\, d\phi$$

$$\hat{p}_n^*(k) = \frac{1}{\pi}\int_0^{2\pi} p_n(\phi)\sin k\phi\, d\phi$$

where $\hat{p}_n(k)$ and $\hat{p}_n^*(k)$ are the Fourier coefficients of the series. Considering only the single Fourier component $\hat{p}_n(m)$ which drives the resonance $Q_r = m/n$, the differential equations for a and φ may be rewritten as

$$\frac{da}{d\phi} = -\frac{n}{m}\hat{p}_n(m)a^{n-1}\cos n(\psi-\varphi)\cos^{n-1}\psi\sin\psi + \\ + 2a\,\delta Q\cos\psi\sin\psi \tag{5.10}$$

$$\frac{d\varphi}{d\phi} = -\frac{n}{m}\hat{p}_n(m)a^{n-2}\cos n(\psi-\varphi)\cos^n\psi + 2\delta Q\cos^2\psi \tag{5.11}$$

using

$$\phi = \frac{\psi-\varphi}{Q_r} = \frac{n}{m}(\psi-\varphi)\,.$$

As $a(\phi)$ and $\varphi(\phi)$ are slowly varying functions of ϕ over a "cycle" 2π, $\dot{a}$ and $\dot{\varphi}$ may be treated as periodic in ψ with period 2π and thus expanded in a Fourier series

$$\dot{a}(\psi) = \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\hat{\dot{a}}(k)e^{ik\psi} + \hat{\dot{a}}(0) \tag{5.12}$$

$$\dot{\varphi}(k) = \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\hat{\dot{\varphi}}(k)e^{ik\psi} + \hat{\dot{\varphi}}(0) \tag{5.13}$$

with

$$\hat{\dot{a}}(k) = \frac{1}{2\pi}\int_0^{2\pi}\dot{a}(\psi)e^{-ik\psi}d\psi \tag{5.14}$$

$$\hat{\dot{\varphi}}(k) = \frac{1}{2\pi}\int_0^{2\pi}\dot{\varphi}(\psi)e^{-ik\psi}d\psi\,. \tag{5.15}$$

Exponential Fourier series have been considered, and the coefficients $\hat{\dot{a}}(0)$ and $\hat{\dot{\varphi}}(0)$ have been explicitly written for convenience. The principle of the averaging method consists in replacing $\dot{a}(\psi)$ and $\dot{\varphi}(\psi)$ by their average parts $\langle\dot{a}\rangle$ and $\langle\dot{\varphi}\rangle$, which is equivalent to assuming that $\dot{a}$ and $\dot{\varphi}$ are not influenced by small rapid oscillations, that is

$$\dot{a} = \langle\dot{a}\rangle + \text{small rapidly oscillating terms} \approx \langle\dot{a}\rangle$$

$$\dot{\varphi} = \langle\dot{\varphi}\rangle + \text{small rapidly oscillating terms} \approx \langle\dot{\varphi}\rangle\,.$$

Hence, taking into account (5.12) to (5.15) we obtain

$$\frac{da}{d\phi} \approx \left\langle \frac{da}{d\phi} \right\rangle = \hat{a}(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{da}{d\phi} d\psi \tag{5.16}$$

$$\frac{d\varphi}{d\phi} \approx \left\langle \frac{d\varphi}{d\phi} \right\rangle = \hat{\varphi}(0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{d\phi} d\psi \tag{5.17}$$

or equivalently, using (5.10) and (5.11)

$$\frac{da}{d\phi} \approx -\frac{n}{2\pi m} \hat{p}_n(m) a^{n-1} \int_0^{2\pi} \cos\, n(\psi - \varphi) \cos^{n-1} \psi \sin\, \psi\, d\psi \tag{5.18}$$

$$\frac{d\varphi}{d\phi} \approx \delta Q - \frac{n}{2\pi m} \hat{p}_n(m) a^{n-2} \int_0^{2\pi} \cos\, n(\psi - \varphi) \cos^n \psi\, d\psi\ . \tag{5.19}$$

In summary, the solution of the nonlinear equation of motion (5.1) when the tune is close to a resonance $Q \approx m/n$ is

$$\eta(\phi) = a(\phi) \cos\left[Q_r \phi + \varphi(\phi)\right]$$

where $a(\phi)$ and $\varphi(\phi)$ are given in the first approximation by the averaged differential equations (5.18) and (5.19).

5.2 The nonlinear sextupole resonance

The third-integer resonance is driven by sextupolar fields

$$Q_r = \frac{m}{3}\ .$$

The equation of motion for the mth harmonic of the perturbation $p_3(\phi)$ is

$$\ddot{\eta} + Q_r^2 \left(1 + 2\frac{\delta Q}{Q_r}\right) \eta = \hat{p}_3(m) \eta^2 \cos\, m\phi\ . \tag{5.20}$$

For Q close to Q_r the solution of the equation of motion is given by (5.4)

$$\eta(\phi) = a(\phi) \cos\left[\frac{m}{3}\phi + \varphi(\phi)\right]\ . \tag{5.21}$$

The averaged differential equations for $a(\phi)$ and $\varphi(\phi)$ are, after integration of (5.18) and (5.19) for $n = 3$.

$$\frac{da}{d\phi} = -\frac{3}{8m} \hat{p}_3(m) a^2 \sin\, 3\varphi \tag{5.22}$$

$$\frac{d\varphi}{d\phi} = \delta Q - \frac{3}{8m} \hat{p}_3(m) a \cos\, 3\varphi \tag{5.23}$$

where the following trigonometric formulae have been used

$$\cos 3(\psi - \varphi) = \cos 3\psi \cos 3\varphi + \sin 3\psi \sin 3\varphi$$
$$\cos 3\psi = -3\cos\psi + 4\cos^3\psi$$
$$\sin 3\psi = 3\sin\psi - 4\sin^3\psi$$

and

$$\int_0^{2\pi} \sin^{2m}\psi \cos^{2n}\psi \, d\psi = \frac{(2m)!(2n)!2\pi}{m!n!(m+n)!2^{2(m+n)}}$$
$$\int_0^{2\pi} \sin^{m}\psi \cos^{n}\psi \, d\psi = 0 \quad \text{if } m \text{ and/or } n \text{ odd} .$$

Eliminating the variable ϕ from (5.22) and (5.23) yields

$$\frac{d\varphi}{da} = \frac{3\hat{p}_3(m) a \cos 3\varphi - 8m\,\delta Q}{3\hat{p}_3(m) a^2 \sin 3\varphi} . \tag{5.24}$$

By means of the change of variable

$$z = a\cos 3\varphi$$

the latter equation transforms into

$$\left[z - \frac{4m\delta Q}{\hat{p}_3(m)}\right] da + \frac{a}{2} dz = 0 ,$$

which can be integrated to give

$$\ln a + \ln \left[z - \frac{4m\delta Q}{\hat{p}_3(m)}\right]^{1/2} = \ln A ,$$

where A is a constant depending on the initial conditions. Transforming back to the original variable φ we find

$$a^3 \left[\cos 3\varphi - \frac{4m\delta Q}{\hat{p}_3(m) a}\right] = A . \tag{5.25}$$

The solutions $\eta(\phi)$ and $\dot{\eta}(\phi)$ may be represented on the phase plane with Cartesian axes η, $\dot{\eta}$. As ϕ increases, $[\eta(\phi), \dot{\eta}(\phi)]$ traces out a phase path. A more appropriate phase plane can be chosen with axes η, p_η, with

$$p_\eta(\phi) \equiv \frac{3}{m}\dot{\eta}(\phi) = -a(\phi)\sin\left[\frac{m}{3}\phi + \varphi(\phi)\right] . \tag{5.26}$$

The phase plane coordinates may be rewritten, using the trigonometric formulae for the sum of the angles

$$\eta = X\cos\frac{m}{3}\phi + Y\sin\frac{m}{3}\phi \tag{5.27}$$

$$p_\eta = -X \sin \frac{m}{3}\phi + Y \cos \frac{m}{3}\phi \tag{5.28}$$

with the new phase plane variables $X(\phi)$, $Y(\phi)$ defined by

$$X(\phi) = a(\phi) \cos \varphi(\phi) \tag{5.29}$$

$$Y(\phi) = -a(\phi) \sin \varphi(\phi) \; . \tag{5.30}$$

The link between the variables $(\eta,\, p_\eta)$ and $(X,\, Y)$ is a rotation of an angle $m\phi/3$. Thus it is sufficient to study the representative phase plane with Cartesian coordinates $(X,\, Y)$ or polar coordinates $(a,\, \varphi)$. In particular, for every multiple of three machine revolutions (i.e. for $\phi = 6k\pi$, with $k = 0, 1, 2...$), one has

$$\eta(6k\pi) = X(6k\pi) \quad \text{and} \quad p_\eta(6k\pi) = Y(6k\pi) \; .$$

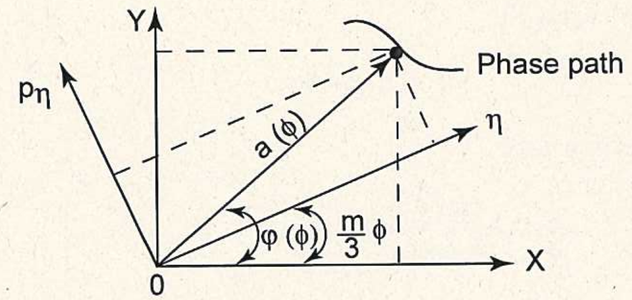

Figure 23: Coordinate systems for the (η, p_η) and (X, Y) phase planes.

Fixed points in phase plane are points for which

$$\frac{dX}{d\phi} = \frac{dY}{d\phi} = 0 \; , \tag{5.31}$$

that is $X(\phi) = \hat{X}$ and $Y(\phi) = \hat{Y}$ where $\hat{X}$ and $\hat{Y}$ are constants. Fixed points are equilibrium points (stable or unstable). It follows from this that

$$\frac{da}{d\phi} = \frac{d\varphi}{d\phi} = 0 \tag{5.32}$$

because

$$\dot{a} = \dot{X} \cos \varphi - \dot{Y} \sin \varphi$$

$$\dot{\varphi} = -\frac{1}{a}(\dot{X} \sin \varphi + \dot{Y} \cos \varphi) \; .$$

Thus, according to (5.22) and (5.23), fixed points are solutions of the equations

$$\sin 3\varphi = 0$$

$$\frac{3}{8m}\hat{p}_3(m)a \cos 3\varphi = \delta Q$$

which gives

$$3\hat{\varphi} = 2k\pi \quad \text{and} \quad 3\hat{\varphi} = (2k+1)\pi \quad k = 0,1,2$$

or

$$\hat{\varphi} = 0, \frac{2\pi}{3}, \frac{4\pi}{3} \quad \text{and} \quad \hat{\varphi} = \frac{\pi}{3}, \pi, \frac{5\pi}{3}\,. \tag{5.33}$$

The first three fixed points give $\cos 3\hat{\varphi} = 1$, the other three give $\cos 3\hat{\varphi} = -1$. It follows that

$$\hat{a} = \pm\frac{8m}{3\hat{p}_3(m)}\delta Q \tag{5.34}$$

where the sign + is chosen when $\delta Q > 0$ and vice versa, since the amplitude $\hat{a}$ cannot be negative [assuming $\hat{p}_3(m)$ positive]. One more fixed point is given considering the trivial case

$$\hat{a} = 0 \text{ (i.e. } \hat{X} = \hat{Y} = 0)\,.$$

Inserting (5.34) into the general solution (5.25) of the averaged equations for a and φ yields

$$a^3\left(\cos 3\varphi \mp \frac{3}{2}\frac{\hat{a}}{a}\right) = A\,. \tag{5.35}$$

Expressed with the variables X and Y this equation becomes, using (5.29) and (5.30),

$$X^3 - 3XY^2 \mp \frac{3}{2}\hat{a}(X^2 + Y^2) = A \tag{5.36}$$

or, solving the quadratic equation,

$$Y = \mp\sqrt{\frac{2X^3 \mp 3\hat{a}X^2 - 2A}{6X \mp 3\hat{a}}} \tag{5.37}$$

with

$$a^3 \cos 3\varphi = a^3(4\cos^3\varphi - 3\cos\psi) = 4X^3 - 3a^2X$$

$$a^2 = X^2 + Y^2\,.$$

The constant A depends on the initial conditions $X(0)$ and $Y(0)$. In particular, the conditions (5.33) and (5.34) for fixed points give

$$\hat{X} = \hat{a},\ -\frac{\hat{a}}{2},\ -\frac{\hat{a}}{2} \quad \text{and} \quad \hat{X} = \frac{\hat{a}}{2},\ -\hat{a},\ \frac{\hat{a}}{2}$$

$$\hat{Y} = 0,\ \frac{\sqrt{3}}{2}\hat{a},\ -\frac{\sqrt{3}}{2} \quad \text{and} \quad \hat{Y} = \frac{\sqrt{3}}{2}\hat{a},\ 0,\ -\frac{\sqrt{3}}{2}\hat{a}\,.$$

Replacing X and Y by $\hat{X}$ and $\hat{Y}$ in (5.36), or better a and φ by $\hat{a}$ and $\hat{\varphi}$ in (5.35), one finds

$$A = \mp \frac{\hat{a}^3}{2}$$

where the minus sign is chosen when $\delta Q > 0$ and vice-versa.

From now on, we shall assume that $\delta Q > 0$. Hence, for the above fixed point conditions, equation (5.36) reduces to

$$X^3 - 3XY^2 - \frac{3}{2}\hat{a}(X^2 + Y^2) = -\frac{\hat{a}^3}{2}$$

which may be factorized to give

$$\left(Y - \frac{X - \hat{a}}{\sqrt{3}}\right)\left(Y + \frac{X - \hat{a}}{\sqrt{3}}\right)\left(X + \frac{\hat{a}}{2}\right) = 0\,. \qquad (5.38)$$

This yields a family of three straight lines, called separatrices. These three separatrices define a triangular area in the phase plane. The three fixed points $(\hat{a},\,0)$, $\left(-\frac{\hat{a}}{2},\,\frac{\sqrt{3}}{2}\hat{a}\right)$, and $\left(-\frac{\hat{a}}{2},\,-\frac{\sqrt{3}}{2}\hat{a}\right)$ are at the intersections of the separatrices.

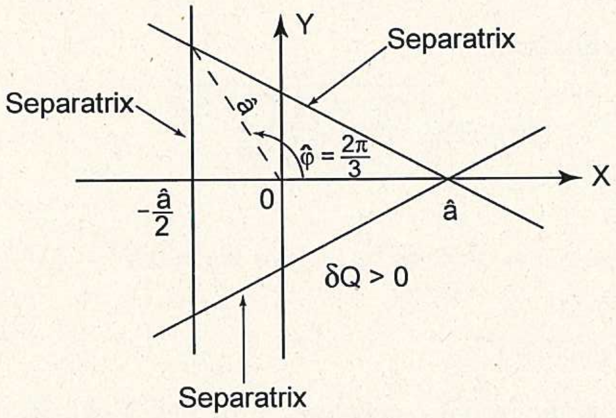

Figure 24: Phase plane separatrices near the third-integer resonance.

Plotting equation (5.37) for different values of the constant A gives the phase plane portrait of the equation of motion (5.20).

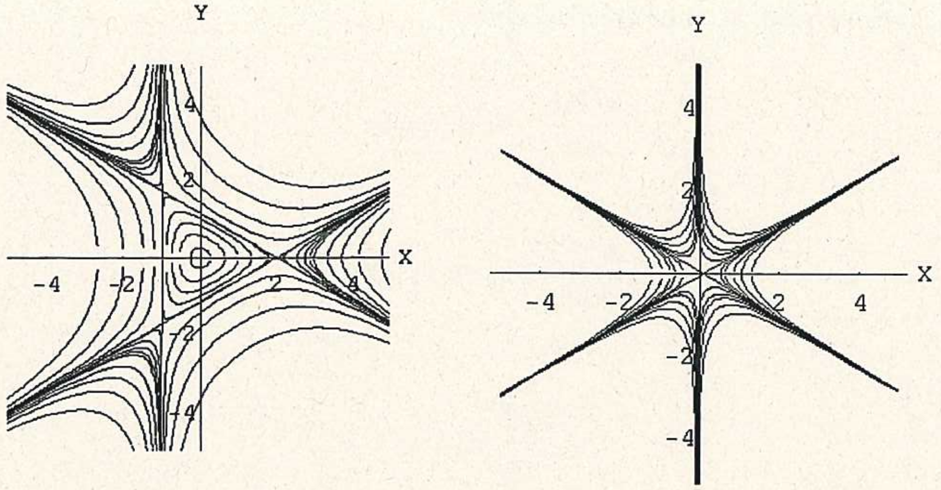

Figure 25: Phase portraits near the third-integer resonance for $\hat{a} = 2$ (left) and $\hat{a} = 1/10$ (right).

The phase path description in phase plane (X, Y) is equivalent to the "stroboscopic" representation of paths in phase plane (η, p_η) in which a sequence of points are plotted at interval $2k\pi (k = 0, 1, 2...)$ corresponding to every machine turn at fixed azimuthal location. The separatrices define a boundary between stable motion (bounded oscillations) and unstable motion (expanding oscillations). The fixed points are equilibrium points (stable or unstable) of period 6π corresponding to three machine turns.

If we had considered a negative tune variation $(\delta Q < 0)$ rather than a positive one $(\delta Q > 0)$, the three fixed points would have been located at $(-\hat{a}, 0)$, $\left(\frac{\hat{a}}{2}, \frac{\sqrt{3}}{2}\hat{a}\right)$, $\left(\frac{\hat{a}}{2}, -\frac{\sqrt{3}}{2}\hat{a}\right)$, yielding a reversed phase portrait with respect to the Y-axis.

When the tune shift δQ is reduced to zero, the phase plane triangle shrinks to a point. The stable area disappears and the separatrix crosses the origin.

The stability about the fixed points may be examined by expanding a as $\hat{a} + \Delta a$ and φ as $\hat{\varphi} + \Delta\varphi$ in (5.22) and (5.23). Keeping only the linear terms in Δa and $\Delta\varphi$, we obtain

$$\begin{aligned} \frac{d}{d\phi}(\hat{a} + \Delta a) &= -\frac{3}{8m}\hat{p}_3(m)\hat{a}^2 \sin 3\hat{\varphi} - \frac{6}{8m}\hat{p}_3(m)\hat{a} \sin 3\hat{\varphi}\Delta a - \\ & - \frac{9}{8m}\hat{p}_3(m)\hat{a}^2 \cos 3\hat{\varphi}\Delta\varphi = -\frac{9}{8m}\hat{p}_3(m)\hat{a}^2\Delta\varphi \end{aligned}$$

since $\cos 3\hat{\varphi} = 1$. Therefore

$$\frac{d\Delta a}{d\phi} = -\frac{9}{8m}\hat{p}_3(m)\hat{a}^2\Delta\varphi \, .$$

Similarly we compute to the first order

$$\frac{d\Delta\varphi}{d\phi} = -\frac{3}{8m}\hat{p}_3(m)\Delta a \, .$$

These two equations may be combined to give

$$\frac{d^2\Delta a}{d\phi^2} - k^2\Delta a = 0 \tag{5.39}$$

$$\frac{d^2\Delta\varphi}{d\phi^2} - k^2\Delta\varphi = 0 \tag{5.40}$$

with

$$k = \frac{3\sqrt{3}}{8m}\hat{p}_3(m)\hat{a} \ .$$

Solutions for Δa and $\Delta\varphi$ near fixed points are of the form

$$\begin{aligned} \Delta a &= A_1 e^{k\phi} + A_2 e^{-k\phi} \\ \Delta\varphi &= B_1 e^{k\phi} + B_2 e^{-k\phi} \ , \end{aligned}$$

where A_1, A_2, B_1, B_2 are constant, depending upon the initial conditions. These solutions show that the motion may either converge to or diverge from the fixed point, depending upon the initial conditions. This kind of fixed point is said to be hyperbolic. Hyperbolic fixed points are unstable because any point near to them will eventually move away from them.

For the trivial fixed point $\hat{a}(\phi) = 0$ one gets

$$\frac{d\Delta a}{d\phi} = 0 \quad \text{and} \quad \frac{d\Delta\varphi}{d\phi} = \delta Q \ ,$$

which gives

$$\Delta a = C_1 \quad \text{and} \quad \Delta\varphi = \delta Q\,\phi + C_2 \ ,$$

where C_1 and C_2 are constants. The motion describes circles around the origin, and the fixed point $a = 0$ is said to be elliptic. Elliptic fixed points are stable because any point near to them remains always in the vicinity of them.

The third-integer resonance may be used for resonant extraction. The principle consists in approaching the third integer resonance by varying the strengths of quadrupoles. Hence, as the triangle area shrinks, the particles in a beam are moved out along the three arms of the separatrices jumping from one arm to the next each turn. When the displacement of a particle reaches the values X_s, it jumps the extraction septum. The septum width Δ_s is equal to the growth in the X-direction in three turns.

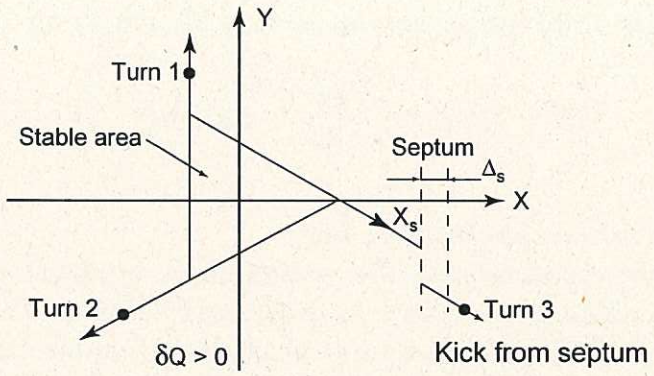

Figure 26: Resonant extraction scheme in phase plane.

Finally, we shall estimate the third-integer resonance width. Suppose that the tune Q is initially set far from the resonance $Q_r = m/3$ so that the beam emittance ϵ may be represented by a phase ellipse entirely located inside the $(\eta,\, p_\eta)$ phase plane triangle defined by the tune difference $\delta Q = Q - Q_r$. The phase ellipse in the $(u,\, u')$ phase plane, given by the Courant–Snyder invariant

$$\gamma(s)u^2 + 2\alpha(s)uu' + \beta(s)u'^2 = \epsilon \tag{5.41}$$

transforms to a phase circle in the phase planes $(\eta,\, p_\eta)$ and $(X,\, Y)$

$$\eta^2 + p_\eta^2 = X^2 + Y^2 = \epsilon \tag{5.42}$$

by (5.27), (5.28), and (4.17), and because

$$u' = \frac{1}{\beta^{1/2}}(p_\eta - \alpha\eta)\,.$$

Let Q slowly approache Q_r, then the triangle area shrinks and the phase circle gradually distorts into a triangular shape, while its area $\pi\epsilon$ is kept conserved. We define the resonance width ΔQ as being twice the tune difference δQ which yields the triangle area equal to $\pi\epsilon$.

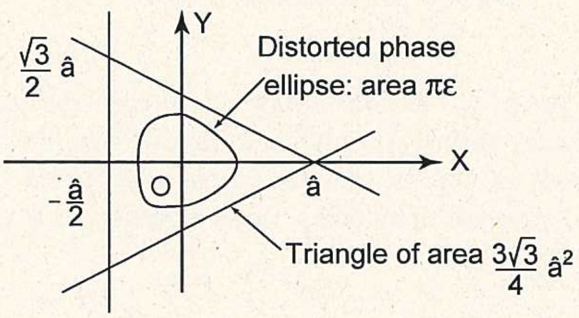

Figure 27: Distorted beam phase ellipse within the stable phase plane region defined by the separatrices near the third-integer resonance.

By (5.34), the width of the third-integer resonance may be written as

$$\Delta Q \equiv 2|\delta Q| = \frac{(\sqrt{3}\pi\epsilon)^{1/2}}{2m}\hat{p}_3(m)\,. \tag{5.43}$$

5.3 Numerical experiment: sextupolar kick

We shall study more quantitatively the motion of an on-momentum particle in a circular accelerator subject to a periodic sextupolar perturbation. The sextupole field is assumed to be a "point-like" perturbation at location s_0 (i.e. the particle position is assumed not to vary as the particle traverses the field). Thus, at each turn the local magnetic field gives a "kick" to the particle, deflecting it from its unperturbed trajectory. This method of description is referred to as a "kick" model.

Written in the normalized coordinate (η, ϕ) the one-dimensional equation of motion is, according to (4.31),

$$\ddot{\eta} + Q^2\eta = A\sum_{m=0}^{\infty}\eta^2(\phi)\delta(\phi - 2\pi m) \tag{5.44}$$

where Q is the machine tune, $\delta(\phi)$ is the Dirac distribution, and A is the strength of the impulse. The right-hand side of (5.44) corresponds to the periodic perturbation term $p_3(\phi)$ in (4.31), with period 2π. The kick strength A may be written as

$$A = Q^2\beta_0^{5/2}\frac{1}{2}S_0\ell \tag{5.45}$$

in which β_0 is the betatron function at s_0 (i.e. at $\phi = 0$), the factor $S_0\ell$ is the integrated sextupole strength (in thin-lens approximation the sextupole length $\ell \to 0$ while keeping constant the product of the sextupole strength S_0 by ℓ). Between kicks the right-hand side of (5.44) is zero, and the unperturbed solution is

$$\eta(\phi) = a_n\cos(Q\phi + \varphi_n) \quad \text{for} \quad 2\pi(n-1) < \phi \le 2\pi n \tag{5.46}$$

where a_n and φ_n are integration constants, and

$$\dot{\eta}(\phi) = -a_nQ\sin(Q\phi + \varphi_n) \tag{5.47}$$

where $n = 1, 2, \ldots$.

For each value of n, at $\phi = 2\pi n$, the angle $\dot{\eta}$ changes discontinuously while the position η is not affected since the perturbation is point-like. The increase of $\dot{\eta}$ at the passage of the sextupolar kick on the nth machine turn is obtained by integrating (5.44) from $2\pi n - \epsilon$ to $2\pi n + \epsilon$, ϵ being an arbitrarily small number, and then taking the limit $\epsilon \to 0$

$$\begin{aligned} \Delta\dot{\eta} &= \int_{2\pi n-\epsilon}^{2\pi n+\epsilon} \ddot{\eta}(\phi)d\phi = -\int_{2\pi n-\epsilon}^{2\pi n+\epsilon} \left[Q^2\eta(\phi) - A\sum_{m=0}^{\infty} \eta^2(\phi)\delta(\phi-2\pi m)\right] d\phi \\ &= A\eta^2(2\pi n) \equiv A\eta_n^2 \ , \end{aligned}$$

where η_n denotes the limit of $\eta(\phi)$ for $\phi \to 2\pi n$. Defining the new phase plane variable $p_\eta(\phi)$ as

$$p_\eta(\phi) = \frac{1}{Q}\dot{\eta}(\phi) \ ,$$

the discontinuous change in "angle" $p_\eta(\phi)$reads

$$\Delta p_n \equiv \Delta p_\eta(2\pi n) = \frac{A}{Q}\eta_n^2 \ . \tag{5.48}$$

According to (5.46) we have for $n+1$

$$\begin{aligned} \eta(\phi) &= a_{n+1}\cos\left(Q\phi + \varphi_{n+1}\right) \quad \text{for} \quad 2\pi n < \phi \le 2\pi(n+1) \\ p_\eta(\phi) &= -a_{n+1}\sin\left(Q\phi + \varphi_{n+1}\right) \ , \end{aligned} \tag{5.49}$$

where a_{n+1} and φ_{n+1} are other integration constants. In particular

$$\begin{aligned} \eta_n &= a_n \cos\left(2\pi n Q + \varphi_n\right) \\ p_n &= -a_n \sin\left(2\pi n Q + \varphi_n\right) \ , \end{aligned} \tag{5.50}$$

where p_n denotes the left-limit of $p_\eta(\phi)$ for $\phi \to 2\pi n$ from the left [also written as $p_\eta(2\pi n-)$].

Similarly, the right-limits of $\eta(\phi)$ and $p_\eta(\phi)$ for $\phi \to 2\pi n$ from the right, written $\eta(2\pi n+)$ and $p_\eta(2\pi n+)$, read

$$\begin{aligned} \eta(2\pi n+) &= a_{n+1}\cos\left(2\pi n Q + \varphi_{n+1}\right) \\ p_\eta(2\pi n+) &= -a_{n+1}\sin\left(2\pi n Q + \varphi_{n+1}\right) \ . \end{aligned} \tag{5.51}$$

Hence, since $\eta(\phi)$ does not change between $2\pi n - \epsilon$ and $2\pi n + \epsilon$ (with $\epsilon \to 0$) while $p_\eta(\phi)$ changes according to (5.48), we must have

$$\begin{aligned} \eta(2\pi n+) &= \eta_n \\ p_\eta(2\pi n+) &= p_n + \Delta p_n \end{aligned} \tag{5.52}$$

that is

$$\begin{aligned} a_{n+1}\cos(2\pi nQ+\varphi_{n+1}) &= a_n\cos(2\pi nQ+\varphi_n) \\ -a_{n+1}\sin(2\pi nQ+\varphi_{n+1}) &= -a_n\sin(2\pi nQ+\varphi_n)+\frac{A}{Q}\eta_n^2\,. \end{aligned} \tag{5.53}$$

Multiplying the first equation (5.53) by $\cos 2\pi nQ$ and the second by $\sin 2\pi nQ$, and then summing the results, we find, using the formulae

$$\begin{aligned} \cos(2\pi nQ+\varphi_n) &= \cos 2\pi nQ\cos\varphi_n-\sin 2\pi nQ\sin\varphi_n \\ \sin(2\pi nQ+\varphi_n) &= \sin 2\pi nQ\cos\varphi_n+\cos 2\pi nQ\sin\varphi_n \end{aligned} \tag{5.54}$$

the relationship between the coefficients in the nth interval and those in the $(n+1)$th interval

$$a_{n+1}\cos\varphi_{n+1}=a_n\cos\varphi_n-\frac{A}{Q}\eta_n^2\sin 2\pi nQ\,. \tag{5.55}$$

Similarly, we can derive

$$a_{n+1}\sin\varphi_{n+1}=a_n\sin\varphi_n-\frac{A}{Q}\eta_n^2\cos 2\pi nQ\,. \tag{5.56}$$

Now, by equations (5.49), we can write for $\phi\to 2\pi(n+1)$

$$\begin{aligned} \eta_{n+1} &= a_{n+1}\cos[2\pi(n+1)Q+\varphi_{n+1}] \\ p_{n+1} &= -a_{n+1}\sin[2\pi(n+1)Q+\varphi_{n+1}]\,. \end{aligned} \tag{5.57}$$

Expanding (5.57) using (5.54), and inserting (5.55) and (5.56) into the result yields after some sorting

$$\begin{aligned} \eta_{n+1} &= \eta_n\cos 2\pi Q+\left(p_n+\frac{A}{Q}\eta_n^2\right)\sin 2\pi Q \\ p_{n+1} &= -\eta_n\sin 2\pi Q+\left(p_n+\frac{A}{Q}\eta_n^2\right)\cos 2\pi Q\,. \end{aligned} \tag{5.58}$$

These equations give the particle position and "angle" just before the phase $2\pi(n+1)$ (i.e. just before the passage of the sextupolar field on turn $n+1$) in terms of position and angle just before the phase $2\pi n$. Equations (5.58) form a quadratic mapping in phase plane (η, p_η), called Hénon mapping. Mappings with different initial conditions (η_0, p_0) have been computed and plotted in Figs. 28–29 for various values of Q (with $A/Q=-1$). This yields a stroboscopic representation of phase-space trajectories on every machine turn at s_0 (i.e. at the phase values of $\phi=0, 2\pi, 4\pi, ...$).

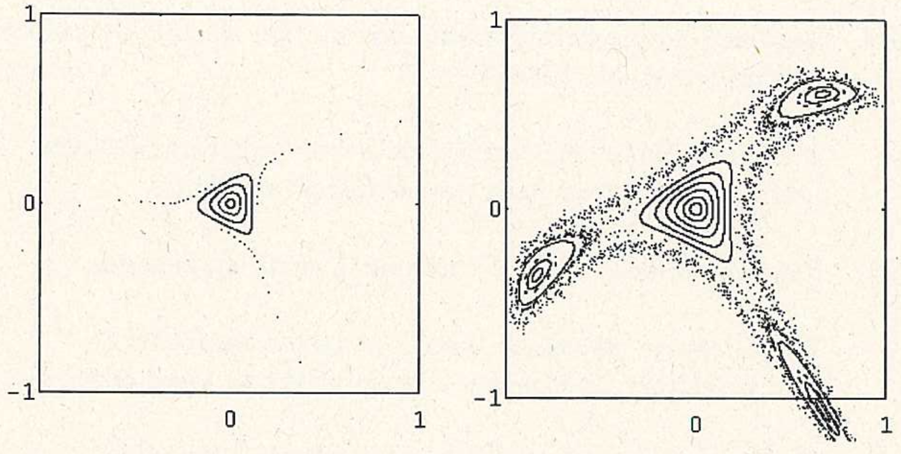

Figure 28: Phase plane plots caused by a single thin sextupole for $Q = 0.324$ (close to 1/3) and for $Q = 0.320$ (close to 1/3).

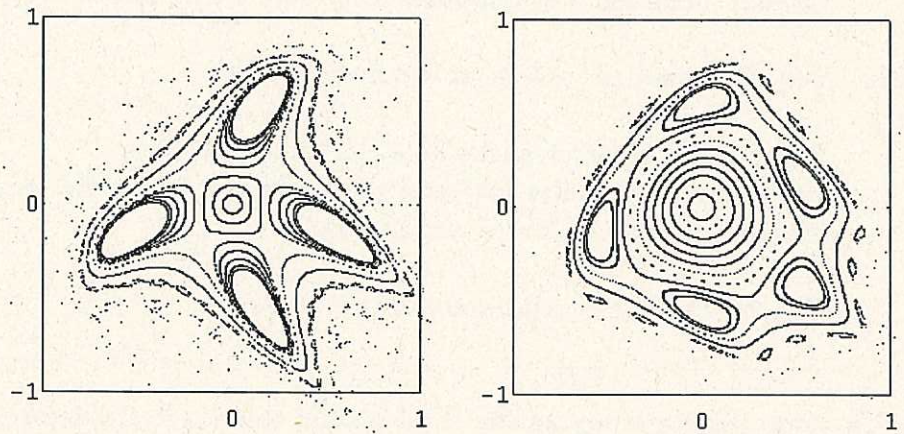

Figure 29: Phase plane plots caused by a single thin sextupole for $Q = 0.252$ (close to 1/4) and for $Q = 0.211$ (close to 1/5).

Tune	Characteristics of the motion
$Q = 0.324$ close to $\frac{1}{3}$	Regular (i.e. predictable) bounded motion at small 'amplitudes' (within the stable triangle).
	Divergent (unstable) regular motion at larger amplitudes (outward trajectory near the outermost contour).
$Q = 0.320$ close to $\frac{1}{3}$	Regular bounded (stable) motion at small amplitudes.
	Three regular resonance islands at larger amplitudes (the trajectory breaks into a chain of three closed curves).
	Eight secondary regular resonance islands are visible within each of the three primary islands.
	Chaotic (i.e. unpredictable) bounded motion at larger amplitudes.
$Q = 0.252$ close to $\frac{1}{4}$	Regular bounded motion at small amplitude.
	Four regular resonance islands at larger amplitudes, surrounded by regular trajectories.
	Chaotic bounded and unbounded motion at still larger amplitudes.
$Q = 0.211$ close to $\frac{1}{5}$	Regular bounded motion at small amplitudes.
	Five regular resonance islands at larger amplitudes, surrounded by regular trajectories. A chain of 16 smaller regular islands is visible at larger amplitudes.
	Chaotic motion at still larger amplitudes.

The analytic approach for studying the third-integer resonance trajectories by perturbation techniques (e.g. the averaging method) is useful when the fractional part of the machine tune is close to 1/3. Indeed, the triangular phase-plane boundary between stable and unstable motions as well as the separatrices may be determined. However, unlike the kick model, the large amplitude description of the motion is incorrect: islands are not revealed and divergent trajectories, which might return, do not.

In general, the equations of motion in the presence of nonlinear fields are untractable for any but the simplest situations. An alternative approach to study phase plane trajectories and determine the regions that are stable consists to perform numerical particle "tracking" experiments using computer programs. Tracking consists to simulate particle motion in circular accelerators in the presence of nonlinear fields. Test particles with initial phase plane coordinates at a fixed azimuthal location along the accelerator are tracked for many turns through the lattice using various numerical techniques. The particle coordinates at the end of one turn may be viewed as a nonlinear mapping of the initial conditions, that specifies the motion for one turn. The resulting stroboscopic phase plane plot of circulating particles for many turns (obtained by iterative mapping) thus reveals the largest surviving oscillation amplitudes (that enclose a central stable phase plane region), which then define the dynamic aperture. A simple technique used in tracking programs consists to describe the mapping by means of the kick model exposed above: any nonlinear magnet is treated in the "point-like" approximation, the motion in all other elements of the lattice is assumed to be linear. Alternative methods of description of the mapping are available (e.g. the "higher-order" matrix method based on Taylor series expansion, the canonical integration method which numerically integrate the equations of motion, and the Lie algebraic formalism).

6 CHROMATICITY

6.1 Chromaticity effect in a closed lattice

The focal properties of lattice elements depend upon the momentum deviation, since the equations of motion of an off-momentum particle are, from (4.9) and (4.10), without skew magnets and sextupoles, and in a zero-curvature region

$$\begin{aligned} x'' + K_0(1-\delta)x &= 0 \\ y'' - K_0(1-\delta)y &= 0 \end{aligned} \tag{6.1}$$

where δ is the relative momentum deviation from the design momentum p_0: $\delta = \Delta p/p_0$.

This momentum dependence of the focusing, which, in turn, causes tune changes, is called chromatic effect. The variation of the tune Q with the momentum is called the chromaticity and is defined as (Q stands for the horizontal or the vertical tune)

$$Q' = \frac{\Delta Q}{\Delta p/p_0}\,. \tag{6.2}$$

The relative chromaticity is defined as

$$\xi = \frac{\Delta Q/Q}{\Delta p/p_0}\,. \tag{6.3}$$

The control of the chromaticity is important for two reasons:

1) To avoid shifting the beam on resonances due to tune changes induced by chromatic effects.
2) To prevent some transverse instabilities (head–tail instabilities).

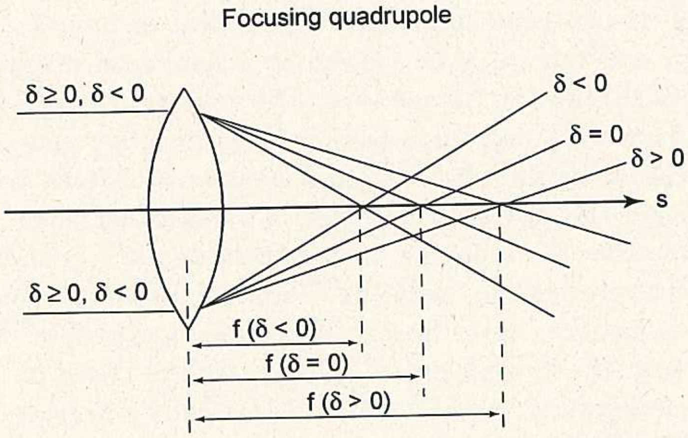

Figure 30: Schematic representation of chromatic effect in a dispersion-free region.

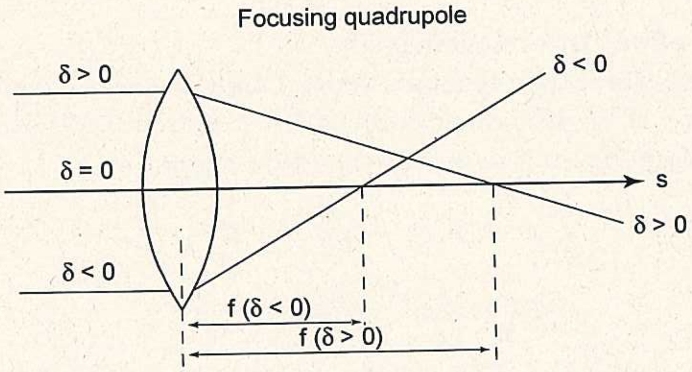

Figure 31: Schematic representation of chromatic effect in a region with dispersion.

Particles with different momenta are spread where there is dispersion. Higher-momentum particles are focused less than particles with the design momentum p_0, lower-momentum particles are focused more. Indeed, if $f(\delta)$ and $K(\delta)$ are the focal length and normalized gradient, at momentum $p = p_0(1+\delta)$, of a quadrupole of length ℓ, with

$$K(\delta) = \frac{e}{p}\frac{\partial B_y}{\partial x} \qquad K_0 = \frac{e}{p_0}\frac{\partial B_y}{\partial x}$$

we find

$$\frac{1}{f(\delta)} = \frac{K_0 \ell}{1+\delta}\,. \tag{6.4}$$

Therefore, once the particles are spread by momentum in a region with dispersion, we can apply focusing corrections depending on the momentum using a sextupole magnet.

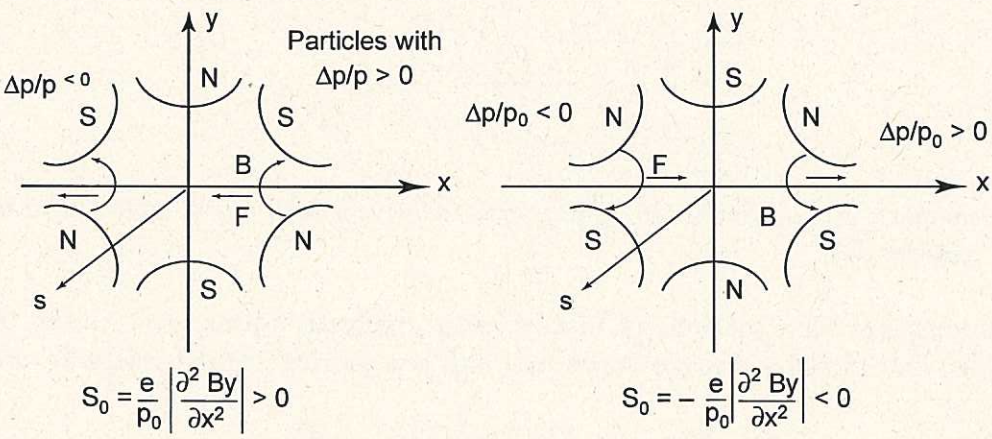

Figure 32: Sextupole fields and forces in a region with dispersion (positive particles approach the reader).

Here, the sextupole is focusing for the higher-momentum particles and defocusing for the lower-momentum particles. Hence, it can be used to correct the chromatic focusing errors in a region with non-zero dispersion.

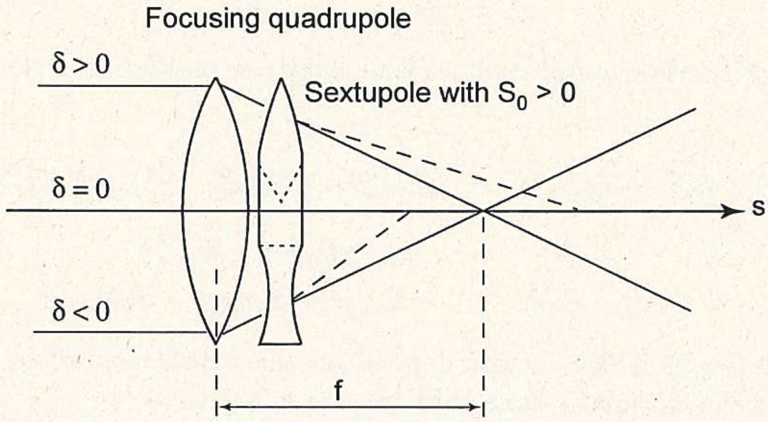

Figure 33: Schematic representation of chromaticity correction with a sextupole added beside a focusing quadrupole.

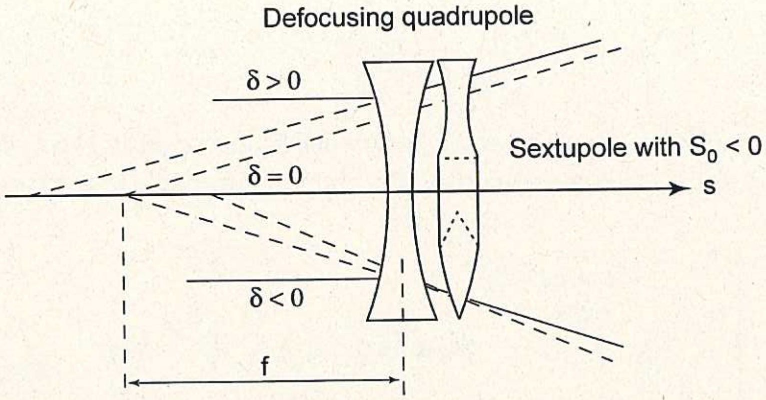

Figure 34: Schematic representation of chromaticity correction with a sextupole added beside a defocusing quadrupole.

Let us consider the equations of motion with quadratic terms in δ, x, and y. We get, using (4.9) and (4.10) with zero curvature, and considering regular magnets only

$$\begin{aligned} x'' + K_0 x &= K_0 x\delta - \frac{1}{2} S_0 (x^2 - y^2) \\ y'' - K_0 y &= -K_0 y \delta + S_0 xy \, . \end{aligned} \tag{6.5}$$

Substituting the total coordinates for the off-momentum particle, assuming no dispersion in the vertical plane,

$$\begin{aligned} x &= x_\beta + D_x \delta \\ y &= y_\beta \end{aligned} \tag{6.6}$$

where x_β and y_β are the horizontal and vertical betatron oscillations, the equations of motion (6.5) become

$$\begin{aligned} x''_\beta + K_0 x_\beta + (D''_x + K_0 D_x)\delta &= K_0 x_\beta \delta - \frac{1}{2} S_0 (x^2_\beta - y^2_\beta) + K_0 D_x \delta^2 - \\ &\quad - S_0 x_\beta D_x \delta - \frac{1}{2} S_0 D_x^2 \delta^2 \\ y''_\beta - K_0 y_\beta &= -K_0 y_\beta \delta + S_0 x_\beta y_\beta + S_0 D_x y_\beta \delta \, . \end{aligned}$$

Discarding the terms which do not depend on the betatron motion, because they do not contribute to the chromatic tune shift, we are left with

$$\begin{aligned} x''_\beta + K_0 x_\beta &= K_0 x_\beta \delta - S_0 D_x x_\beta \delta - \frac{1}{2} S_0 (x^2_\beta - y^2_\beta) \\ y''_\beta - K_0 y_\beta &= -K_0 y_\beta \delta + S_0 D_x y_\beta \delta + S_0 x_\beta y_\beta \, . \end{aligned} \tag{6.7}$$

Ignoring the non-chromatic terms of second order (i.e. terms in x^2_β, y^2_β and $x_\beta y_\beta$) yields

$$x''_\beta + K_0 x_\beta = (K_0 - S_0 D_x) x_\beta \delta$$

$$y''_\beta - K_0 y_\beta = -(K_0 - S_0 D_x) y_\beta \delta \, . \tag{6.8}$$

Introducing the normalized coordinates (4.27) and (4.28)

$$\eta = \frac{x}{\sqrt{\beta_x}} \qquad \xi = \frac{y}{\sqrt{\beta_y}}$$

$$\phi = \frac{1}{Q_{x,y}} \int_{s_0}^{s} \frac{dt}{\beta_{x,y}(t)}$$

transforms the equations (6.8) into

$$\begin{aligned} \ddot{\eta} + Q_x^2 \eta &= Q_x^2 \beta_x^2 [K_0(\phi) - S_0(\phi) D_x] \delta \eta \\ \ddot{\xi} + Q_y^2 \xi &= -Q_y^2 \beta_y^2 [K_0(\phi) - S_0(\phi) D_x] \delta \xi \end{aligned} \tag{6.9}$$

in which K_0, S_0, $\beta_{x,y}$ and D_x are periodic functions of ϕ with period 2π. These equations may be written in the form:

$$\begin{aligned} \ddot{\eta} + Q_x^2 \eta &= Q_x p_{x\delta}(\phi) \eta \\ \ddot{\xi} + Q_y^2 \xi &= -Q_y p_{y\delta}(\phi) \xi \end{aligned} \tag{6.10}$$

with

$$p_{x,y\delta}(\phi) = \beta_{x,y}^2 Q_{x,y} [K_0(\phi) - S_0(\phi) D_x] \delta \, . \tag{6.11}$$

Let us solve the perturbed equation of motion for η. Expanding $p_{x\delta}(\phi)$ in Fourier series yields

$$p_{x\delta}(\phi) = \sum_{m=-\infty}^{\infty} \hat{p}_{x\delta}(m) e^{im\phi} \tag{6.12}$$

in which the $\hat{p}_{x\delta}(m)$ are the Fourier coefficients given by

$$\hat{p}_{x\delta}(m) = \frac{1}{2\pi} \int_0^{2\pi} p_{x\delta}(\phi) e^{-im\phi} d\phi \, . \tag{6.13}$$

From this we obtain

$$\ddot{\eta} + Q_x^2 \left[1 - \frac{1}{Q_x} \sum_{m=-\infty}^{\infty} \hat{p}_{x\delta}(m) e^{im\phi} \right] \eta = 0 \, . \tag{6.14}$$

which is a differential equation with periodic coefficients of the Hill type.

Furthermore, since

$$\sum_{m=-\infty}^{\infty} \hat{p}_{x\delta}(m) e^{im\phi} = \hat{p}_{x\delta}(0) + \sum_{m \neq 0} \hat{p}_{x\delta}(m) e^{im\phi}$$

equation (6.14) may be written as

$$\ddot{\eta} + \tilde{Q}_x^2(\phi)\eta = 0 \tag{6.15}$$

where $\tilde{Q}_x(\phi)$ has a static and an oscillatory part

$$\tilde{Q}_x^2(\phi) = Q_x^2\left[1 - \frac{\hat{p}_{x\delta}(0)}{Q_x}\right] - Q_x \sum_{m\neq 0} \hat{p}_{x\delta}(m)e^{im\phi} . \tag{6.16}$$

The oscillatory part of the tune $\tilde{Q}_x$ averages to zero over one period 2π. Replacing $\tilde{Q}_x(\phi)$ by its average part $\langle\tilde{Q}_x\rangle$ (see the averaging method) gives the static tune shift due to chromatic effect. Then equation (6.15) transforms into

$$\ddot{\eta} + [Q_x^2 - Q_x\hat{p}_{x\delta}(0)]\eta = 0 , \tag{6.17}$$

which may be rewritten as

$$\ddot{\eta} + (Q_x + \Delta Q_x)^2\eta = 0 , \tag{6.18}$$

where ΔQ_x is the chromatic tune shift. It follows that for a small tune shift

$$Q_x^2 - Q_x\hat{p}_{x\delta}(0) = (Q_x + \Delta Q_x)^2 \approx Q_x^2 + 2Q_x\Delta Q_x .$$

Hence, from (6.11) and (6.13) we obtain the tune shift as

$$\begin{aligned} \Delta Q_x &= -\frac{1}{2}\hat{p}_{x\delta}(0) = \\ &= -\frac{1}{4\pi}\int_0^{2\pi} Q_x\beta_x^2(\phi)[K_0(\phi) - S_0(\phi)D_x(\phi)]\delta\, d\phi . \end{aligned} \tag{6.19}$$

Returning to the original variables x and s we find

$$\Delta Q_x = -\frac{\delta}{4\pi}\int_{s_0}^{s_0+C} \beta_x(s)[K_0(s) - S_0(s)D_x(s)]ds \tag{6.20}$$

where C is the machine circumference. Similarly, we compute

$$\Delta Q_y = \frac{\delta}{4\pi}\int_{s_0}^{s_0+C} \beta_y(s)[K_0(s) - S_0(s)D_x(s)]ds . \tag{6.21}$$

Introducing these equations into the definition (6.3) of the chromaticity yields, replacing s_0 by s

$$\begin{aligned} \xi_x &= -\frac{1}{4\pi Q_x}\int_s^{s+C} \beta_x(t)[K_0(t) - S_0(t)D_x(t)]dt \\ \xi_y &= \frac{1}{4\pi Q_y}\int_s^{s+C} \beta_y(t)[K_0(t) - S_0(t)D_x(t)]dt . \end{aligned} \tag{6.22}$$

The contribution to chromaticity arising from pure quadrupole elements (and also pure dipoles) is called the natural chromaticity. Thus, the natural chromaticity of a lattice is given by

$$\begin{aligned}\xi_{x0} &= -\frac{1}{4\pi Q_x}\int_s^{s+C} \beta_x(t)K_0(t)dt \\ \xi_{y0} &= +\frac{1}{4\pi Q_y}\int_s^{s+C} \beta_y(t)K_0(t)dt \,. \end{aligned} \tag{6.23}$$

6.2 The natural chromaticity of a FODO cell

As an example we compute the natural chromaticity of a synchrotron formed only of thin-lens FODO cells of length 2L.

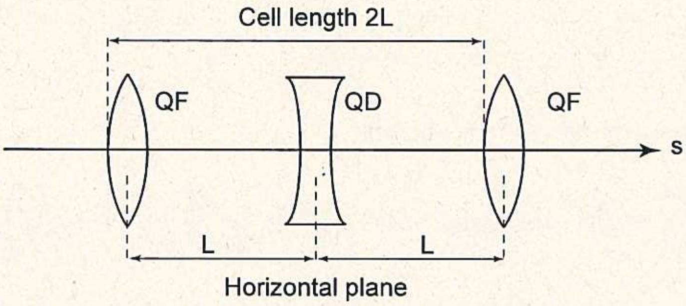

Figure 35: FODO cell (QF: focusing quadrupole, QD: defocusing quadrupole.)

The transfer matrix of a thin-lens FODO cell has already been derived to be

$$M_{\text{FODO}} = \begin{pmatrix} 1 - \frac{L}{f} - \frac{L^2}{f^2} & 2L + \frac{L^2}{f} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} \end{pmatrix} \tag{6.24}$$

where $f^{-1} = K_0\ell > 0$, K_0 and ℓ are the quadrupole strength and length. The FODO transfer matrix may be identified with the general transfer matrix through any section (from s_1 to s_2)

$$M(s_2/s_1) = \begin{bmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos\Delta\mu + \alpha_1 \sin\Delta\mu) & \sqrt{\beta_1\beta_2}\sin\Delta\mu \\ \frac{\alpha_1-\alpha_2}{\sqrt{\beta_1\beta_2}}\cos\Delta\mu - \frac{1+\alpha_1\alpha_2}{\sqrt{\beta_1\beta_2}}\sin\Delta\mu & \sqrt{\frac{\beta_1}{\beta_2}}(\cos\Delta\mu - \alpha\sin\Delta\mu) \end{bmatrix} \tag{6.25}$$

where $\Delta\mu = \mu(s_2) - \mu(s_1)$ is the phase advance per cell. However, the lattice being composed only of FODO cells, periodic conditions yield $\beta_1 = \beta_2 = \beta_x^{QF}$ and $\alpha_1 = \alpha_2 = \alpha_x^{QF}$. Hence, over one FODO cell the latter expression reduces to

$$M(s_2/s_1) = M_{\text{FODO}} = \begin{pmatrix} \cos\Delta\mu_x + \alpha_x^{QF}\sin\Delta\mu_x & \beta_x^{QF}\sin\Delta\mu_x \\ -\gamma_x^{QF}\sin\Delta\mu_x & \cos\Delta\mu_x - \alpha_x^{QF}\sin\Delta\mu_x \end{pmatrix} \tag{6.26}$$

where

$$\gamma_x^{QF} = \frac{1+\alpha_x^{QF^2}}{\beta_x^{QF}}\,.$$

Identifying the matrix components, we obtain

$$2L + \frac{L^2}{f} = \beta_x^{QF}\sin\Delta\mu_x \quad \text{or} \quad \beta_x^{QF} = \frac{2L}{\sin\Delta\mu_x}\left(1+\frac{L}{2f}\right)$$

$$\text{Tr}\,(M_{\text{FODO}}) = 2 - \frac{L^2}{f^2} = 2\cos\Delta\mu_x \quad \text{or} \quad \frac{L^2}{4f^2} = \frac{1}{2}(1-\cos\Delta\mu_x) = \sin^2\frac{\Delta\mu_x}{2}\,.$$

Hence

$$\frac{1}{f} = \frac{2}{L}\sin\frac{\Delta\mu_x}{2} \tag{6.27}$$

and

$$\beta_x^{QF} = \frac{2L}{\sin\Delta\mu_x}\left(1+\sin\frac{\Delta\mu_x}{2}\right)\,. \tag{6.28}$$

Here, β_x^{QF} denotes the betatron function of a horizontally thin focusing quadrupole, for which $f^{QF} = |f| > 0$, with $f^{-1} = K_0\ell$. We can derive the betatron function β_x^{QD} of a horizontally thin defocusing quadrupole, for which $f^{QD} = -|f| < 0$, by starting the cell by a defocusing quadrupole instead of a focusing one. In that case we obtain the transfer matrix of a DOFO cell, in which the matrix components are those of a FODO cell, where f is replaced by $-f$. Hence, by identification with the general transfer matrix with periodic conditions $\beta_1 = \beta_2 = \beta_x^{QD}$ and $\alpha_1 = \alpha_2 = \alpha_x^{QD}$, we find

$$\beta_x^{QD} = \frac{2L}{\sin\Delta\mu_x}\left(1-\sin\frac{\Delta\mu_x}{2}\right)\,. \tag{6.29}$$

From these results, the natural relative chromaticity of a FODO cell may be computed as

$$\begin{aligned}
\xi_{x0} &= -\frac{1}{4\pi Q_x}\int_s^{s+C}\beta_x(t)K_0(t)dt = \\
&= -\frac{1}{4\pi Q_x}\left(\beta_x^{QF}\int K^{QF}ds + \beta_x^{QD}\int K^{QD}ds\right) = \\
&= -\frac{1}{4\pi Q_x}(\beta_x^{QF}K^{QF}\ell + \beta_x^{QD}K^{QD}\ell) = -\frac{1}{4\pi Q_x}\left(\frac{\beta_x^{QF}}{f^{QF}} + \frac{\beta_{\hat{x}}^{QD}}{f^{QD}}\right) = \\
&= -\frac{1}{4\pi Q_x}\frac{\beta_x^{QF} - \beta_x^{QD}}{f},
\end{aligned}$$

where, according to the thin-lens approximation

$$\int K^{QF,D}ds = K^{QF,D}\ell = \frac{1}{f^{QF,D}}\,.$$

Introducing $\beta_x^{QF,D}$ and the focal length f in the expression (6.23) yields, using the trigonometric formula,

$$\sin\Delta\mu_x = 2\cos\frac{\Delta\mu_x}{2}\sin\frac{\Delta\mu_x}{2}$$

the natural chromaticity

$$\xi_{x0} = -\frac{1}{\pi Q_x}\tan\left(\frac{\Delta\mu_x}{2}\right), \tag{6.30}$$

where $\Delta\mu_x$ is the horizontal phase advance for the FODO cell. If the full lattice is made of N similar FODO cells, the natural chromaticity of the machine becomes

$$\xi_{x0} = -\frac{N}{\pi Q_x}\tan\left(\frac{\Delta\mu_x}{2}\right)\,. \tag{6.31}$$

Furthermore, since

$$Q_x = \frac{1}{2\pi}[\mu_x(s_0 + C) - \mu_x(s_0)]\,,$$

where C is the machine circumference, we can write

$$Q_x = \frac{N\Delta\mu_x}{2\pi}\,.$$

Thus, from (6.31)

$$\xi_{x0} = -\frac{2}{\Delta\mu_x}\tan\left(\frac{\Delta\mu_x}{2}\right)\,. \tag{6.32}$$

To compute the natural vertical chromaticity we use the fact that $\beta_y^{QF} = \beta_x^{QD}$ (with $f^{QF} = |f| > 0$), and $\beta_y^{QD} = \beta_x^{QF}$ (with $f^{QD} = -|f| < 0$). Therefore

$$\xi_{y0} = \frac{1}{4\pi Q_y}\left(\frac{\beta_y^{QF}}{f^{QF}} + \frac{\beta_y^{QD}}{f^{QD}}\right) = \frac{1}{4\pi Q_y}\frac{\beta_y^{QF} - \beta_y^{QD}}{f} = -\frac{1}{4\pi Q_y}\frac{\beta_x^{QF} - \beta_x^{QD}}{f}\,.$$

Similarly, with

$$Q_y = \frac{N\Delta\mu_y}{2\pi},$$

where $\Delta\mu_y$ is the vertical phase advance for the FODO cell, we find

$$\xi_{y_0} = -\frac{2}{\Delta\mu_y}\tan\left(\frac{\Delta\mu_y}{2}\right). \tag{6.33}$$

Thus, the natural chromaticities of a synchrotron made up of FODO cells are negative. For instance, if $\Delta\mu_{x,y} = \pi/2$ we obtain $\xi_{x_0} = \xi_{y_0} = -4/\pi$. More generally the natural chromaticities are always negative since for higher-momentum particles (i.e. $\delta > 0$) the focusing is less effective, and then the tune is reduced (i.e. $\Delta Q < 0$).

6.3 Chromaticity correction

The chromaticity equations suggest the insertion of sextupoles close to each quadrupole, where the dispersion function is non-zero, in order to correct the chromaticity. Thus, for the chromaticity to vanish, the sextupole strength S_0 would be, from (6.22)

$$S_0\ell_s = \frac{K_0\ell_Q}{D_x} \tag{6.34}$$

where ℓ_Q, ℓ_S are the quadrupole and sextupole lengths, K_0 is the quadrupole strength, and D_x is the dispersion function value at the sextupole location.

Unfortunately, such localized connections are seldom feasible. A standard way of adjusting both the horizontal and the vertical chromaticities is to use families of sextupoles with moderate strength, distributed around the ring. Rewriting the equations for the chromaticity as

$$\begin{aligned} \xi_x &= \xi_{x_0} + \frac{1}{4\pi Q_x}\int_s^{s+C} \beta_x(t)S_0(t)D_x(t)dt \\ \xi_y &= \xi_{y_0} - \frac{1}{4\pi Q_x}\int_s^{s+C} \beta_y(t)S_0(t)D_x(t)dt \end{aligned} \tag{6.35}$$

and using the thin-lens approximation, we obtain

$$\begin{aligned} \xi_x &= \xi_{x_0} + \frac{1}{4\pi Q_x}\sum_{i=1}^{N}\beta_{x_i}S_{0_i}D_{x_i}\ell_{s_i} \\ \xi_y &= \xi_{y_0} - \frac{1}{4\pi Q_x}\sum_{i=1}^{N}\beta_{y_i}S_{0_i}D_{s_i}\ell_{s_i} \end{aligned} \tag{6.36}$$

in which the sextupoles are located at s_i, their strengths and lengths being S_{0_i} and ℓ_{s_i}, and N is the total number of sextupoles. If the chromaticities have to be adjusted to the values ξ_x and ξ_y, the sextupole strengths are obtained by solving the linear system of equations

$$\sum_{i=1}^{N} \beta_{x_i} S_{0_i} D_{x_i} \ell_{s_i} = -4\pi Q_x \Delta\xi_x$$
$$\sum_{i=1}^{N} \beta_{y_i} S_{0_i} D_{x_i} \ell_{s_i} = 4\pi Q_y \Delta\xi_y \tag{6.37}$$

with $\Delta\xi_{x,y} = \xi_{x_0,y_0} - \xi_{x,y}$. Assuming there are only two sextupoles in the ring, the linear system of equations reduces to

$$(\beta_{x_1} S_{0_1} D_{x_1} + \beta_{x_2} S_{0_2} D_{x_2})\ell_s = -4\pi Q_x \Delta\xi_x$$
$$(\beta_{y_1} S_{0_1} D_{x_1} + \beta_{y_2} S_{0_2} D_{x_2})\ell_s = 4\pi Q_y \Delta\xi_y \ , \tag{6.38}$$

where we have assumed that the two sextupoles have the same lengths ℓ_s. Solving these two equations yields

$$S_{0_1} = -\frac{4\pi}{\ell_s D_{x_1}} \left(\frac{\beta_{y_2} Q_x \Delta\xi_x + \beta_{x_2} Q_y \Delta\xi_y}{\beta_{x_1}\beta_{y_2} - \beta_{x_2}\beta_{y_1}} \right)$$
$$S_{0_2} = \frac{4\pi}{\ell_s D_{x_2}} \left(\frac{\beta_{y_1} Q_x \Delta\xi_x + \beta_{x_1} Q_y \Delta\xi_y}{\beta_{x_1}\beta_{y_2} - \beta_{x_2}\beta_{y_1}} \right) \ . \tag{6.39}$$

The sextupoles have to be placed where the dispersion function is high and the betatron functions well separated to minimize the sextupole strengths (e.g. $\beta_x \gg \beta_y$ at one sextupole and $\beta_x \ll \beta_y$ at the other sextupole). Furthermore, the strengths S_{0_1} and S_{0_2} are opposite in sign (i.e. $S_{0_1} S_{0_2} < 0$). In the case of a two-family sextupole with N_1 sextupoles of strength S_{0_1} and N_2 sextupoles of strength S_{0_2}, the terms in (6.37) may be grouped in two sums and solved for S_{0_1} and S_{0_2}. The "two-family sextupole" chromaticity correction scheme yields sextupole strengths lower than those based on the "two sextupoles in a ring" scheme.

APPENDIX A:
HILL'S EQUATION

A.1 Linear equations

Consider a linear differential equation of the form, called Hill's equation

$$u'' + K(s)u = 0 , \tag{A.1}$$

where $K(s)$ is a periodic function with period L,

$$K(s + L) = K(s) . \tag{A.2}$$

A prime denotes the derivative with respect to the variable s. For any second-order differential equation there exist two independent solutions, $C(s)$ and $S(s)$, called a fundamental set of solutions and such that every solution $u(s)$ is a linear combination of these two:

$$u(s) = c_1C(s) + c_2S(s) , \tag{A.3}$$

where c_1 and c_2 are arbitrary constants. However, there is only one solution $u(s)$ that meets the initial conditions $u(s_0) = u_0$ and $u'(s_0) = u'_0$ at s_0. Assume that a fundamental set of solutions $C_0(s)$ and $S_0(s)$—cosine-like and sine-like solutions—of (A.1), whether $K(s)$ is periodic or not, have been found satisfying the initial conditions

$$\begin{array}{ll} C_0(s_0) = 1 & C'_0(s_0) = 0 \\ S_0(s_0) = 0 & S'_0(s_0) = 1 . \end{array} \tag{A.4}$$

Then, the solution $u(s)$ of Hill's equation with initial conditions u_0 and u'_0 at s_0 can be expressed as a linear combination of $C_0(s)$ and $S_0(s)$,

$$\begin{aligned} u(s) &= u_0C_0(s) + u'_0S_0(s) , \\ u'(s) &= u_0C'_0(s) + u'_0S'_0(s) , \end{aligned}$$

or equivalently,

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \begin{pmatrix} C_0(s) & S_0(s) \\ C'_0(s) & S'_0(s) \end{pmatrix} \begin{pmatrix} u(s_0) \\ u'(s_0) \end{pmatrix} . \tag{A.5}$$

The above matrix is called a transformation matrix or transfer matrix from s_0 to s, written as

$$M(s/s_0) = \begin{pmatrix} C_0(s) & S_0(s) \\ C'_0(s) & S'_0(s) \end{pmatrix} \tag{A.6}$$

Equations (A.2) to (A.5) hold whether $K(s)$ is periodic or not. The name cosine- and sine-like solutions of $C_0(s)$ and $S_0(s)$ come from the case where $K(s) = K$ is a positive constant for which a fundamental set of solutions is $C_0(s) = \cos\sqrt{K}s$ and $S_0(s) = \sin\sqrt{K}s/\sqrt{K}$, with $C_0(0) = 1$, $C_0'(0) = 0$, and $S_0(0) = 0$, $S_0'(0) = 1$. The determinant of the transfer matrix $M(s/s_0)$ is called the Wronskian. Differentiation of the Wronskian yields

$$\frac{d}{ds}|M(s/s_0)| = \frac{d}{ds}[C_0(s)S'(s) - C_0'(s)S_0(s)] = C_0(s)S_0''(s) - C_0''(s)S_0(s) .$$

Since both $C_0(s)$ and $S_0(s)$ are solutions of Hill's equation, we find

$$\frac{d}{ds}|M(s/s_0)| = 0$$

and thus the Wronskian is a constant. Consequently, since by (A.4) $|M(s_0/S_0)| = 1$, we get the general result, true whether $K(s)$ is periodic or not:

$$|M(s/s_0)| = \begin{vmatrix} C_0(s) & S_0(s) \\ C_0'(s) & S_0'(s) \end{vmatrix} = 1 . \tag{A.7}$$

The transfer matrix from $s_1 \neq s_0$ to s, written as $M(s/s_1)$, is built with the cosine- and sine-like functions $C_1(s)$ and $S_1(s)$, which are generally different from $C_0(s)$ and $S_0(s)$. The conditions (A.4) are not generally satisfied by $C_1(s)$ and $S_1(s)$. However,

$$\begin{aligned} C_1(s_1) = 1 \quad &\text{and} \quad C_1'(s_1) = 0 \\ S_1(s_1) = 0 \quad &\text{and} \quad S_1'(s_1) = 1 . \end{aligned}$$

Using (A.5) in which $u(s)$, $u'(s)$ are replaced by $C_0(s)$, $C_0'(s)$— $S_0(s)$, $S_0'(s)$, respectively— and $u(s_0)$, $u'(s_0)$ are replaced by $C_0(s_1)$, $C_0'(s_1)$—$S_0(s_1)$, $S_0'(s_1)$, respectively—we obtain with $C_1(s)$ and $S_1(s)$ the fundamental set of solutions

$$\begin{pmatrix} C_0(s) & S_0(s) \\ C_0'(s) & S_0'(s) \end{pmatrix} = \begin{pmatrix} C_1(s) & S_1(s) \\ C_1'(s) & S_1'(s) \end{pmatrix} \begin{pmatrix} C_0(s_1) & S_0(s_1) \\ C_0'(s_1) & S_0'(s_1) \end{pmatrix} ,$$

where we have combined two matrix equations in one. By the definition of a transfer matrix this result may be written as

$$M(s/s_0) = M(s/s_1)M(s_1/s_0) , \tag{A.8}$$

with

$$M(s/s_1) = \begin{pmatrix} C_1(s) & S_1(s) \\ C_1'(s) & S_1'(s) \end{pmatrix} .$$

From now on the subscript in $C_0(s)$ and $S_0(s)$, or in $C_1(s)$ and $C_2(s)$, will be omitted, since the reference point s_0 already appears in the notation $M(s/s_0)$. When $K(s)$ is periodic the transfer matrix $M(s + L/s)$ over one period L will be written as $M(s)$. By (A.8),

$$M(s) \equiv M(s + L/s) = M(s + L/s_0)M(s/s_0)^{-1} . \tag{A.9}$$

A.2 Equations with periodic coefficients (Floquet theory)

If $K(s)$ is periodic of period L, and since $C(s)$ and $S(s)$ are a fundamental set of solutions of Hill's equation, $C(s+L)$ and $S(s+L)$ are also a fundamental set of solutions of the same equation, because

$$\begin{aligned}\frac{d^2C(s+L)}{ds^2} + K(s)C(s+L) &= \frac{d^2C(s+L)}{ds^2} + K(s+L)C(s+L)\\ &= \frac{d^2C(t)}{dt^2} + K(t)C(t) = 0\end{aligned}$$

with the change of variables $t = s + L$.

Hence we may write alternatively the cosine- and sine-like solutions $C(s+L)$ and $S(s+L)$ as a linear combination of $C(s)$ and $S(s)$ using (A.3):

$$\begin{aligned}C(s+L) &= a_{11}C(s) + a_{12}S(s),\\ S(s+L) &= a_{21}C(s) + a_{22}S(s)\,,\end{aligned} \tag{A.10}$$

where a_{ij} are the components of a constant matrix A. Similarly, any general solution of Hill's equation is a linear combination of the fundamental set $C(s)$ and $S(s)$

$$u(s) = c_1C(s) + c_2S(s)\,. \tag{A.11}$$

The solution $u(s)$ is not necessarily periodic, although $K(s)$ is periodic. However, let us try to find a solution $u(s)$ with the property

$$u(s+L) = \lambda u(s)\,. \tag{A.12}$$

Using (A.10) to (A.12) we obtain

$$\begin{aligned}u(s+L) &= c_1C(s+L) + c_2S(s+L)\\ &= c_1(a_{11}C(s) + a_{12}S(s)) + c_2(a_{21}C(s) + a_{22}S(s))\end{aligned}$$

and

$$u(s+L) = \lambda[c_1C(s) + c_2S(s)]\,.$$

Comparison of the last two equations gives

$$[(a_{11}-\lambda)c_1 + a_{21}c_2]\,C(s) + [a_{12}c_1 + (a_{22}-\lambda)c_2]\,S(s) = 0\,.$$

Since $C(s)$ and $S(s)$ are linearly independent, the coefficients of the last equation must vanish, yielding the system

$$\begin{aligned}(a_{11}-\lambda)c_1 + a_{21}c_2 &= 0\\ a_{12}c_1 + (a_{22}-\lambda)c_2 &= 0\,.\end{aligned}$$

The condition for nonvanishing solution c_1, c_2 is

$$\begin{vmatrix} a_{11}-\lambda & a_{21} \\ a_{12} & a_{22}-\lambda \end{vmatrix} = |A - \lambda I| = 0\,, \tag{A.13}$$

or equivalently,

$$\lambda^2 - \mathrm{Tr}\,(A)\lambda + |A| = 0\,. \tag{A.14}$$

This expression is called the characteristic equation. Now, we want to express (A.14) in terms of the transfer matrix, rather than in terms of the matrix A with coefficients a_{ij}. Rewriting (A.10) in matrix formulation

$$\begin{pmatrix} C(s+L) \\ S(s+L) \end{pmatrix} = A \begin{pmatrix} C(s) \\ S(s) \end{pmatrix}$$

and transposing this expression gives

$$[C(s+L) \quad S(s+L)] = [C(s) \quad S(s)]A^t\,.$$

This equation and its derivative may be written into the compact form

$$\begin{pmatrix} C(s+L) & S(s+L) \\ C'(s+L) & S'(s+L) \end{pmatrix} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}\,,$$

or

$$M(s + L/s_0) = M(s/s_0)A^t\,. \tag{A.15}$$

Introducing the transfer matrix $M(s)$ over one period L, using (A.8) and the last expression we get

$$M(s) = M(s/s_0)A^t M(s/s_0)^{-1} \tag{A.16}$$

Since the determinant of a transfer matrix is equal to unity, it follows immediately that

$$|A^t| = |A| = 1\,. \tag{A.17}$$

Similarly, the characteristic equation of $M(s)$ is

$$|M(s) - \lambda I| = \lambda^2 - \mathrm{Tr}\,[M(s)]\lambda + 1 = 0\,.$$

then

$$\begin{aligned} |M(s) - \lambda I| &= |M(s/s_0)(A^t - \lambda I)M(s/s_0)^{-1}| = |A^t - \lambda I| \\ &= \lambda^2 - \mathrm{Tr}\,(A)\lambda + 1 = 0\,, \end{aligned}$$

since the trace of a matrix is equal to the trace of its transpose. Identifying these last two expressions yields

$$\mathrm{Tr}\,[M(s)] = \mathrm{Tr}\,(A)\,. \tag{A.18}$$

The trace $M(s)$ is independent of the reference point s, due to A being a constant matrix. Hence the roots λ_1 and λ_2 of the characteristic equation (A.14) are

$$\lambda_{1,2} = \frac{1}{2}\mathrm{Tr}\,[(M(s)] \pm \sqrt{\mathrm{Tr}\,[M(s)]^2 - 4}\,. \tag{A.19}$$

These roots are related by

$$\lambda_1 \lambda_2 = 1 \ . \tag{A.20}$$

Let $u_1(s)$ and $u_2(s)$ be the solution $u(s)$ with the property (A.12), in which $\lambda = \lambda_1$ and $\lambda = \lambda_2$, respectively. Define $w_i(s)$ for $i = 1, 2$ by

$$w_i(s) = e^{-\frac{\ln \lambda_i}{L} s} u_i(s) \ ,$$

then, using (A.12), we get

$$w_i(s + L) = e^{-\frac{\ln \lambda_i}{L}(s+L)} u_i(s + L) = e^{-\frac{\ln \lambda_i}{L} s} \frac{1}{\lambda_i} \lambda_i u_i(s) = w_i(s) \ ,$$

so that $w_i(s)$ is periodic with period L:

$$w_i(s + L) = w_i(s) \ . \tag{A.21}$$

Hence we may write

$$u_i(s) = w_i(s) e^{\frac{\mu_i}{L} s} \ , \tag{A.22}$$

where

$$\mu_i = \ln \lambda_i \ . \tag{A.23}$$

The numbers μ_i are called the characteristic exponents of Hill's equation. They need not be real numbers. If λ_2 and λ_2 are distinct, the solutions $u_1(s)$ and $u_2(s)$ are a fundamental set of solutions with the property

$$u_i(s + L) = e^{\mu_i} u_i(s) \ . \tag{A.24}$$

This result is known as Floquet's theorem. When $\lambda_1 = \lambda_2 = \lambda$, we have $\lambda = \pm 1$ by means of (A.20). There exists a fundamental set of solutions of the form

$$\begin{aligned} u_1(s) &= w_1(s) e^{\frac{\mu}{L} s} \\ u_2(s) &= \left(w_2(s) + \frac{s}{\lambda L} w_1(s) \right) e^{\frac{\mu}{L} s} \ , \end{aligned} \tag{A.25}$$

with $\mu = 0$ for $\lambda = 1$ and $\mu = i\pi$ for $\lambda = -1$. In this case the solution $u_1(s)$ is periodic, with period L when $\lambda = 1$ and period $2L$ when $\lambda = -1$. The main difficulty in the Floquet analysis is that the fundamental set of solutions $C(s)$ and $S(s)$, from which we derived λ_1 and λ_2, is generally unknown.

Instead of defining the Floquet solution $u_1(s)$ and $u_2(s)$ by (A.22), we may alternatively express these solutions in the generalized Floquet form

$$u_i(s) = w_i(s) e^{\mu_i(s)} \ , \tag{A.26}$$

such that $w_i(s)$ are periodic,

$$w_i(s + L) = w_i(s) \ , \tag{A.27}$$

and

$$\mu_i(s+L) - \mu_i(s) \equiv \ln \lambda_i \ , \tag{A.28}$$

μ_1 and μ_2 being the characteristic exponents (assuming λ_1 and λ_2 as distinct). Thus the derivative $\mu_i'(s)$ is periodic with period L. Indeed, owing to $u_i(s)$ having the property (A.12), we define

$$w_i(s) = e^{-\mu_i(s)} u_i(s)$$

and then

$$w_i(s+L) = e^{-\mu_i(s+L)} u_i(s+L) = e^{-[\mu_i(s+L)-\mu_i(s)]} \frac{1}{\lambda_i} \lambda_i u_i(s) = w_i(s) \ ,$$

since $\mu_i = \ln \lambda_i$, and provided (A.28) is satisfied. However, such an equation (A.28) exists since λ_1 and λ_2 are independent of s by (A.18) and (A.19). The Floquet solutions $u_1(s)$ and $u_2(s)$ are obviously different from the cosine- and sine-like solutions $C(s)$ and $S(s)$ from which they are derived.

A.3 Stability of solutions

In plain words, a given solution of Hill's equation is stable (in the Liapunov sense) if any other solution coming near it remains in its neighborhood. Otherwise it is said to be unstable. It can be shown that a solution is stable if, and only if, it is bounded. Some deductions about the stability of the solution $u(s)$ of Hill's equation can be made from the roots λ_1 and λ_2 of the characteristics equation. By (A.19) and (A.20) we see that:

1) The roots may be complex conjugate on the unit circle

$$\lambda_{1,2} = e^{\pm i\mu} \ , \tag{A.29}$$

representing stable solutions

$$u(s) = c_1 w_1(s) e^{i\frac{\mu}{L}s} + c_2 w_2(s) e^{-i\frac{\mu}{L}s} \ , \tag{A.30}$$

with

$$|\mathrm{Tr}\ [M(s)]| = |2\cos\mu| < 2 \ ., \tag{A.31}$$

2) The roots may be real reciprocals

$$\lambda_{1,2} = e^{\pm\mu} \;,, \qquad \text{(A.32)}$$

representing growing and decaying solutions

$$u(s) = c_1 w_1(s) e^{\frac{\mu}{L}s} + c_2 w_2(s) e^{-\frac{\mu}{L}s} \;,, \qquad \text{(A.33)}$$

with

$$|\mathrm{Tr}\,[M(s)]| = |2\cosh\mu| > 2, \qquad \text{(A.34)}$$

where the unstable solutions are divided into two possibilities: $\mathrm{Tr}\,[M(s)] > 2$ (for $\lambda_1 > 1$) and $\mathrm{Tr}\,[M(s)] < 2$ (for $\lambda_1 < -1$).

3) The roots may be equal,

$$\lambda_1 = \lambda_2 = \pm 1 \;,, \qquad \text{(A.35)}$$

representing transition between the stable and unstable cases with

$$|\mathrm{Tr}\,[M(s)]| = 2 \;., \qquad \text{(A.36)}$$

According to (A.25) there is one stable solution of period L for $\lambda_1 = \lambda_2 = 1$ (the other is unstable), and one stable solution of period $2L$ for $\lambda_1 = \lambda_2 = -1$ (the other also being unstable).

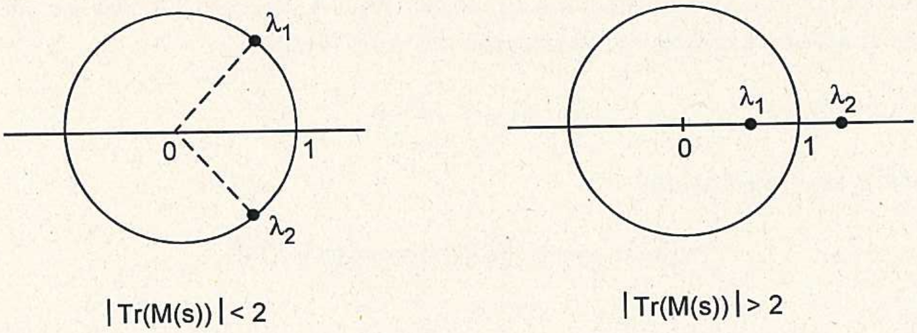

Figure 36: Stability conditions for a one-period transfer matrix: location of eigenvalues in the complex plane (left figure: stable motion, right figure: unstable motion).

As an example, consider again a symmetric thin-lens FODO cell of length $2L$ whose transfer matrix has been derived to be

$$M_{\mathrm{FODO}}(2L/0) = \begin{pmatrix} 1 - \frac{L}{f} - \frac{L^2}{f^2} & 2L + \frac{L^2}{f} \\ -\frac{L}{f^2} & 1 + \frac{L}{f} \end{pmatrix}$$

The stability criterion (A.31) gives the condition

$$|\mathrm{Tr}\,[M_{\mathrm{FODO}}(2L/0)]| = \left|2 - \frac{L^2}{f^2}\right| < 2$$

or equivalently

$$-1 < 1 - \frac{1}{2}\frac{L^2}{f^2} < 1$$

yielding

$$0 < \frac{L}{2f} < 1$$

The stability is thus obtained for distances L between the quadrupoles up to twice their focal length f.

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