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SINGLE PARTICLE DYNAMICS AND NONLINEAR RESONANCES IN CIRCULAR ACCELERATORS*

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1. INTRODUCTION

The purpose of this paper is to introduce the reader to single particle dynamics in circular accelerators with an emphasis on nonlinear resonances. In several sections we follow Ref. 1 closely although the treatment given here is in some cases more general.

We begin with the Hamiltonian and the equations of motion in the neighborhood of the design orbit. In the linear theory this yields linear *betatron oscillations* about a *closed orbit.* It is useful then to introduce the action-angle variables of the linear problem.

Next we discuss the nonlinear terms which are present in an actual accelerator, and in particular, we motivate the inclusion of sextupoles to cure chromatic effects. To study the effects of the nonlinear terms, we next discuss canonical perturbation theory which leads us to nonlinear resonances. After showing a few examples of perturbation theory, we abandon it when very close to a resonance.

This leads to the study of an isolated resonance in one degree of freedom with a 'time'-dependent Hamiltonian. We see the familiar resonance structure in phase space which is simply closed islands when the nonlinear amplitude dependence of the frequency or 'tune' is included. To show the limits of the validity of the isolated resonance approximation, we discuss two criteria for the onset of chaotic motion.

Finally, we study an isolated coupling resonance in two degrees of freedom with a 'time'-dependent Hamiltonian and calculate the two invariants in this case. This leads to a surface of section which is a 2-torus in 4-dimensional phase space. However, we show that it remains a 2-torus when projected into particular 3-dimensional subspaces and thus can be viewed in perspective.

2. THE MOTION OF A PARTICLE IN AN ACCELERATOR

2.1 THE HAMILTONIAN AND THE EQUATIONS OF MOTION

The motion of a particle in a circular accelerator is governed by the Lorentz force equation,

$$\frac{d\mathbf{P}}{dt} = e\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \ , \tag{2.1}$$

where $\mathbf{P}$ is the relativistic kinetic momentum and $\mathbf{v}$ is the velocity. It is convenient to cast these equations in Hamiltonian form. If we introduce the vector and scalar potentials,

$$\left\{\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A} \ , \end{aligned}\right. \tag{2.2}$$

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then the Hamiltonian is given by

$$\boxed{H = e\phi + c\,[m^2c^2 + (\mathbf{p} - e\mathbf{A}/c)^2]^{1/2}} \ , \tag{2.3}$$

where $\mathbf{p}$ is the canonical momentum. In terms of the kinetic momentum and the vector potential

$$\boxed{\mathbf{p} = \mathbf{P} + \frac{e}{c}\mathbf{A}(\mathbf{x}, t)} \ . \tag{2.4}$$

The equations of motion can then be written in terms of Hamilton's equations,

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{x}} \ , \quad \frac{d\mathbf{x}}{dt} = \frac{\partial H}{\partial \mathbf{p}} \ . \tag{2.5}$$

It is useful to use a coordinate system based on a closed planar reference curve as shown in Fig. 1.1. This reference curve is taken to be the closed trajectory of a particle with some reference momentum p_0 in the guiding magnetic field. The coordinate system (x, s, y) is similar to a cylindrical system, however, the radius of curvature may vary along the curve. If $\mathbf{r}$ is the coordinate of a particle in space, and $\mathbf{r}_0$ is the point on the reference curve closest to $\mathbf{r}$, then

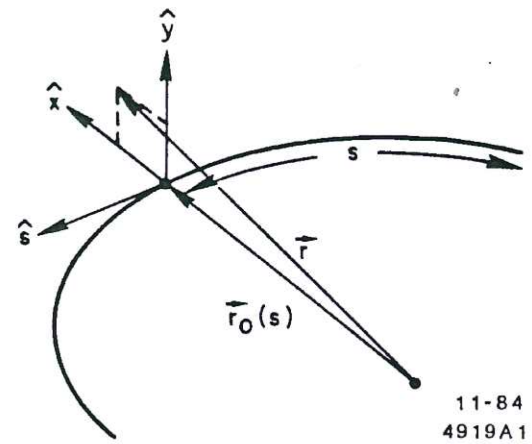

Fig. 1.1. The coordinate system.

s = distance along the curve to the point $\mathbf{r}_0$ from a fixed origin somewhere on the curve,
x = horizontal projection of the vector $\mathbf{r} - \mathbf{r}_0$,
y = vertical projection of the vector $\mathbf{r} - \mathbf{r}_0$,
ρ = local radius of curvature.

The Hamiltonian written in terms of these coordinates is[2]

$$H = e\phi + c\left[m^2c^2 + \frac{(p_s - \frac{e}{c}A_s)^2}{(1 + \frac{x}{\rho})^2} + \left(p_x - \frac{e}{c}A_x\right)^2 + \left(p_y - \frac{e}{c}A_y\right)^2\right]^{1/2} \tag{2.6}$$

where p_x and p_y are projections of $\mathbf{p}$ onto the x and y direction and

$\left\{\begin{array}{l} P_x = eB_0 r_c \\ P_{x0} = eB_0 R_0 \end{array}\right.$ $\quad \delta \equiv \frac{P_x - P_{x0}}{P_x} = \frac{x}{r_0}$

$r_c = r_0 + x$

$$p_s = (\mathbf{p} \cdot \hat{s})\left(1 + \frac{x}{\rho}\right) \cong (\vec{p} \cdot \hat{s})(1 + \delta) \quad \text{off-momentum particle} \tag{2.7}$$

We will call the vector potential used in Eq. (2.6) the *canonical* vector potential since A_s, A_x, and A_y are defined analogously to the canonical momenta. In particular note that

$$A_s = (\mathbf{A} \cdot \hat{s})\left(1 + \frac{x}{\rho}\right) \ . \tag{2.8}$$

Instead of using the Hamiltonian above, it is useful to change the independent variable to s rather than t. This can be done provided that s is monotonic in t. This is a standard transformation and can be accomplished by defining another Hamiltonian,

$$\boxed{\mathcal{H} \equiv -p_s(x, p_x, y, p_y, t, -H)} . \tag{2.9}$$

That is, we solve Eq. (2.6) for p_s. With this new Hamiltonian and new independent variable, Hamilton's equations become

$$\left\{ \begin{aligned} \frac{dx}{ds} &= \frac{\partial \mathcal{H}}{\partial p_x}, & \frac{dp_x}{ds} &= -\frac{\partial \mathcal{H}}{\partial x} \\ \frac{dy}{ds} &= \frac{\partial \mathcal{H}}{\partial p_y}, & \frac{dp_y}{ds} &= -\frac{\partial \mathcal{H}}{\partial y} \\ \frac{dt}{ds} &= \frac{\partial \mathcal{H}}{\partial(-H)}, & \frac{d(-H)}{ds} &= -\frac{\partial \mathcal{H}}{\partial t} \end{aligned} \right. . \tag{2.10}$$

(s, p_s)

Note that $(t, -H)$ now play the role of the third coordinate and conjugate momentum.

To be specific we will specialize to the case of no electric field and a constant magnetic field given by

$$\left\{ \begin{aligned} B_y &= -B_0(s) + B_1(s)\, x + \cdots \\ B_x &= B_1(s)\, y + \cdots \end{aligned} \right. . \tag{2.11}$$

The main bending field $B_0(s)$ is chosen so that a particle at the reference momentum p_0 will bend with a local radius of curvature $\rho(s)$. Thus, we set

$$B_0(s) = \frac{p_0 c}{e\rho(s)} . \tag{2.12}$$

$B_1(s)$ in Eq. (2.11) is simply the gradient of the magnetic field. It is conventional and useful to scale the gradient to obtain the focusing function,

$$K_1(s) = \frac{eB_1(s)}{p_0 c} . \tag{2.13}$$

Using Eqs. (2.12) and (2.13) the canonical vector potential which yields the above magnetic field is

$$A_s = -\frac{p_0 c}{e}\left[\frac{x}{\rho} + \left(\frac{1}{\rho^2} - K_1\right)\frac{x^2}{2} + \frac{K_1 y^2}{2}\right] + \cdots . \tag{2.14}$$

The new Hamiltonian from Eq. (2.9) is $(A_x = A_y = 0 \;;\; \phi = 0)$

$$\mathcal{H} = (-p_s) = \frac{-e\,A_s}{c} - \left(1 + \frac{x}{\rho}\right)\left[\frac{H^2}{c^2} - m^2c^2 - p_x^2 - p_y^2\right]^{1/2} . \tag{2.15}$$

Since there is no time dependence, H is a constant of the motion which we call E (the energy). In an actual accelerator the magnetic fields do change in time, and there are longitudinal electric fields to accelerate the particles. However, the acceleration process is slow and can be considered adiabatic for our purposes. In addition, the longitudinal electric fields cause longitudinal oscillations which are omitted here. These are discussed in Ref. 3 in these proceedings.

To continue we expand the square root in Eq. (2.15) and substitute the vector potential from Eq. (2.14) to obtain

$$\boxed{\mathcal{H} = (p_0 - p)\frac{x}{\rho} + p_0\left[\left(\frac{1}{\rho^2} - K_1\right)\frac{x^2}{2} + K_1\frac{y^2}{2}\right] + \frac{p_x^2}{2p} + \frac{p_y^2}{2p} + \cdots} \ , \tag{2.16}$$

where p is the total kinetic momentum of the particle,

$$p = [E^2/c^2 - m^2c^2]^{1/2} \ , \tag{2.17}$$

which may be somewhat different from the reference momentum. The expansion of the square root is a good approximation provided that

$$\left|\frac{p_{x,y}}{p}\right| \ll 1 \ , \tag{2.18}$$

which is typically the case. From Hamilton's equations and the Hamiltonian in Eq. (2.16) we find

$$\begin{aligned}
\frac{dx}{ds} &= \frac{p_x}{p} \quad , \quad \frac{dp_x}{ds} = -p_0\left(\frac{1}{\rho^2} - K_1\right)x + \frac{(p - p_0)}{\rho} \\
\frac{dy}{ds} &= \frac{p_{\cancel{x}y}}{p} \quad , \quad \frac{dp_y}{ds} = -p_0 K_1 y \ .
\end{aligned} \tag{2.19}$$

In terms of x and y Eqs. (2.19) become

$$\boxed{\begin{aligned}
x'' + \frac{p_0}{p}\left(\frac{1}{\rho^2} - K_1\right)x &= \frac{p - p_0}{p}\frac{1}{\rho} \\
y'' + \frac{p_0 K_1}{p}y &= 0 \quad ,
\end{aligned}} \tag{2.20}$$

where prime denotes differentiation with respect to s. Equations (2.20) yield the motion of particles near the reference orbit. Because K_1 and ρ are *periodically* dependent on s with period C , the circumference, these equations are Hill's equations.

2.2 Betatron Oscillations

Before proceeding to discuss the nonlinear terms which have so far been neglected, it is useful to discuss the linear equations of motion. Since Eqs. (2.20) are inhomogeneous, we construct a general solution by a linear combination of a particular solution of the inhomogeneous equation and the general solution of the homogeneous equation. It is conventional and useful to take the particular solution to be the periodic solution or *closed orbit*.

Let us assume that we have this periodic solution to Eq. (2.20), and let us denote it by $[x_\epsilon(s), p_\epsilon(s)]$. The periodic solution in the y direction is simply $y = 0$. (In the presence of errors the vertical closed orbit is nonzero and must also be calculated.) Now perform a canonical transformation which shifts the origin of phase space to (x_ϵ, p_ϵ). The transformation $(x, p) \mapsto (x_\beta, p_\beta)$ can be performed with the generating function

$$F_2(x, p_\beta) = (x - x_\epsilon(s))(p_\beta + p_\epsilon(s)) \ , \tag{2.21}$$

which yields the transformation equations

$$\begin{aligned}
x &= x_\beta + x_\epsilon(s) \\
p &= p_\beta + p_\epsilon(s) \\
\mathcal{H}_\beta &= \mathcal{H} + \partial F_2/\partial s \ ,
\end{aligned} \tag{2.22}$$

where the identity transformation for y and p_y has been suppressed. Substituting into $\mathcal{H}$, the new Hamiltonian is given by

$$\mathcal{H}_\beta = p_0 \left[\left(\frac{1}{\rho^2} - K_1 \right) \frac{x_\beta^2}{2} + K_1 \frac{y^2}{2} \right] + \frac{p_\beta^2}{2p} + \frac{p_y^2}{2p} + \cdots \quad . \tag{2.23}$$

Thus, we are left a Hamiltonian with terms which are quadratic and higher order. In the nonlinear case a similar transformation can be performed; however, in this case we must use the periodic solutions to the full nonlinear equations.

The linear differential equations which are obtained from the Hamiltonian in Eq. (2.23) are of the form

$$z'' + K(s)z = 0 \ , \tag{2.24}$$

with

$$K(s) = K(s + C) \ , \tag{2.25}$$

where z stands for either x_β or y, and C is the circumference. The periodicity of K is that of the closed orbit; however, there may also be stronger periodicity imposed by design.

Equation (2.24) is Hill's equation and has a solution of the form

$$z = A\beta^{1/2} \cos(\psi(s) + \delta) \ , \tag{2.26}$$

where

$$\psi = \int_0^s \frac{ds'}{\beta(s')} \ , \tag{2.27}$$

and $\beta(s)$, the Courant-Snyder amplitude function,[2] is the periodic solution of

$$\beta''' + 4K\beta' + 2K'\beta = 0 \ , \tag{2.28}$$

with the additional condition

$$\beta\beta''/2 - (\beta')^2/4 + K\beta^2 = 1 \ . \tag{2.29}$$

Both A and δ are constants.

This solution is well known and constitutes a pseudo-harmonic oscillation with a periodically varying amplitude and wavelength. This motion is called *betatron oscillations* after the early betatron accelerators although in that case the transverse equations of motion reduced to two simple harmonic oscillator equations.

For stability, the tune ν,

$$\nu \equiv \frac{1}{2\pi} \int_0^C \frac{ds}{\beta(s)} \ , \tag{2.30}$$

must be non-integer. In the case of piecewise constant K, it is useful to use a matrix mapping technique to calculate both ν and $\beta(s)$.[2] This technique is used extensively in the design of magnetic lattices for circular accelerators.