A SHORT DEMONSTRATION OF LIOUVILLE'S THEOREM*

M. Weiss
CERN, Geneva, Switzerland

ABSTRACT
A brief demonstration of Liouville's Theorem is given by applying the Hamiltonian.

An ensemble of particles evolving in a system of external forces (space and velocity dependent) and self forces (space charge) is described by two families of canonically conjugated variables (coordinates) q and p. The equations of motion form a system of first-order differential equations of the coordinates

$$\dot{q} \text{ and } \dot{p} ,$$

where the dot indicates derivatives with respect to time.

If the system is non-dissipative, one can obtain the equations of motion from a function called Hamiltonian:

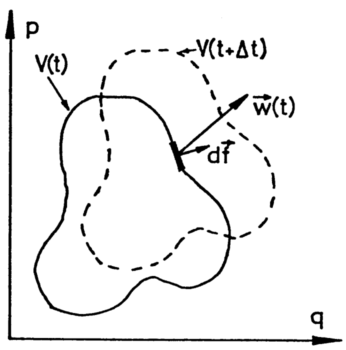
$$\dot{q} = \frac{\partial H}{\partial p}$$

$$\dot{p} = - \frac{\partial H}{\partial q} .$$

The Hamiltonian is in general also a function of time:

$$H(q,p,t) .$$

An ensemble of particles, at a given moment t, occupies a volume V(t) in the (q,p) space called the phase space.



$d\vec{f}$ ... vector of surface element

$\vec{w}(t)$ ... phase space velocity of surface element:

$$\vec{w} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

* Derivation shown at a discussion session

At the time $t + \Delta t$, the particles occupy another volume $V(t+\Delta t)$. It can easily be shown that these volumes are the same:

$$\frac{dV(t)}{dt} = \int \vec{w}\cdot d\vec{f} = \int (\nabla\cdot\vec{w})\,dv = \int \left(\frac{\partial}{\partial q}\dot{q} + \frac{\partial}{\partial p}\dot{p}\right) dv = 0$$

surface integral → volume integral

(Gauss Theorem)

$$\left(\frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q}\right) = 0$$

(Hamilton)

The volume $V(t)$ remains constant, if the motion can be represented by a Hamiltonian. This is true also when H is an explicit function of time. We conclude:

In non-dissipative systems, the particles move like an incompressible fluid in <u>phase space</u>. This is Liouville's Theorem.