Linear Dynamics, Lecture 3

The Accelerator Hamiltonian in a Curved Coordinate System

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What we Learned in the Previous Lecture

In the previous lecture, we derived a Hamiltonian for the motion of a particle through an electromagnetic field, with dynamical variables appropriate for a particle accelerator. For particles close to the reference trajectory, and with energy close to the reference energy, the values of the dynamical variables are expected to remain small as the particle moves through the accelerator.

Since the dynamical variables take small values, we can make approximations to the Hamiltonian to construct linear maps. We saw how this could be applied to the map for a field-free region (a *drift space*).

So far we have assumed that the reference trajectory is a straight line.

Course Outline

Part I (Lectures 1 – 5): Dynamics of a relativistic charged particle in the electromagnetic field of an accelerator beamline.

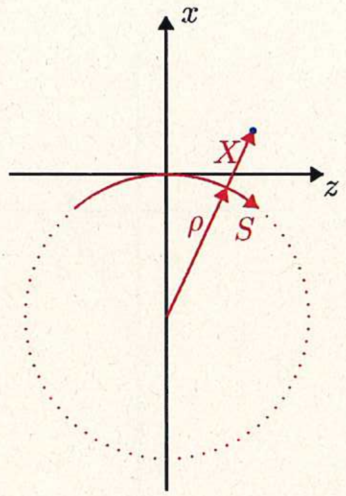
1. Review of Hamiltonian mechanics
2. The accelerator Hamiltonian in a straight coordinate system
3. The Hamiltonian for a relativistic particle in a general electromagnetic field using accelerator coordinates
4. Dynamical maps for linear elements
5. Three loose ends: edge focusing; chromaticity; beam rigidity.

Goals of This Lecture

In this lecture, we shall see how to modify the Hamiltonian to deal with cases where the reference trajectory is curved. This will allow us to deal with dipole magnets, where all particles follow curved paths.

Using a curved reference trajectory in dipole magnets allows us to maintain small values for the dynamical variables, even where the deflection from the dipole is large. This means we can continue to use series expansion approximations for the Hamiltonian in such cases.

Ultimately, we shall derive the linear transfer map (transfer matrix) for a dipole.



Old coordinates are (x, y, z); new coordinates are (X, Y, S):

$$x = (\rho + X)\cos\frac{S}{\rho} - \rho \quad (1)$$

$$y = Y \quad (2)$$

$$z = (\rho + X)\sin\frac{S}{\rho} \quad (3)$$

We can construct a generating function to find the conjugate momenta in the new coordinate system:

$$F_3(X, p_x, Y, p_y, S, p_z) = -\left[(\rho + X)\cos\frac{S}{\rho} - \rho\right] p_x - Y p_y - \left[(\rho + X)\sin\frac{S}{\rho}\right] p_z \quad (4)$$

The old and new coordinates and momenta are related by:

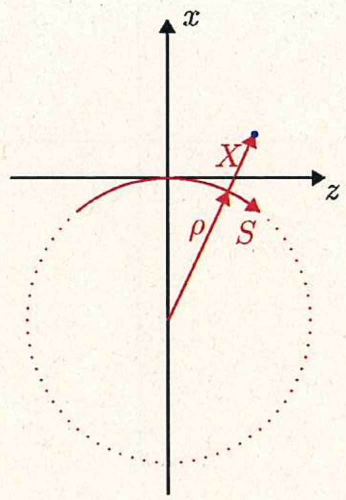
$$x_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial X_i} \quad (5)$$

The coordinates transform as required:

$$x = (\rho + X)\cos\frac{S}{\rho} - \rho \quad (6)$$

$$y = Y \quad (7)$$

$$z = (\rho + X)\sin\frac{S}{\rho} \quad (8)$$



The new transverse momenta are given by:

$$P_X = p_x \cos\frac{S}{\rho} + p_z \sin\frac{S}{\rho} \quad (9)$$

$$P_Y = p_y \quad (10)$$

The curvature of the trajectory has a surprising effect on the longitudinal component of the momentum: P_S is *not* just the tangential component of the momentum in Cartesian coordinates!

$$P_S = p_z\left(1+\frac{X}{\rho}\right)\cos\frac{S}{\rho} - p_x\left(1+\frac{X}{\rho}\right)\sin\frac{S}{\rho} \quad (11)$$

To complete the transformation, we also need to express the components of the vector potential in the new coordinate system:

$$A_X = A_x \cos\frac{S}{\rho} - A_z \sin\frac{S}{\rho} \quad (12)$$

$$A_Y = A_y \quad (13)$$

$$A_S = A_z \cos\frac{S}{\rho} + A_x \sin\frac{S}{\rho} \quad (14)$$

Recall the general form for the Hamiltonian for a relativistic particle in Cartesian coordinates, in an electromagnetic field:

$$H = \sqrt{(\mathbf{p} - q\mathbf{A})^2 c^2 + m^2c^4} + q\phi \tag{15}$$

The transformation into "accelerator variables" in a curvilinear coordinate system follows exactly the same lines as the transformations in a straight coordinate system. The only difference is that when we change the independent variable from t to s (and switch the Hamiltonian from H to $-P_S$), we pick up a factor $1 + x/\rho$ from equation (11).

The result – our final "Accelerator Hamiltonian" – is:

$$\begin{aligned} H \;&=\; -(1+hx)\sqrt{\left(\frac{1}{\beta_0} + \delta - \frac{q\phi}{P_0 c}\right)^2 - (p_x - a_x)^2 - (p_y - a_y)^2 - \frac{1}{\beta_0^2\gamma_0^2}} \\ &\quad - (1+hx)\, a_s + \frac{\delta}{\beta_0} \end{aligned} \tag{16}$$

where we have (as usual) renamed our variables so as to tidy up the notation; and we have defined the "curvature":

$$h = \frac{1}{\rho} \tag{17}$$

Note that, from the figures shown in the previous slides, the curvature h is *positive* for a bend moving towards the *negative* x direction. This is simply a convention.

We are now in a position to write down the equations of motion, with a curved reference trajectory, for a relativistic particle moving through any field for which we know the potentials ϕ and $\mathbf{a}$.

Before writing down and solving the equations of motion for a particle travelling through various kinds of magnet, RF cavity etc., we should know something about the fields generated by these devices.

Recall that the fields are the derivatives of the potentials:

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{18}$$

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{19}$$

Allowed physical fields must be solutions of Maxwell's equations...

James Clerk Maxwell, 1831-1879

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{20}$$

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$$

Finding solutions to Maxwell's equations for a given set of boundary conditions is in general no easy task. Significant effort has been devoted to developing computer codes to solve this problem accurately and efficiently. Such codes have many important applications in accelerator physics.

Fortunately, for linear beam dynamics, we are interested in a few simple cases. In particular, we note that we can write the field in a "long straight multipole" magnet as:

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r_0} \right)^{n-1} \quad (21)$$

where b_n and a_n are arbitrary coefficients (chosen to give the correct field map), and r_0 is an arbitrary "reference radius". It is readily shown that the field of (21) satisfies Maxwell's equations (20).

The magnetic multipole field expansion is:

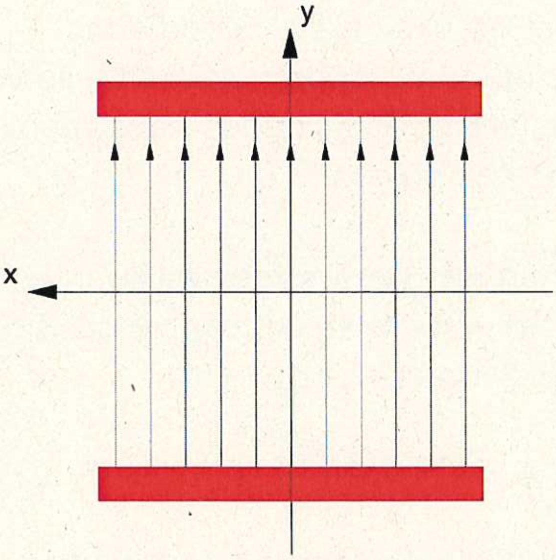
$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r_0} \right)^{n-1} \quad (22)$$

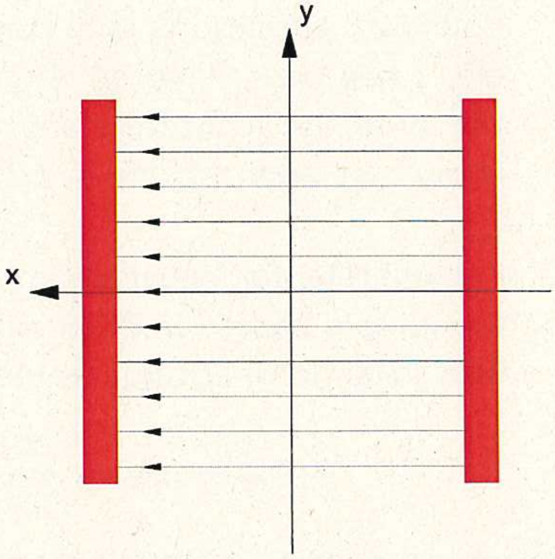
The "multipole components" are indexed by the value of n: so $n = 1$ is a dipole; $n = 2$ is a quadrupole; $n = 3$ is a sextupole, etc.

An ideal multipole has coefficients a_n and b_n equal to zero, for all except one value of n.

A "normal multipole" has $a_n = 0$ for all values of n; a "skew" multipole has $b_n = 0$ for all values of n.

Normal and Skew Dipole Fields





Normal dipole

$$B_x = 0, \quad B_y = b_1.$$

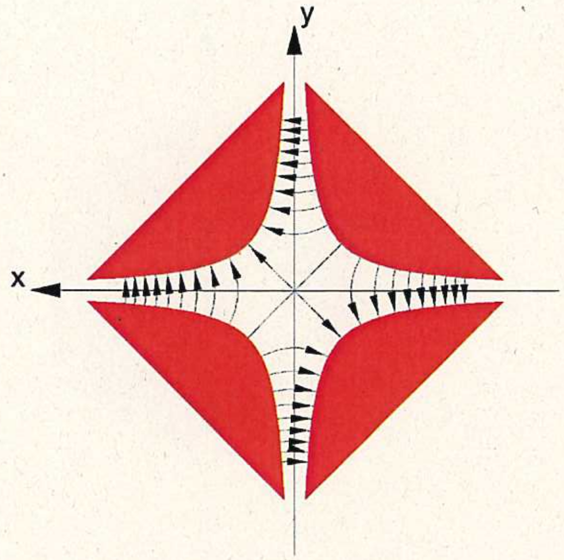
Skew dipole

$$B_x = a_1, \quad B_y = 0.$$

Normal and Skew Dipole Fields

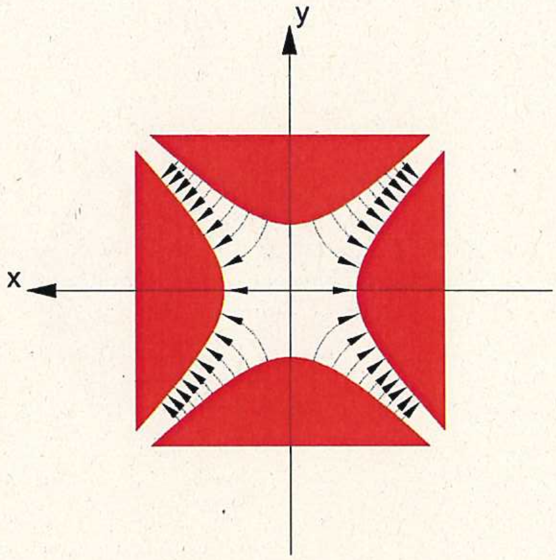
Dipole magnet being installed in the Australian synchrotron.

Normal and Skew Quadrupole Fields



Normal quadrupole

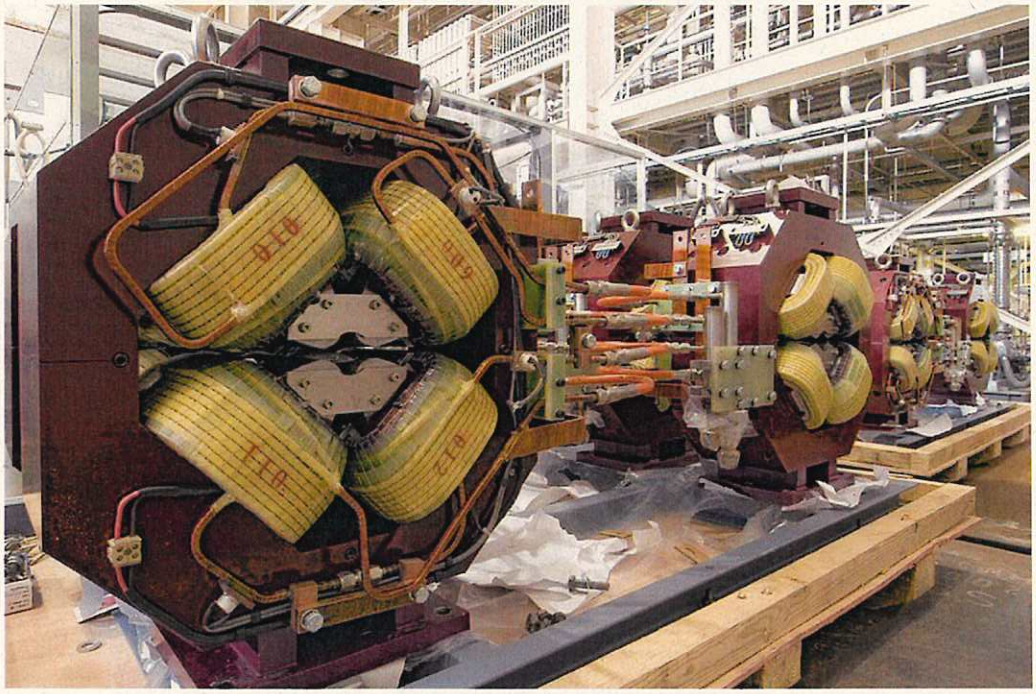
$$B_x = b_2 \frac{y}{r_0}, \quad B_y = b_2 \frac{x}{r_0}.$$



Skew quadrupole

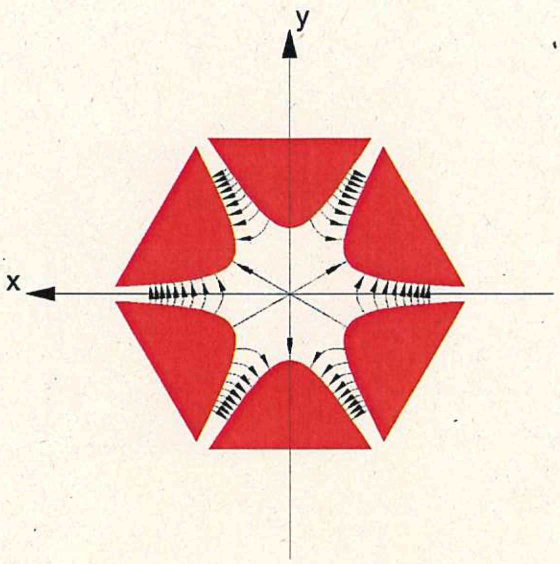
$$B_x = a_2 \frac{x}{r_0}, \quad B_y = -a_2 \frac{y}{r_0}.$$

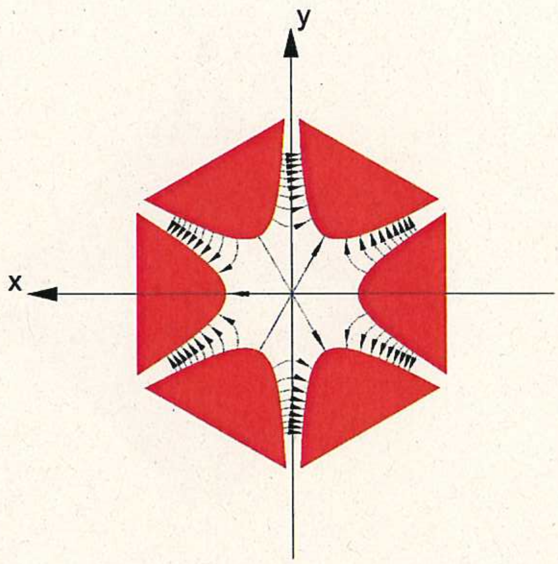
Normal and Skew Quadrupole Fields



Quadrupole magnets (from IHEP, Beijing, China) awaiting installation in ATF2 (KEK, Tsukuba, Japan).

Normal and Skew Sextupole Fields





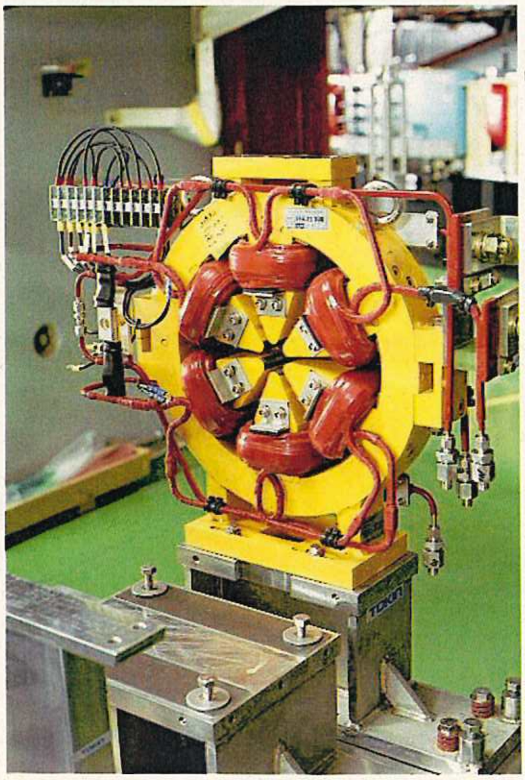
Normal sextupole

$$B_x = 2b_3\frac{xy}{r_0^2}, \quad B_y = b_3\frac{x^2-y^2}{r_0^2}.$$

Skew sextupole

$$B_x = a_3\frac{x^2-y^2}{r_0^2}, \quad B_y = -2a_3\frac{xy}{r_0^2}.$$

Normal and Skew Sextupole Fields



Sextupole magnet from the ATF (KEK, Tsukuba, Japan).

Let us write down the magnetic vector potential:

$$A_x = 0, \qquad A_y = 0, \qquad A_z = -\Re \sum_{n=1}^{\infty} (b_n + i a_n) \frac{(x+iy)^n}{n r_0^{n-1}} \tag{23}$$

We find from the standard relation between the magnetic field and the vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{24}$$

that the potential (23) gives the magnetic multipole field (22):

$$B_y + iB_x = -\frac{\partial A_z}{\partial x} + i\frac{\partial A_z}{\partial y} = \sum_{n=1}^{\infty} (b_n + i a_n) \left(\frac{x+iy}{r_0} \right)^{n-1} \tag{25}$$

Although there are many possible vector potentials that give the same field (25) (and all give the same equations of motion!) the particular choice (23), is convenient, because the transverse components are zero, and there is no dependence on the longitudinal coordinate.

Let's consider first the dipole field. This should be easy: it's just a uniform field perpendicular to the reference trajectory. But there's a catch...

... a dipole field will lead to a curved trajectory for the reference particle. In other words, we will need to use a curved reference trajectory, so when writing down the magnetic vector potential $\mathbf{A}$, we have to take into account the fact we are using curvilinear coordinates.

Our vector potential should satisfy:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{26}$$

with

$$B_x = 0, \qquad B_y = B_0, \qquad B_s = 0 \tag{27}$$

In general curvilinear coordinates (q_1, q_2, q_3), the curl of a vector field can be written:

$$[\nabla \times \mathbf{A}]_1 = \frac{1}{Q_2 Q_3}\left(\frac{\partial}{\partial q_2} Q_3 A_3 - \frac{\partial}{\partial q_3} Q_2 A_2\right) \quad (28)$$

$$[\nabla \times \mathbf{A}]_2 = \frac{1}{Q_3 Q_1}\left(\frac{\partial}{\partial q_3} Q_1 A_1 - \frac{\partial}{\partial q_1} Q_3 A_3\right) \quad (29)$$

$$[\nabla \times \mathbf{A}]_3 = \frac{1}{Q_1 Q_2}\left(\frac{\partial}{\partial q_1} Q_2 A_2 - \frac{\partial}{\partial q_2} Q_1 A_1\right) \quad (30)$$

where

$$Q_i^2 = \left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2 \quad (31)$$

In our coordinates (1), (2), (3), we find that the curl is given by:

$$[\nabla \times \mathbf{A}]_x = \frac{\partial A_s}{\partial y} - \frac{1}{(1+hx)}\frac{\partial A_y}{\partial s} \quad (32)$$

$$[\nabla \times \mathbf{A}]_y = \frac{1}{(1+hx)}\frac{\partial A_x}{\partial s} - \frac{h}{(1+hx)} A_s - \frac{\partial A_s}{\partial x} \quad (33)$$

$$[\nabla \times \mathbf{A}]_s = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \quad (34)$$

Using these expressions we find that the vector potential in our curvilinear coordinates:

$$A_x = 0 \qquad A_y = 0 \qquad A_s = -B_0\left(x - \frac{hx^2}{2(1+hx)}\right) \quad (35)$$

gives the magnetic field:

$$B_x = 0 \qquad B_y = B_0 \qquad B_s = 0 \quad (36)$$

as desired.

Using the vector potential (35), and the general accelerator Hamiltonian (16) we construct the Hamiltonian for a dipole:

$$H = \frac{\delta}{\beta_0} - (1 + hx)\sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + (1 + hx)\, k_0 \left(x - \frac{hx^2}{2(1 + hx)}\right) \tag{37}$$

Note that the normalised dipole field strength is given by:

$$k_0 = \frac{q}{P_0} B_0 \tag{38}$$

where q is the charge of the reference particle, and P_0 is the reference momentum.

The full Hamiltonian for a dipole (37) looks rather intimidating. We shall resort to the same technique we used to get a linear map for a drift space, and expand the Hamiltonian to second-order in the dynamical variables. As before, this is valid as long as the dynamical variables remain small.

The second-order Hamiltonian is:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}hk_0x^2 - \frac{h}{\beta_0}x\delta + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \tag{39}$$

We can tell a good deal already just by looking at this Hamiltonian...

The second-order Hamiltonian is (39):

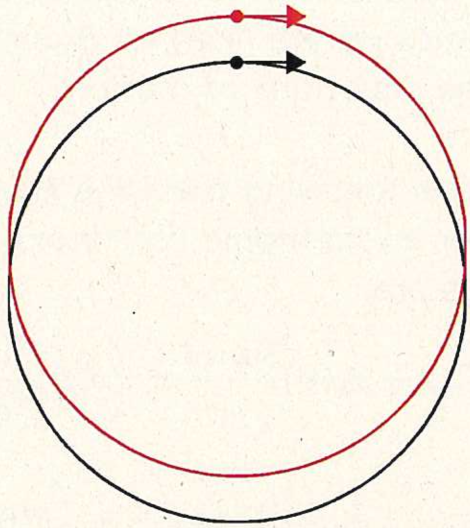
$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}hk_0x^2 - \frac{h}{\beta_0}x\delta + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \qquad (40)$$

Note the term $(k_0 - h)x$. A term in the Hamiltonian that is first order in one of the variables results in a zeroth-order term in the map for the conjugate variable. In this case, we expect to see a horizontal deflection – a change in p_x. This happens if the curvature of the reference trajectory is not matched to the magnetic field of the dipole. If $k_0 = h$, then the curvature is properly matched, and this term vanishes: a particle initially on the reference trajectory and having the reference energy stays on the reference trajectory through the dipole.

The second-order Hamiltonian is (39):

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}hk_0x^2 - \frac{h}{\beta_0}x\delta + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \qquad (41)$$

Next, note the term $\frac{1}{2}hk_0x^2$. This looks like a "focusing term" – recall the potential energy term in the Hamiltonian for an harmonic oscillator. It appears that in moving through the dipole, particles will *oscillate* about the reference trajectory. This is perhaps unexpected. How do we understand this effect?



In a uniform magnetic field, the trajectories of two particles with some small initial offset will "oscillate" around each other.

The second-order Hamiltonian is (39):

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}hk_0x^2 - \frac{h}{\beta_0}x\delta + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \quad (42)$$

Finally, note the term $\frac{h}{\beta_0}x\delta$. This contains the product of two dynamical variables, the horizontal coordinate x, and the energy deviation δ. The result of this term will be a *coupling* of the horizontal and longitudinal motion. For example, there will be a horizontal deflection depending on the particle's energy. This is called "dispersion", and is a consequence of the fact that for relativistic particles, the higher the particle's energy, the higher its mass, and the less effect there is on its trajectory from the Lorentz force.

Now we have the Hamiltonian for a dipole, and have considered some of the dynamics we are likely to expect from it. What are the solutions to the equations of motion?

Hamilton's equations following from the Hamiltonian (39) are essentially those for an harmonic oscillator. In the horizontal plane, the solutions are:

$$x(s) = x(0)\cos\omega s + p_x(0)\frac{\sin\omega s}{\omega} + \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{(1-\cos\omega s)}{\omega^2} \tag{43}$$

$$p_x(s) = -x(0)\omega\sin\omega s + p_x(0)\cos\omega s + \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{\sin\omega s}{\omega} \tag{44}$$

where:

$$\omega = \sqrt{hk_0} \tag{45}$$

In the vertical plane, the solutions are:

$$y(s) = y(0) + p_y(0)s \tag{46}$$

$$p_y(s) = p_y(0) \tag{47}$$

which is the same as for a drift space: there is no weak focusing in the vertical plane.

In the longitudinal plane, the solutions are:

$$z(s) = z(0) - x(0)\frac{h}{\beta_0}\frac{\sin\omega s}{\omega} - p_x(0)\frac{h}{\beta_0}\frac{(1-\cos\omega s)}{\omega^2} + \delta(0)\frac{s}{\beta_0^2\gamma_0^2} - \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{h}{\beta_0}\frac{(\omega s - \sin\omega s)}{\omega^3} \tag{48}$$

$$\delta(s) = \delta(0) \tag{49}$$

Equations (43)–(49) constitute the dynamical map for a dipole. Since the equations are linear, we can write them in the form of a transfer matrix, R. Let us consider the case that the reference trajectory is matched to the dipole strength, i.e. $\omega = h = k_0$: this is the situation that we normally design in an accelerator. In this case, the transfer matrix for a dipole of length L is:

$$R = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & \frac{1-\cos\omega L}{\omega\beta_0} \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & \frac{\sin\omega L}{\beta_0} \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{\sin\omega L}{\beta_0} & -\frac{1-\cos\omega L}{\omega\beta_0} & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} - \frac{\omega L - \sin\omega L}{\omega\beta_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (50)$$

Note that we have not yet included end effects - the edges of the dipole have their own dynamical effects on the beam!

To keep the values of the dynamical variables small in dipole magnets, we use a curved reference trajectory. Generally, we choose a reference trajectory that follows the path of a particle having the reference momentum. We need to define the variables in the curved coordinate system carefully: this can be achieved using a canonical transformation.

The dynamics in dipoles displays some interesting features. These include *dispersion* (variation in trajectory with energy) and *weak focusing*. The effect of weak focusing in a horizontal bending magnet is to keep the horizontal coordinate of a particle close to the reference trajectory: in the horizontal plane, particles oscillate around the reference trajectory with period equal to the period of the circular motion in the field of the magnet.

Linear Dynamics, Lecture 4

Dynamical Maps for "Linear" Elements

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November, 2012

What we Learned in the Previous Lecture

In the previous lecture, we derived a Hamiltonian for the motion of a particle through an accelerator. This Hamiltonian included a general electromagnetic field, allowed a curved reference trajectory, and used dynamical variables that remain small for particles following a trajectory close to the reference trajectory.

We applied this Hamiltonian to the case of a dipole (bending magnet). To obtain a linear dynamical map, we made an approximation by making a series expansion of the Hamiltonian to second order in the dynamical variables.

There were several interesting effects that we saw arising from the Hamiltonian: these included dispersion (variation of the bending angle with the energy of the particle) and weak focusing.

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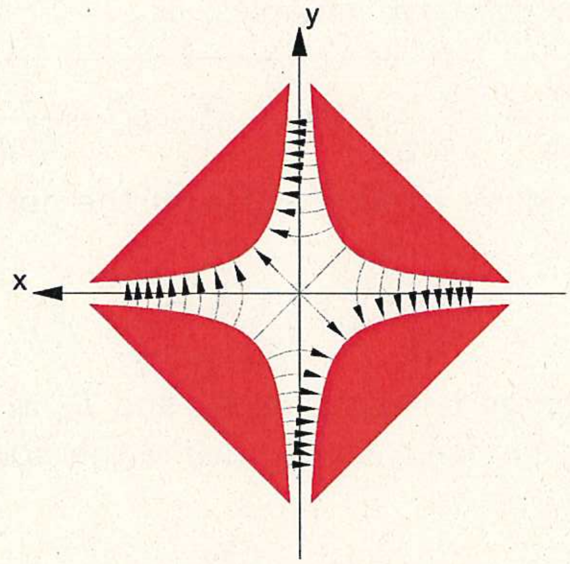
5. Three loose ends: edge focusing; chromaticity; beam rigidity.

Goals of This Lecture

In this lecture, we shall continue our derivation of dynamical maps for "linear" beamline elements. To the drift space and dipole, we shall add the quadrupole, the RF cavity, and the solenoid.

Note that all elements are in fact nonlinear. By "linear" elements, we refer to those whose principle effects on the beam may be obtained by expanding the Hamiltonian to second order in the dynamical variables. We shall make extensive use of this approximation - usually called the *paraxial approximation*.

Recall the magnetic field inside a normal quadrupole magnet:



Normal quadrupole

$B_x = b_2 \frac{y}{r_0}, \quad B_y = b_2 \frac{x}{r_0}.$

The field inside a normal quadrupole magnet in Cartesian coordinates may be written:

$$B_x = b_2 \frac{y}{r_0} \quad (1)$$

$$B_y = b_2 \frac{x}{r_0} \quad (2)$$

$$B_s = 0 \quad (3)$$

Note that on the axis of the quadrupole, the field strength is zero. Therefore, we can choose the reference trajectory to lie along the axis, in which case there is no curvature: we can work in a straight coordinate system.

The above field may be derived from the potential:

$$A_x = 0 \quad (4)$$

$$A_y = 0 \quad (5)$$

$$A_s = -\frac{1}{2}\frac{b_2}{r_0}\left(x^2 - y^2\right) \quad (6)$$

The Hamiltonian describing the motion inside a quadrupole, using the usual accelerator variables, is:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} - a_s \quad (7)$$

where the longitudinal component a_s of the normalised vector potential is:

$$a_s = q\frac{A_s}{P_0} = -\frac{1}{2}\frac{q}{P_0}\frac{b_2}{r_0}\left(x^2 - y^2\right) \quad (8)$$

where q is the charge on the particle, and P_0 is the reference momentum. For convenience, we define the *normalised quadrupole gradient*:

$$k_1 = \frac{q}{P_0}\frac{b_2}{r_0} \quad (9)$$

In terms of the normalised quadrupole gradient (9) the Hamiltonian can be written:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} + \frac{1}{2}k_1\left(x^2 - y^2\right) \quad (10)$$

Expanding the Hamiltonian (10) to second order in the dynamical variables (making the paraxial approximation) we construct the Hamiltonian:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}k_1 x^2 - \frac{1}{2}k_1 y^2 + \frac{1}{2\beta_0^2\gamma_0^2}\delta^2 \quad (11)$$

Note that this looks very much like the harmonic oscillator equation; for $k_1 > 0$ we have a "focusing" potential in x, and a "defocusing" potential in y. In z there is no focusing of any kind.

Solving the equations of motion for the Hamiltonian (11) we find the transfer matrix for a quadrupole of length L ($k_1 > 0$):

$$R = \begin{pmatrix} \cos\omega L & \frac{\sin\omega L}{\omega} & 0 & 0 & 0 & 0 \\ -\omega\sin\omega L & \cos\omega L & 0 & 0 & 0 & 0 \\ 0 & 0 & \cosh\omega L & \frac{\sinh\omega L}{\omega} & 0 & 0 \\ 0 & 0 & \omega\sinh\omega L & \cosh\omega L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (12)$$

where

$$\omega = \sqrt{k_1} \qquad (13)$$

Note that the field, if *focusing* in x is *defocusing* in y, and vice-versa. This is a direct consequence of the constraints on the magnetic field from Maxwell's equations: it is not possible to build a quadrupole that focuses or defocuses in both transverse planes simultaneously.

A skew quadrupole is obtained from a normal quadrupole by rotating the magnet 90° about the magnetic axis. The skew multipole field components are given by the c_n coefficients in the multipole expansion:

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r_0} \right)^{n-1} \qquad (14)$$

For a skew quadrupole, all coefficients are zero except for a_2:

$$B_x = a_2 \frac{x}{r_0} \qquad B_y = -a_2 \frac{y}{r_0} \qquad (15)$$

The magnetic vector potential is given by:

$$A_x = 0 \qquad A_y = 0 \qquad A_s = a_2 xy \qquad (16)$$

Hamiltonian for a Skew Quadrupole

If we define:

$$k_{1s} = -\frac{q}{P_0}\frac{a_2}{r_0} \tag{17}$$

where P_0 is the reference momentum, and r_0 is the reference radius of the magnet, then the normalised vector potential is:

$$a_s = -k_{1s}xy \tag{18}$$

and the Hamiltonian for a skew quadrupole is:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} + k_{1s}xy \tag{19}$$

Making the paraxial approximation, we find the second-order Hamiltonian:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + k_{1s}xy + \frac{1}{2\beta_0^2\gamma_0^2}\delta^2 \tag{20}$$

Note the term in xy: this leads to *coupling* of the horizontal and vertical motion. The skew quadrupole gives a horizontal kick proportional to the vertical offset of the particle, and vice-versa.

Transfer Matrix for a Skew Quadrupole

Hamilton's equations with the second-order skew quadrupole Hamiltonian (20) may be solved as for the normal quadrupole. The resulting map is linear, and so it may be written as a transfer matrix, R (for $k_{1s} > 0$):

$$\begin{pmatrix} \frac{1}{2}(\cos\omega L + \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L + \sinh\omega L) & \frac{1}{2}(\cos\omega L - \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L - \sinh\omega L) & 0 & 0 \\ -\frac{1}{2}\omega(\sin\omega L - \sinh\omega L) & \frac{1}{2}(\cos\omega L + \cosh\omega L) & -\frac{1}{2}\omega(\sin\omega L + \sinh\omega L) & \frac{1}{2}(\cos\omega L - \cosh\omega L) & 0 & 0 \\ \frac{1}{2}(\cos\omega L - \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L - \sinh\omega L) & \frac{1}{2}(\cos\omega L + \cosh\omega L) & \frac{1}{2\omega}(\sin\omega L + \sinh\omega L) & 0 & 0 \\ -\frac{\omega}{2}(\sin\omega L + \sinh\omega L) & \frac{1}{2}(\cos\omega L - \cosh\omega L) & -\frac{\omega}{2}(\sin\omega L - \sinh\omega L) & \frac{1}{2}(\cos\omega L + \cosh\omega L) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2\gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

where

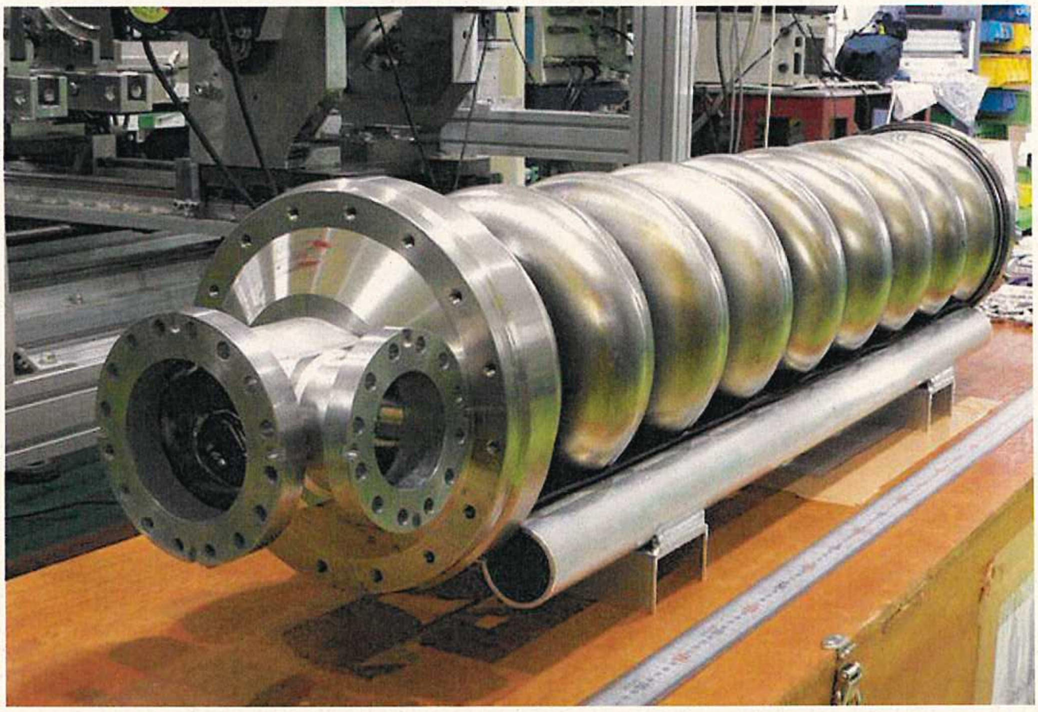
$$\omega = \sqrt{k_{1s}} \tag{22}$$

Now we know how to focus the beam horizontally (dipole, or quadrupole with $k_1 > 0$ and vertically (quadrupole with $k_1 < 0$). But nothing we have seen so far produces any longitudinal focusing. If we want to control the bunch size in all three dimensions, some kind of longitudinal focusing will be necessary. This can be provided by an RF cavity.

An RF cavity contains an electromagnetic field that has a sinusoidal dependence on time. The dependence of the field strength on the spatial coordinates (x, y, s) is in general quite complicated; but in simple cases it can be broken down into a set of modes – just like the magnetic field in a multipole magnet can be broken down into a set of multipoles.

For the simplest RF cavity, we only need consider a single mode – the TM_{010} mode.

RF cavity.



Superconducting 9-cell RF cavity.

The TM_{010} Mode in an RF Cavity

In the TM_{010} mode in an RF cavity, the electric field has components in cylindrical coordinates:

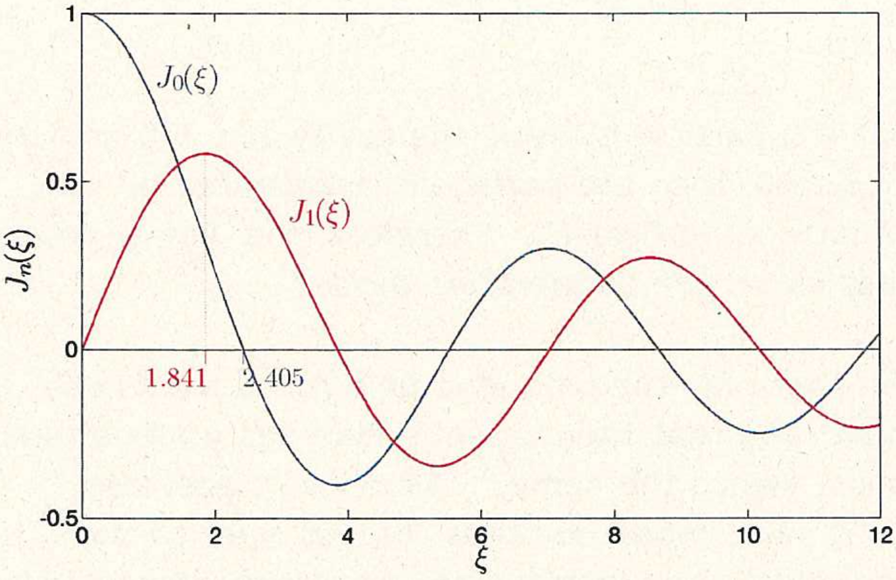
$$
\begin{aligned}
E_\rho &= 0 \\
E_\phi &= 0 \\
E_s &= \hat{E}_s J_0(k\rho) \sin(\omega_{RF} t + \phi_0)
\end{aligned}
\tag{23}
$$

(where $\rho = \sqrt{x^2 + y^2}$) and the magnetic field is:

$$
\begin{aligned}
B_\rho &= 0 \\
B_\phi &= \frac{k}{\omega} \hat{E}_s J_1(k\rho) \cos(\omega_{RF} t + \phi_0) \\
B_s &= 0
\end{aligned}
\tag{24}
$$

where J_n are Bessel functions of the first kind, ω_{RF} is the RF frequency, and ϕ_0 is an arbitrary phase. It can be shown that for $\omega_{RF}/k = c$, the above fields satisfy Maxwell's equations, so they are valid electromagnetic fields.

The TM_{010} Mode in an RF Cavity



Bessel functions are solutions of the differential equation:

$$\xi^2 \frac{d^2 J_n}{d\xi^2} + \xi \frac{dJ_n}{d\xi} + (\xi^2 - n^2) J_n = 0 \tag{25}$$

for real n. Note that $J_0(\xi) = 0$ for $\xi \approx 2.405$.

The TM_{010} Mode in an RF Cavity

If the cavity consists of a conducting cylinder of radius ρ_0 with axis along the reference trajectory, then the boundary conditions require the longitudinal component E_s to vanish at $\rho = \rho_0$.

Hence, the frequency of the electromagnetic field in the cavity is determined by the cavity radius:

$$k\rho_0 \approx 2.405 \tag{26}$$

Since the function $J_0(\xi)$ has multiple zeroes, there are (infinitely) many other modes that may exist in the cavity. These *higher-order modes* have undesired effects, and are a general problem in cavity design. Significant efforts are made in the design and construction of RF cavities in accelerators to suppress or "damp" higher-order modes.

Note that if a particle is inside the cavity at $t = 0$ and the RF phase is $\phi_0 = 0$, then the particle is *accelerated* by the longitudinal electric field E_s. Therefore, the TM_{010} mode is sometimes called the *accelerating mode*.

Note also that only the magnetic field has a transverse component; and that the magnetic field has no longitudinal component. Hence the name "TM" (for "transverse magnetic"). The mode numbers $(0, 1, 0)$ refer to the azimuthal, radial, and longitudinal directions, respectively.

The Hamiltonian in a TM_{010} RF Cavity

The TM_{010} mode fields may be derived from the time-dependent magnetic vector potential:

$$A_x = 0 \tag{27}$$

$$A_y = 0 \tag{28}$$

$$A_s = \frac{\hat{E}_s}{\omega} J_0(k\rho) \cos(\omega_{RF} t + \phi_0) \tag{29}$$

Now, in the accelerator Hamiltonian, we use the path length s as the independent variable, rather than the time t. The relationship between the two involves the dynamical variable z:

$$ct = \frac{s}{\beta_0} - z \tag{30}$$

Therefore, we can write the Hamiltonian in the TM_{010} fields:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2 \gamma_0^2}} - \frac{q}{P_0} \frac{\hat{E}_s}{\omega} J_0(k\rho) \cos\left(\frac{k}{\beta_0} s - kz + \phi_0\right) \tag{31}$$

where (for the fields to satisfy Maxwell's equations) $\omega_{RF}/k = c$.

The Hamiltonian (31) has an unpleasant feature that we have so far managed to avoid: it has an explicit dependence on the independent variable s. This is allowed, but in this case makes the equations of motion very difficult to solve, and the paraxial approximation does not get us out of trouble.

To simplify the problem, we therefore *average* the Hamiltonian in s over the length of the cavity:

$$\langle H \rangle = \frac{1}{L} \int_{-L/2}^{L/2} H ds \qquad (32)$$

where L is the length of the cavity. The fields we have written down in (23) and (24) have no dependence on s, so we can in principle make the cavity any length we like; however, for technical reasons, it is usual to make the cavity length $L = \pi/k$, i.e. half the wavelength of radiation of frequency ω_{RF}.

The Hamiltonian in a TM_{010} RF Cavity

Using $L = \pi/k$, we can perform the integral in (32) and we find:

$$\langle H \rangle = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - p_x^2 - p_y^2 - \frac{1}{\beta_0^2\gamma_0^2}} - \frac{\alpha}{\pi} J_0(k\rho) \cos(\phi_0 - kz) \qquad (33)$$

where

$$\alpha = \pi \frac{q}{P_0} \frac{\hat{E}_s}{\omega_{\text{RF}}} T = \frac{q \hat{E}_s L}{P_0 c} T \qquad (34)$$

and the *transit time factor*, T is given by:

$$T = \frac{2\beta_0}{\pi} \sin \frac{\pi}{2\beta_0} \qquad (35)$$

Normally, we define the *cavity voltage*, $\hat{V}$ such that:

$$\frac{\hat{V}}{L} = \hat{E}_s T \qquad (36)$$

so:

$$\alpha = \frac{q\hat{V}}{P_0 c} \qquad (37)$$

Making the paraxial approximation, we find the Hamiltonian:

$$\begin{aligned} \langle H_2 \rangle \; = \; & \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{\alpha}{4\pi}\cos(\phi_0)k^2\left(x^2+y^2\right) - \\ & \frac{\alpha}{\pi}\sin(\phi_0)kz + \frac{\alpha}{2\pi}\cos(\phi_0)k^2z^2 + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \end{aligned} \qquad (38)$$

Note first the transverse focusing term: it is focusing in both the horizontal plane and the vertical plane simultaneously. This is something we could not achieve by the use of static magnetic fields. In this case, it arises from the azimuthal component of the magnetic field in the TM_{010} mode. To make use of it, we have to choose a phase ϕ_0 close to zero.

For an RF cavity, we will use the Hamiltonian in the paraxial approximation (38):

$$\begin{aligned} \langle H_2 \rangle \; = \; & \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{\alpha}{4\pi}\cos(\phi_0)k^2\left(x^2+y^2\right) - \\ & \frac{\alpha}{\pi}\sin(\phi_0)kz + \frac{\alpha}{2\pi}\cos(\phi_0)k^2z^2 + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \end{aligned}$$

Note next the appearance of a term linear in z: this will result in a change in the energy deviation independent of z, as long as the phase $\phi_0 \neq 0$ (and $\phi_0 \neq \pi$). This is the term that describes the acceleration of the particle.

Finally, note the term quadratic in z: this is the longitudinal focusing we were looking for.

Solving the equations of motion in the transverse plane, we find that the solutions have zeroth-order as well as first-order terms:

$$\vec{x}(L) = R \cdot \vec{x}(0) + \vec{m} \tag{39}$$

The transfer matrix R is given by:

$$R = \begin{pmatrix} \cos\psi_\perp & \frac{L}{\psi_\perp}\sin\psi_\perp & 0 & 0 & 0 & 0 \\ -\frac{\psi_\perp}{L}\sin\psi_\perp & \cos\psi_\perp & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\psi_\perp & \frac{L}{\psi_\perp}\sin\psi_\perp & 0 & 0 \\ 0 & 0 & -\frac{\psi_\perp}{L}\sin\psi_\perp & \cos\psi_\perp & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\psi_\parallel & \frac{1}{\beta_0^2\gamma_0^2}\frac{L}{\psi_\parallel}\sin\psi_\parallel \\ 0 & 0 & 0 & 0 & -\beta_0^2\gamma_0^2\frac{\psi_\parallel}{L}\sin\psi_\parallel & \cos\psi_\parallel \end{pmatrix} \tag{40}$$

where:

$$\psi_\perp = \sqrt{\frac{\pi\alpha\cos\phi_0}{2}} \qquad \psi_\parallel = \frac{\sqrt{\pi\alpha\cos\phi_0}}{\gamma_0\beta_0} \tag{41}$$

The zeroth-order transverse terms in the solutions to the equations of motion are all identically zero. The zeroth-order longitudinal terms are:

$$m_z = \frac{2}{\pi} L \sin^2\left(\frac{\psi_\parallel}{2}\right)\tan\phi_0 \tag{42}$$

$$m_\delta = \alpha\frac{\sin\psi_\parallel}{\psi_\parallel}\sin\phi_0 \tag{43}$$

For small α (high energy particle in a cavity with a weak field), the map for the energy error δ becomes:

$$\Delta\delta \approx \frac{q\hat{V}}{P_0 c}(\sin\phi_0 - kz_0\cos\phi_0) \tag{44}$$

where $z_0 = z(0)$.

Vector Potential and Hamiltonian for a Solenoid

Solenoids are important components in accelerators. For example, detectors in colliding beam machines usually sit inside strong solenoids. A solenoid has a uniform magnetic field in the longitudinal direction:

$$B_x = 0, \qquad B_y = 0, \qquad B_s = B_0. \tag{45}$$

It is not possible to derive this field from a vector potential having zero transverse components. A suitable potential is:

$$A_x = -\frac{1}{2}B_0 y, \qquad A_y = \frac{1}{2}B_0 x, \qquad A_s = 0. \tag{46}$$

This leads to the Hamiltonian:

$$H = \frac{\delta}{\beta_0} - \sqrt{\left(\frac{1}{\beta_0} + \delta\right)^2 - (p_x + k_s y)^2 - (p_y - k_s x)^2 - \frac{1}{\beta_0^2\gamma_0^2}} \tag{47}$$

where the normalised solenoid field strength k_s is given by:

$$k_s = \frac{1}{2}\frac{q}{P_0}B_0 \tag{48}$$

Second-Order Hamiltonian for a Solenoid

The fact that the vector potential has non-zero transverse components (unlike the other linear elements we have looked at) means that we have to be particularly careful with the meaning of the canonical momenta p_x and p_y. But let us proceed with solving the equations of motion in the Hamiltonian (47), which we do by making the usual paraxial approximation:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}k_s^2 x^2 + \frac{1}{2}k_s^2 y^2 - \frac{1}{2}k_s x p_y + \frac{1}{2}k_s p_x y + \frac{\delta^2}{2\beta_0^2\gamma_0^2} \tag{49}$$

Note the terms in x^2 and y^2: a solenoid provides horizontal and vertical focusing, rather than focusing in one plane and defocusing in the other. Note also the coupling terms in xp_y and $p_x y$: motion lying initially in just one plane becomes (at least partially) transferred into the other plane.

Transfer Matrix for a Solenoid

We can solve the equations of motion from the Hamiltonian (49). The resulting map can be expressed as a transfer matrix:

$$R = \begin{pmatrix} \cos^2 \omega L & \frac{\sin 2\omega L}{2\omega} & \frac{1}{2}\sin 2\omega L & \frac{\sin^2 \omega L}{\omega} & 0 & 0 \\ -\frac{\omega}{2}\sin 2\omega L & \cos^2 \omega L & -\omega \sin^2 \omega L & \frac{1}{2}\sin 2\omega L & 0 & 0 \\ -\frac{1}{2}\sin 2\omega L & -\frac{\sin^2 \omega L}{\omega} & \cos^2 \omega L & \frac{\sin 2\omega L}{2\omega} & 0 & 0 \\ \omega \sin^2 \omega L & -\frac{1}{2}\sin 2\omega L & -\frac{\omega}{2}\sin 2\omega L & \cos^2 \omega L & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{L}{\beta_0^2 \gamma_0^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{50}$$

where:

$$\omega = k_s = \frac{1}{2}\frac{q}{P_0}B_0 \tag{51}$$

Combined Function Magnets

Multipole fields can be superposed on each other. In the multipole field expansion:

$$B_y + iB_x = \sum_{n=1}^{\infty} (b_n + ia_n) \left(\frac{x + iy}{r_0} \right)^{n-1} \tag{52}$$

superposed fields are described by having more than one non-zero coefficient b_n and/or a_n. A magnet with superposed magnetic fields is generally called a "combined function" magnet. Examples of combined function magnets widely used in accelerators are dipoles (bending magnet) with superposed quadrupole fields, and sextupoles with superposed skew quadrupole fields. Generally, combined function magnets are used to help reduce the length (and therefore the cost) of a beamline, but they can also help to improve the dynamical properties of a lattice.

For linear dynamics, the most important combined function magnets are dipoles with superposed quadrupole fields. In Cartesian coordinates, the field is:

$$B_y = b_1 + b_2\frac{x}{r_0}, \quad B_x = b_2\frac{y}{r_0}, \quad B_z = 0. \tag{53}$$

In bending magnets, we generally want to use a curved reference trajectory; however, using curvilinear coordinates complicates the description of the magnetic field in a combined function bend.

The magnetic field in a combined function bend may be derived from the vector potential:

$$A_x = 0 \tag{54}$$

$$A_y = 0 \tag{55}$$

$$A_s = -B_0\left(x - \frac{hx^2}{2(1+hx)}\right) - B_1\left(\frac{1}{2}\left(x^2-y^2\right) - \frac{h}{6}x^3 + \frac{h^2}{24}\left(4x^4-y^4\right) + \cdots\right) \tag{56}$$

Note that the higher-order terms (x^3, x^4, y^4 etc.) arise from the curvature of the reference trajectory. The higher-order terms are important for nonlinear dynamics, but do not contribute to the linear effects.

Hamiltonian for a Combined Function Bend

Using the vector potential (55) in the Hamiltonian, and making the paraxial approximation (expanding to second order) we have:

$$H_2 = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + (k_0 - h)x + \frac{1}{2}(hk_0 + k_1)x^2 - \frac{1}{2}k_1 y^2 - \frac{h}{\beta_0}x\delta - \frac{\delta^2}{2\beta_0^2\gamma_0^2} \tag{57}$$

where the normalised field strengths are defined as usual:

$$k_0 = \frac{q}{P_0}b_1, \qquad k_1 = \frac{q}{P_0}\frac{b_2}{r_0} \tag{58}$$

The effect of the superposed gradient k_1 in the Hamiltonian is as expected: it simply provides additional transverse focusing.

Dynamical Map for a Combined Function Bend

Hamilton's equations with the Hamiltonian (56) can be solved. In the horizontal plane, the solutions are:

$$x(s) = x(0)\cos\omega_x s + p_x(0)\frac{\sin\omega_x s}{\omega_x} + \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{(1-\cos\omega_x s)}{\omega_x^2} \tag{59}$$

$$p_x(s) = -x(0)\omega_x\sin\omega_x s + p_x(0)\cos\omega_x s + \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{\sin\omega_x s}{\omega_x} \tag{60}$$

where:

$$\omega_x = \sqrt{hk_0 + k_1} \tag{61}$$

In the vertical plane, the map for the combined function bend is:

$$y(s) = y(0)\cosh\omega_y s + p_y(0)\frac{\sinh\omega_y s}{\omega_y} \quad (62)$$

$$p_y(s) = y(0)\omega_y\sinh\omega_y s + p_y(0)\cosh\omega_y s \quad (63)$$

where

$$\omega_y = \sqrt{k_1} \quad (64)$$

The map in the vertical plane for a combined function bend is the same as for a quadrupole: the only focusing in the vertical plane comes from the quadrupole gradient.

In the longitudinal plane, the solutions are:

$$\begin{aligned} z(s) &= z(0) - x(0)\frac{h}{\beta_0}\frac{\sin\omega_x s}{\omega_x} - p_x(0)\frac{h}{\beta_0}\frac{(1-\cos\omega_x s)}{\omega_x^2} + \delta(0)\frac{s}{\beta_0^2\gamma_0^2} \\ &\quad - \left(\delta(0)\frac{h}{\beta_0} + h - k_0\right)\frac{h}{\beta_0}\frac{(\omega_x s - \sin\omega_x s)}{\omega_x^3} \end{aligned} \quad (65)$$

$$\delta(s) = \delta(0) \quad (66)$$

A Word About Fringe Fields

So far, we have only considered the dynamics of a particle *within* a given electromagnetic field: we have not thought about how to get particles in and out of the fields. For example, Maxwell's equations forbid us from moving abruptly from a drift (field-free) region into a multipole or solenoid field. There has to be some "transition region" within which there are non-zero fields that are not described by the usual multipole formulae. The transition regions at either end of a magnet are usually called the "fringe fields".

Fringe fields have significant, and sometimes complicated, effects. For linear dynamics, the most important fringe fields are those at the ends of dipoles and solenoids. Fringe fields at the ends of quadrupoles lead to (usually small) higher-order effects.

A Word About Fringe Fields

The precise effects of fringe fields depend on the design details of the magnet, e.g. the gap between the poles in a dipole. To do things properly, one should construct the dynamical map from a detailed field description. This often requires significant effort, and the techniques involved are beyond the scope of this course. However, in many cases, we can make simple approximations that provide a good description of the gross effects. These approximations are one of the topics covered in the next lecture.

Summary

We have now derived linear dynamical maps for:

- separated and combined function dipoles
- solenoids
- normal and skew quadrupoles
- RF cavities

For each of these elements, we made the *paraxial approximation* by expanding the Hamiltonian to second order in the dynamical variables. This allowed us to find a linear map for each element. The linear map may be expressed as a transfer matrix.