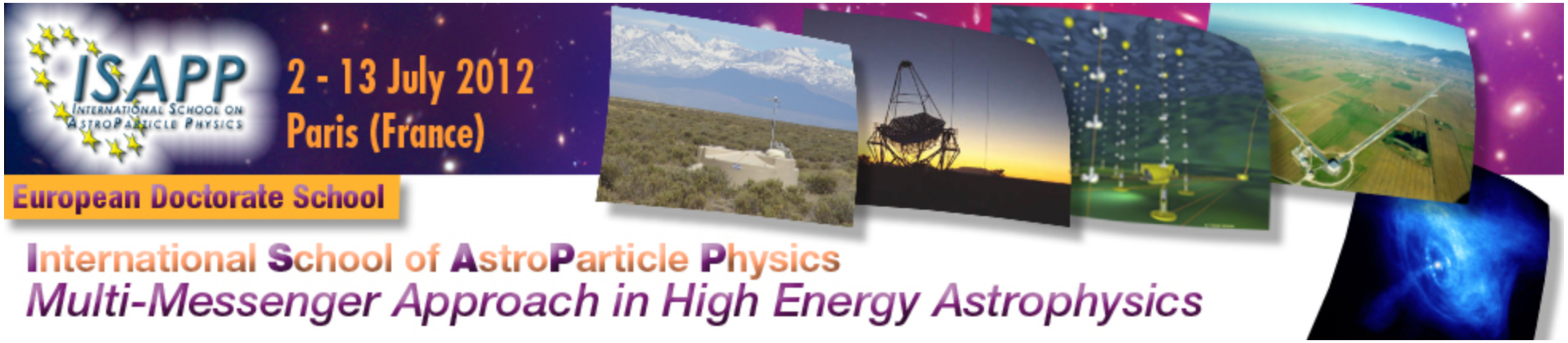
Astrofisica Nucleare e Subnucleare

Radiation Processes



Programme

Introductory courses

- From particle physics to astroparticle physics (3h) : Pierre Salati (slides 1, 2)
- Radiation mechanisms (3h) : Malcolm S. Longair (slides : 1, 2, 3)
- Multi-messenger astronomy (3h) : Jacques Paul (slides : 1, 2)
- High energy showers (4h) : Ralph Engel (slides : 1, 2, 3)

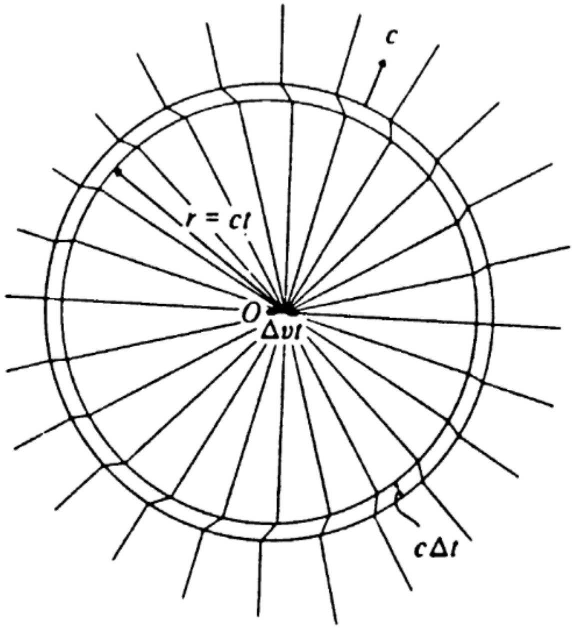
Basic Radiation Concepts

Much of what we need to understand radiation processes in X-ray and γ-ray astronomy can be derived using classical electrodynamics and central to that development is the physics of the radiation of accelerated charged particles. The central relation is the *radiation loss rate of an accelerated charged particle* in the non-relativistic limit

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} = \frac{|\ddot{\boldsymbol{p}}|^2}{6\pi\varepsilon_0 c^3} = \frac{q^2|\ddot{\boldsymbol{r}}|^2}{6\pi\varepsilon_0 c^3}. \tag{1}$$

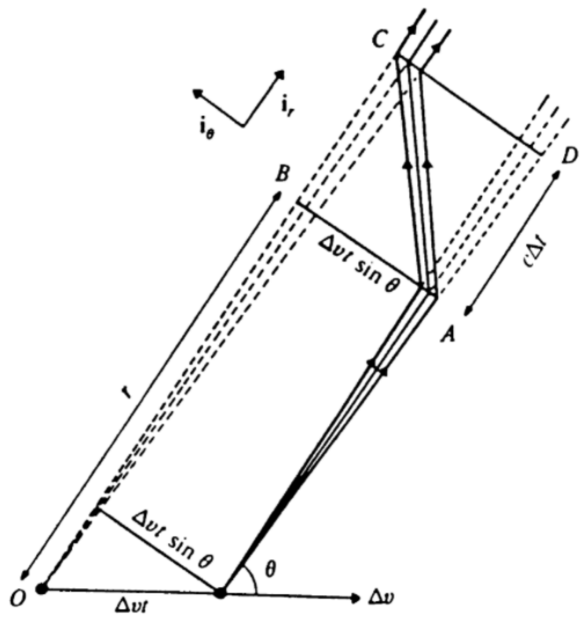
$\boldsymbol{p} = q\boldsymbol{r}$ is the *dipole moment* of the accelerated electron with respect to some origin. This formula is very closely related to the radiation rate of a dipole radio antenna and so is often referred to as the radiation loss rate for *dipole radiation*. Note that I will use *SI units* in all the derivations, although it will be necessary to convert the results into the conventional units used in X-ray and γ-ray astronomy when they are confronted with observations. Thus, I will normally use metres, kilograms, teslas and so on.

The radiation of an accelerated charged particle
J.J. Thomson's treatment (1906, 1907)



Consider a charge q stationary at the origin O of some inertial frame of reference S at time $t = 0$. The charge then suffers a small acceleration to velocity Δv in the short time interval Δt. After a time t, we can distinguish between the field configuration inside and outside a sphere of radius $r = ct$ centred on the origin of S. Outside this sphere, the field lines do not yet know that the charge has moved away from the origin and so the field lines are radial, centred on O. Inside this sphere, the field lines are radial about the origin of the frame of reference centred on the moving charge. Between these two regions, there is a thin shell of thickness $c\,\Delta t$ in which we join up corresponding electric field lines.

Radiation of an accelerated charged particle (2)



There must be a component of the electric field in the $\boldsymbol{i}_\theta$ direction. This 'pulse' of electromagnetic field is propagated away from the charge at the speed of light and is the energy loss of the accelerated charged particle.

The increment in velocity Δv is very small, $\Delta v \ll c$, and therefore it can be assumed that the field lines are radial at $t = 0$ and also at time t in the frame of reference S.

Consider a small cone of electric field lines at angle θ with respect to the acceleration vector of the charge at $t = 0$ and at some later time t when the charge is moving at a constant velocity Δv. We join up electric field lines through the thin shell of thickness $c\,\mathrm{d}t$ as shown in the diagram.

The radiation of an accelerated charged particle (3)

The strength of the E_θ component of the field is given by number of field lines per unit area in the $\boldsymbol{i}_\theta$ direction. From the geometry of the diagram, the E_θ field component is given by the relative sizes of the sides of the rectangle $ABCD$, that is

$$E_\theta / E_r = \Delta v \, t \sin\theta / c\Delta t. \tag{2}$$

E_r is given by Coulomb's law,

$$E_r = q/4\pi\varepsilon_0 r^2 \quad \text{where} \quad r = ct, \tag{3}$$

and so

$$E_\theta = \frac{q(\Delta v/\Delta t)\sin\theta}{4\pi\varepsilon_0 c^2 r}. \tag{4}$$

$\Delta v/\Delta t$ is the acceleration $\ddot{r}$ of the charge and hence

$$E_\theta = \frac{q\ddot{r}\sin\theta}{4\pi\varepsilon_0 c^2 r}. \tag{5}$$

The radiation of an accelerated charged particle (4)

Notice that the radial component of the field decreases as r^{-2}, according to Coulomb's law, but the field in the pulse decreases only as r^{-1} because the field lines become more and more stretched in the E_θ-direction, as can be seen from (2). Alternatively we can write $p = qr$, where p is the dipole moment of the charge with respect to some origin, and hence

$$E_\theta = \frac{\ddot{p}\sin\theta}{4\pi\varepsilon_0 c^2 r}. \tag{6}$$

This is a pulse of electromagnetic radiation and hence the energy flow per unit area per second at distance r is given by the Poynting vector $\boldsymbol{E} \times \boldsymbol{H} = E^2/Z_0$, where $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$ is the impedance of free space. The rate loss of energy through the solid angle $\mathrm{d}\Omega$ at distance r from the charge is therefore

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} \mathrm{d}\Omega = \frac{|\ddot{\boldsymbol{p}}|^2 \sin^2\theta}{16\pi^2 Z_0 \varepsilon_0^2 c^4 r^2} r^2 \mathrm{d}\Omega = \frac{|\ddot{\boldsymbol{p}}|^2 \sin^2\theta}{16\pi^2 \varepsilon_0 c^3} \mathrm{d}\Omega. \tag{7}$$

The radiation of an accelerated charged particle (5)

To find the total radiation rate, we integrate over all solid angles, that is, we integrate over θ with respect to the direction of the acceleration. Integrating over solid angle means integrating over $\mathrm{d}\Omega = 2\pi \sin\theta \, \mathrm{d}\theta$ and so

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} = \int_0^{\pi} \frac{|\ddot{\boldsymbol{p}}|^2 \sin^2\theta}{16\pi^2 \varepsilon_0 c^3} 2\pi \sin\theta \, \mathrm{d}\theta. \tag{8}$$

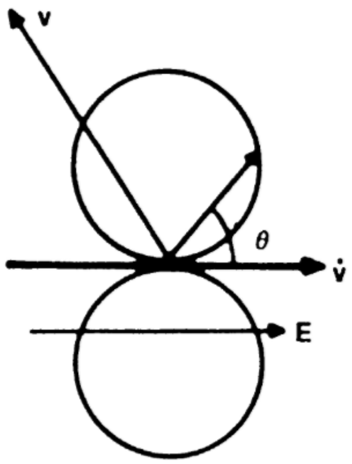
We find the key result

$$\boxed{-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rad}} = \frac{|\ddot{\boldsymbol{p}}|^2}{6\pi\varepsilon_0 c^3} = \frac{q^2|\ddot{\boldsymbol{r}}|^2}{6\pi\varepsilon_0 c^3}.} \tag{9}$$

This result is sometimes called *Larmor's formula* – precisely the same result comes out of the full theory. These formulae embody the three essential properties of the radiation of an accelerated charged particle.

The Properties of Dipole Radiation

$$-\left(\frac{dE}{dt}\right)_{\text{rad}} = \frac{q^2|\ddot{\boldsymbol{r}}|^2}{6\pi\varepsilon_0 c^3}\,.$$



1. The total radiation rate is given by Larmor's formula. The acceleration is the *proper acceleration* of the particle and the loss rate is measured in its instantaneous rest frame.
2. The *polar diagram* of the radiation is of *dipolar* form, that is, the electric field strength varies as $\sin\theta$ and the power radiated per unit solid angle varies as $\sin^2\theta$, where θ is the angle with respect to the acceleration vector of the particle. Notice that there is no radiation along the acceleration vector and the field strength is greatest at right angles to the acceleration vector.
3. The radiation is *polarised* with the electric field vector lying in the direction of the acceleration vector of the particle, as projected onto a sphere at distance r from the charged particle.

A useful relativistic invariant

The energy loss rate by radiation $\mathrm{d}E/\mathrm{d}t$ is a Lorentz invariant between inertial frames.

Expert version. The total energy emitted in the form of radiation is the 'time' component of the momentum four-vector $[E/c, \boldsymbol{p}]$ and $\mathrm{d}t$ is the time component of the displacement four-vector $[\mathrm{d}t, \mathrm{d}\boldsymbol{r}]$. Therefore, both the energy $\mathrm{d}E$ and the time interval $\mathrm{d}t$ transform in the same way between inertial frames of reference and so their ratio $\mathrm{d}E/\mathrm{d}t$ is also an invariant.

Gentler version. In the *instantaneous rest frame* of the accelerated charged particle, dipole radiation is emitted with zero net momentum, as may be seen from the polar diagram of dipole radiation. Therefore its four-momentum can be written $[\mathrm{d}E'/c, 0]$. This radiation is emitted in the interval of proper time $\mathrm{d}t'$ which has four vector $[\mathrm{d}t', 0]$. We may now use the inverse Lorentz transformation to relate $\mathrm{d}E'$ and $\mathrm{d}t'$ to $\mathrm{d}E$ and $\mathrm{d}t$.

$$\mathrm{d}E = \gamma\,\mathrm{d}E' \qquad \mathrm{d}t = \gamma\,\mathrm{d}t' \,, \tag{28}$$

and hence

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E'}{\mathrm{d}t'} \,.} \tag{29}$$

Radiation of Accelerated Relativistic Electron

We can derive from this result the radiation rate as observed by the external observer who measures the velocity and acceleration of the electron to be $\boldsymbol{a}$ and $\boldsymbol{v}$ respectively, the proper acceleration measured in the instantaneous rest frame of the electron being $\boldsymbol{a}_0$. Then, from the above results,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}E'}{\mathrm{d}t'} = \frac{e^2|\boldsymbol{a}_0|^2}{6\pi\varepsilon_0 c^3}. \tag{30}$$

To relate $\boldsymbol{a}_0$, $\boldsymbol{a}$ and $\boldsymbol{v}$, it is simplest to equate the norms of the four-accelerations of the accelerated electron in the frames S and S′. I leave it as an exercise to the reader to show that

$$\boldsymbol{a}_0^2 = \gamma^4 \left[\boldsymbol{a}^2 + \gamma^2 \left(\frac{\boldsymbol{v}\cdot\boldsymbol{a}}{c}\right)^2\right] \tag{31}$$

and so

$$\boxed{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{in S}} = \frac{e^2\gamma^4}{6\pi\varepsilon_0 c^3} \left[\boldsymbol{a}^2 + \gamma^2 \left(\frac{\boldsymbol{v}\cdot\boldsymbol{a}}{c}\right)^2\right] .} \tag{32}$$

Radiation of Accelerated Relativistic Electron (2)

Another useful exercise is to resolve $\boldsymbol{a}$ parallel and perpendicular to $\boldsymbol{v}$ so that

$$\boldsymbol{a} = a_{\parallel}\boldsymbol{i}_{\parallel} + a_{\perp}\boldsymbol{i}_{\perp} \tag{33}$$

and then to show that the radiation rate is

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\text{in S}} = \frac{e^2\gamma^4}{6\pi\varepsilon_0 c^3}\left(|a_{\perp}|^2 + \gamma^2|a_{\parallel}|^2\right) \; . \tag{34}$$

I have shown how these relations are obtained in *HEA3*. This is a useful expression for obtaining the loss rate due to synchrotron radiation very quickly (see later).

Parseval's theorem and the spectral distribution of the radiation of an accelerated electron

We need to be able to decompose the radiation field of the electron into its spectral components. *Parseval's theorem* provides an elegant method of relating the dynamical history of the particle to its radiation spectrum. We introduce the Fourier transform of the acceleration of the particle through the Fourier transform pair, which I write in symmetrical form:

$$\dot{\boldsymbol{v}}(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \dot{\boldsymbol{v}}(\omega) \exp(-\mathrm{i}\omega t)\, \mathrm{d}\omega \quad (35)$$

$$\dot{\boldsymbol{v}}(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \dot{\boldsymbol{v}}(t) \exp(\mathrm{i}\omega t)\, \mathrm{d}t. \quad (36)$$

According to Parseval's theorem, $\dot{\boldsymbol{v}}(\omega)$ and $\dot{\boldsymbol{v}}(t)$ are related by the following integrals:

$$\int_{-\infty}^{\infty} |\dot{\boldsymbol{v}}(\omega)|^2\, \mathrm{d}\omega = \int_{-\infty}^{\infty} |\dot{\boldsymbol{v}}(t)|^2\, \mathrm{d}t. \quad (37)$$

This is proved in all textbooks on Fourier anlaysis.

Parseval's Theorem

We can therefore apply this relation to the energy radiated by a particle which has an *acceleration history* $\dot{\boldsymbol{v}}(t)$,

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}E}{\mathrm{d}t} = \int_{-\infty}^{\infty} \frac{e^2}{6\pi\varepsilon_0 c^3} |\dot{\boldsymbol{v}}(t)|^2 \, \mathrm{d}t = \int_{-\infty}^{\infty} \frac{e^2}{6\pi\varepsilon_0 c^3} |\dot{\boldsymbol{v}}(\omega)|^2 \, \mathrm{d}\omega. \tag{38}$$

Now, what we really want is $\int_0^{\infty} \ldots \mathrm{d}\omega$ rather than $\int_{-\infty}^{\infty} \ldots \mathrm{d}\omega$. Since the acceleration is a real function, another theorem in Fourier analysis tells us that

$$\int_0^{\infty} |\dot{\boldsymbol{v}}(\omega)|^2 \, \mathrm{d}\omega = \int_{-\infty}^{0} |\dot{\boldsymbol{v}}(\omega)|^2 \, \mathrm{d}\omega \tag{39}$$

and hence we find

$$\text{Total emitted radiation} = \int_0^{\infty} I(\omega) \, \mathrm{d}\omega = \int_0^{\infty} \frac{e^2}{3\pi\varepsilon_0 c^3} |\dot{\boldsymbol{v}}(\omega)|^2 \, \mathrm{d}\omega. \tag{40}$$

Parseval's Theorem

Therefore

$$I(\omega) = \frac{e^2}{3\pi\varepsilon_0 c^3}|\dot{v}(\omega)|^2. \quad (41)$$

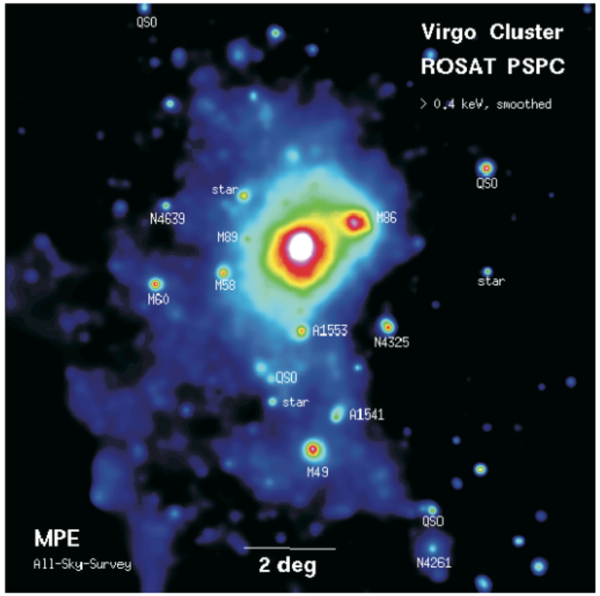
Note that this is the total energy per unit bandwidth emitted throughout the period during which the particle is accelerated. For a distribution of particles, this result must be integrated over all the particles contributing to the radiation at frequency ω.

This is also a very good result which often gives physical insight into the shape of the spectrum of emitted radiation.

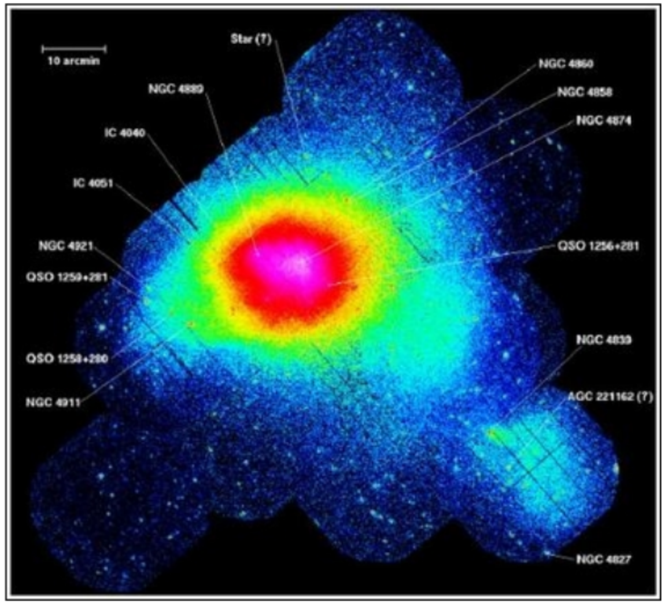
Bremsstrahlung

Bremsstrahlung is the radiation associated with the acceleration of electrons in the electrostatic fields of ions and the nuclei of atoms. In X-ray and γ-ray astronomy, the most important cases are those in which bremmstrahlung is emitted by very hot plasmas at $T \geq 10^6$ K, at which temperatures the hydrogen and helium atoms are fully ionised. We use the tools already introduced to derive classically the expressions for the bremsstrahlung emissivity of a hot plasma.

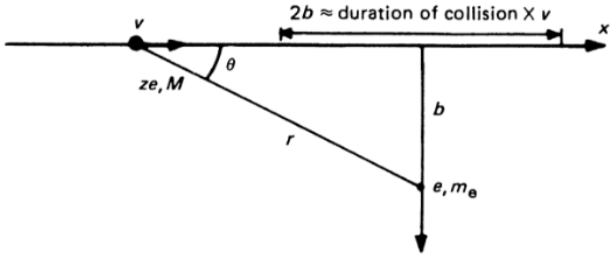
Virgo Cluster in X-rays (ROSAT)



Coma Cluster in X-rays (ROSAT)



Encounters between Charged Particles



The charge of the high energy particle is ze and its mass M.

Let us first study the 'collision' of a high energy proton or nucleus with the electrons of a fully ionised plasma. It is assumed that the nucleus is undeviated in the encounter with the electron; b, the distance of closest approach of the particle to the electron, is called the *collision parameter* of the interaction.

The total *momentum impulse* given to the electron in the encounter is $\int F\,\mathrm{d}t$. By symmetry, the forces parallel to the line of flight of the high energy particle cancel out and so we need only work out the component of force perpendicular to the line of flight.

$$F_{\perp} = \frac{ze^2}{4\pi\varepsilon_0 r^2}\sin\theta \quad ; \quad \mathrm{d}t = \frac{\mathrm{d}x}{v}. \qquad (42)$$

Encounters between Charged Particles

Changing variables to θ, $b/x = \tan\theta$, $r = b/\sin\theta$ and therefore

$$\mathrm{d}x = (-b/\sin^2\theta)\,\mathrm{d}\theta\ . \tag{43}$$

v is effectively constant and therefore

$$\int_{-\infty}^{\infty} F_{\perp}\,\mathrm{d}t = -\int_0^{\pi} \frac{ze^2}{4\pi\varepsilon_0 b^2}\sin^2\theta \frac{b\sin\theta}{v\sin^2\theta}\,\mathrm{d}\theta = -\frac{ze^2}{4\pi\varepsilon_0 bv}\int_0^{\pi}\sin\theta\,\mathrm{d}\theta \tag{44}$$

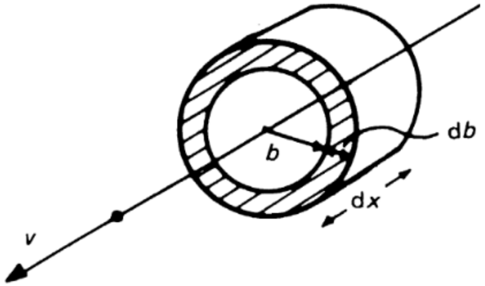
Therefore

$$\text{momentum impulse } p = \frac{ze^2}{2\pi\varepsilon_0 bv} \tag{45}$$

and the kinetic energy transferred to the electron is

$$\boxed{\frac{p^2}{2m_\mathrm{e}} = \frac{z^2e^4}{8\pi^2\varepsilon_0^2 b^2 v^2 m_\mathrm{e}} = \text{energy lost by high energy particle.}} \tag{46}$$

Encounters between Charged Particles



We want the *average energy loss per unit length* and so we need the number of collisions with collision parameters in the range b to $b + \mathrm{d}b$ and integrate over collision parameters. The total energy loss of the high energy particle, $-\mathrm{d}E$, is:

$$\begin{aligned} & (\text{number of electrons in volume } 2\pi b \, \mathrm{d}b \, \mathrm{d}x) \\ \times \; & (\text{energy loss per interaction}) \\ = \; & \int_{b_{\mathrm{min}}}^{b_{\mathrm{max}}} N_{\mathrm{e}} \, 2\pi b \, \mathrm{d}b \, \frac{z^2 e^4}{8\pi^2 \varepsilon_0^2 b^2 v^2 m_{\mathrm{e}}} \, \mathrm{d}x \end{aligned}$$

where N_{e} is the number density of electrons. I have included limits b_{max} and b_{min} to the range of collision parameters in this integral. Then,

$$-\frac{\mathrm{d}E}{\mathrm{d}x} = \frac{z^2 e^4 N_{\mathrm{e}}}{4\pi \varepsilon_0^2 v^2 m_{\mathrm{e}}} \ln \left(\frac{b_{\mathrm{max}}}{b_{\mathrm{min}}} \right) . \qquad (47)$$

This process is closely related to the *ionisation losses* which we will meet again.

Gaunt Factors

Notice how the logarithmic dependence upon $b_{\rm max}/b_{\rm min}$ comes about. The closer the encounter, the greater the momentum impulse, $p \propto b^{-2}$. There are, however, more electrons at large distances $(\propto b\,{\rm d}b)$ and hence, when we integrate, we obtain only a logarithmic dependence of the energy loss rate upon the range of collision parameters.

Why introduce the limits $b_{\rm max}$ and $b_{\rm min}$?

The reason is that the proper sum is very much more complicated and would take account of the acceleration of the electron by the high energy particle and include a proper quantum mechanical treatment of the interaction. Our approximate methods give rather good answers, however, because the limits $b_{\rm max}$ and $b_{\rm min}$ only appear inside the logarithm and hence need not be known very precisely.

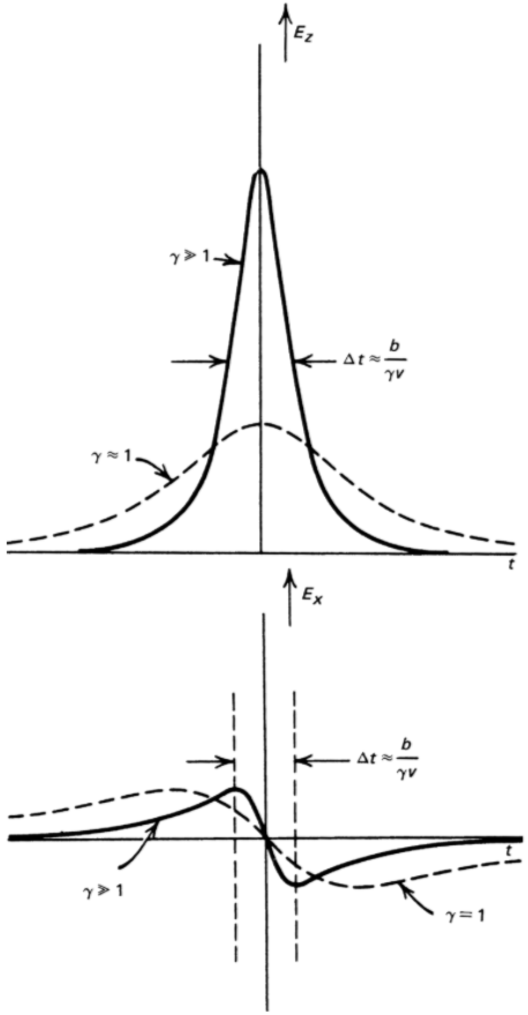
This is the simplest example of the type of calculation which needs to be carried out in working out energy transfers and accelerations of electrons and protons in fully ionised plasmas. The logarithmic term $\ln(b_{\rm max}/b_{\rm min})$ appears in the guise of what are often referred to as *Gaunt factors* and care has to be taken to use the correct values of $b_{\rm max}$ and $b_{\rm min}$ in different physical conditions. Similar forms of Gaunt factor appear in working out the spectrum of bremsstrahlung and the electrical conductivity of a plasma.

Spectrum and Energy Loss Rate of Bremsstrahlung

In the classical limit, bremsstrahlung is the emission of an electron accelerated in an electrostatic encounter with a nucleus. Electrons lose more energy in electron-electron collisions, but these do not result in the emission of dipole radiation since there is no net electric dipole moment associated with these encounters. Hence, the calculation

- Work out an expression for the acceleration of an electron in the electrostatic field of the nucleus. The roles of the particles in the calculation above are reversed – the electron is moving at a high speed past the stationary nucleus.
- Fourier transform of the acceleration of the electron and use Parseval's theorem to work out the spectrum of the emitted radiation.
- Integrate this result over all collision parameters and worry about suitable limits for b_{max} and b_{min}.
- In the case in which the electron is relativistic, transform back into the laboratory frame of reference.
- For a Maxwellian gas, integrate over the Maxwell distribtion.
- For a non-thermal distribution, integrate over the velocity or energy distribution.

Outline Calculation



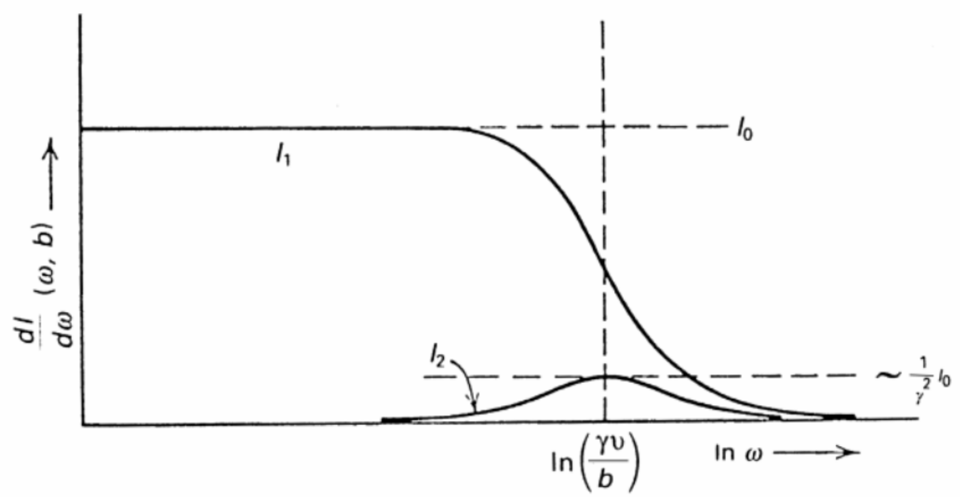
Both the relativistic and non-relativistic cases begin in the same way and so work out both cases simultaneously. The accelerations along the trajectory of the electron, $a_{\|}$, and perpendicular to it, $a_{\perp}$, *in its rest-frame* are given by

$$a_{\|} = \dot{v}_x = -\frac{eE_x}{m_e} = \frac{\gamma Z e^2 v t}{4\pi\varepsilon_0 m_e [b^2 + (\gamma v t)^2]^{3/2}}$$

$$a_{\perp} = \dot{v}_z = -\frac{eE_z}{m_e} = \frac{\gamma Z e^2 b}{4\pi\varepsilon_0 m_e [b^2 + (\gamma v t)^2]^{3/2}}$$

where Ze is the charge of the nucleus (see *HEA3*).

Outline Calculation



The radiation spectrum of the electron in an encounter with a charged nucleus is then

$$\begin{aligned} I(\omega) &= \frac{e^2}{3\pi\varepsilon_0 c^3}\left[|a_{\parallel}(\omega)|^2 + |a_{\perp}(\omega)|^2\right] \\ &= \frac{Z^2 e^6}{24\pi^4 \varepsilon_0^3 c^3 m_e^2} \frac{\omega^2}{\gamma^2 v^2}\left[\frac{1}{\gamma^2} K_0^2\left(\frac{\omega b}{\gamma v}\right) + K_1^2\left(\frac{\omega b}{\gamma v}\right)\right] \quad (48) \end{aligned}$$

where K_0 and K_1 are modified Bessel functions of order zero and one. This is the intensity spectrum which results from a single encounter between an electron and a nucleus with collision parameter b.

The Results

- The impulse perpendicular to the direction of travel contributes the greater intensity, even in the non-relativistic case $\gamma = 1$.

- The perpendicular component results in significant radiation at low frequencies.

- The spectrum is flat because the acceleration perpendicular to the line of flight is a brief impulse. The Fourier transform of a delta function is a flat spectrum.

- The turn-over corresponds roughly to the frequency which is the inverse of the duration of the collision.

Bremsstrahlung

Thus, *at high frequencies*, there is an exponential cut-off to the spectrum

$$I(\omega) = \frac{Z^2 e^6}{48\pi^3 \varepsilon_0^3 c^3 m_e^2 \gamma v^3} \left[\frac{1}{\gamma^2} + 1\right] \exp\left(-\frac{2\omega b}{\gamma v}\right) . \tag{49}$$

The exponential cut-off tells us that there is little power emitted at frequencies greater than $\omega \approx \gamma v / b$.

The *low frequency spectrum* has the form

$$I(\omega) = \frac{Z^2 e^6}{24\pi^4 \varepsilon_0^3 c^3 m_e^2} \frac{1}{b^2 v^2} = K . \tag{50}$$

To an excellent approximation, the low frequency spectrum is flat up to frequency $\omega = \gamma v / b$, above which the spectrum falls off exponentially.

Bremsstrahlung

Finally, we integrate over all collision parameters which contribute to the radiation at frequency ω. If the electron is moving relativistically, the number density of nuclei it observes is enhanced by a factor γ because of relativistic length contraction. Hence, in the moving frame of the electron, $N' = \gamma N$ where N is the number density of nuclei in the laboratory frame of reference. The number of encounters per second is $N'v$ and since, properly speaking, all parameters are measured in the rest frame of the electron, let us add superscript dashes to all the relevant parameters. The radiation spectrum in frame of the electron is therefore

$$I(\omega') = \int_{b'_{\text{min}}}^{b'_{\text{max}}} 2\pi b' \gamma N v K' \, \mathrm{d}b' \tag{51}$$

$$= \frac{Z^2 e^6 \gamma N}{12\pi^3 \varepsilon_0^3 c^3 m_{\text{e}}^2} \frac{1}{v} \ln\left(\frac{b'_{\text{max}}}{b'_{\text{min}}}\right). \tag{52}$$

Relativistic Bremsstrahlung Losses

The formulae we have derived are correct in the rest frame of the electron, namely,

$$I(\omega') = \int_{b'_{\min}}^{b'_{\max}} 2\pi b'(\gamma N) v K' \, \mathrm{d}b' \tag{69}$$

where we have written the number density of nuclei γN because of length contraction. Since the collision parameters b' are perpendicular to the direction of motion, it follows that, since $y = y'$, the same collision parameters appropriate for the laboratory frame of reference can be used. I have given a discussion of the relevant collision parameters in *HEA3* and I will not repeat that discussion here. It suffices to note that we can write the emission spectrum in the frame of the electron

$$I(\omega') = \frac{Z^2 e^6 N \gamma}{12\pi^3 \varepsilon_0^3 c^3 m_e^2} \frac{1}{v} \ln \Lambda. \tag{70}$$

Notice that there is at best a very weak dependence upon frequency ω and so we again obtain the characteristic flat bremsstrahlung intensity spectrum.

Relativistic Bremsstrahlung Losses

On transforming this spectrum to the laboratory frame of reference, we note that the bandwidth changes as $\Delta\omega = \gamma\Delta\omega'$ and so the spectrum becomes

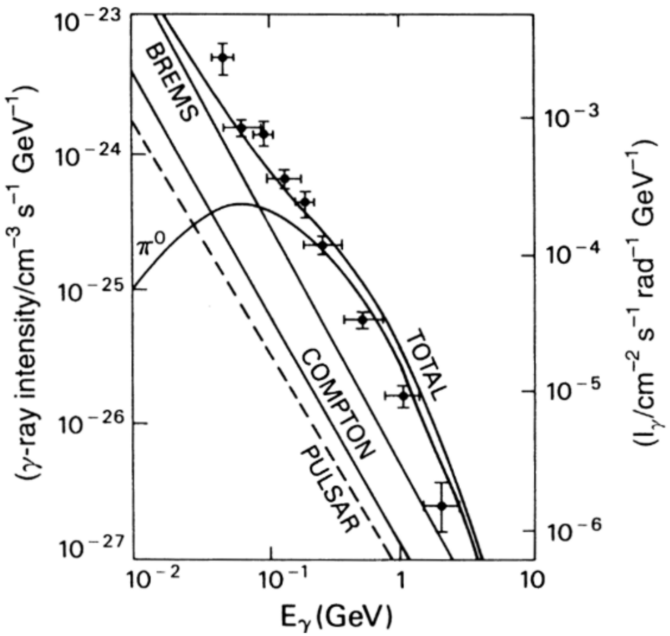
$$I(\omega) = \frac{Z^2 e^6 N}{12\pi^3 \varepsilon_0^3 c^3 m_e^2} \frac{1}{v} \ln \Lambda \qquad (71)$$

where the integral extends up to energies $E = \hbar\omega = \gamma m_e c^2$, where $\gamma \gg 1$. Thus, the rate of loss of energy of the relativistic electron is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{rel}} = \int_0^{E/\hbar} I(\omega)\,\mathrm{d}\omega = \frac{Z^2 e^6 N \bar{g}}{12\pi^3 \varepsilon_0^3 c^4 \hbar} E. \qquad (72)$$

Notice that the dependence of the energy loss rate changes from $E^{1/2}$ to E between the non-relativistic and relativistic cases.

The Galactic Gamma-ray Emission



The diagram shows the γ-ray spectrum of our Galaxy as well as theoretical estimates of the emission by Stecker (1977). At energies $\varepsilon > 70$ MeV, the dominant emission mechanism is the decay of neutral pions created in collisions between cosmic rays and the nuclei of atoms and molecules of the interstellar gas. This spectrum peaks at about 70 MeV and so there must be another mechanism which contributes at the lower energies. Relativistic bremsstrahlung may be the dominant source of emission at these energies. The spectrum labelled 'brems' is derived from an extrapolation of the relativistic electron spectrum in our Galaxy to energies $1 < E < 1000$ MeV.

Synchrotron Radiation

The synchrotron radiation, the emission of very relativistic and ultrarelativistic electrons gyrating in a magnetic field, is the process which dominates much of high energy astrophysics. It was originally observed in early betatron experiments in which electrons were first accelerated to ultrarelativistic energies. This process is responsible for the radio emission from the Galaxy, from supernova remnants and extragalactic radio sources. It is also responsible for the non-thermal optical and X-ray emission observed in the Crab Nebula and possibly for the optical and X-ray continuum emission of quasars.

The word *non-thermal* is used frequently in high energy astrophysics to describe the emission of high energy particles. This an unfortunate terminology since all emission mechanisms are 'thermal' in some sense. The word is conventionally taken to mean 'continuum radiation from particles, the energy spectrum of which is not Maxwellian'. In practice, continuum emission is often referred to as 'non-thermal' if it cannot be described by the spectrum of thermal bremsstrahlung or black-body radiation.

Motion of an Electron in a Uniform, Static Magnetic field

We begin by writing down the equation of motion for a particle of rest mass m_0, charge ze and Lorentz factor $\gamma = (1 - v^2/c^2)^{-1/2}$ in a uniform static magnetic field $\boldsymbol{B}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\gamma m_0 \boldsymbol{v}) = ze(\boldsymbol{v} \times \boldsymbol{B}) \tag{1}$$

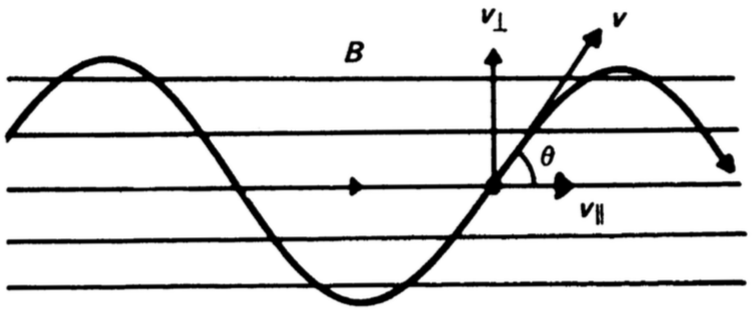
We recall that the left-hand side of this equation can be expanded as follows:

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \boldsymbol{v}) = m_0 \gamma \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + m_0 \gamma^3 \boldsymbol{v} \frac{(\boldsymbol{v} \cdot \boldsymbol{a})}{c^2} \tag{2}$$

because the Lorentz factor γ should be written $\gamma = (1 - \boldsymbol{v} \cdot \boldsymbol{v}/c^2)^{-1/2}$. In a magnetic field, the three-acceleration $\boldsymbol{a} = \mathrm{d}\boldsymbol{v}/\mathrm{d}t$ is always perpendicular to $\boldsymbol{v}$ and consequently $\boldsymbol{v} \cdot \boldsymbol{a} = 0$. As a result,

$$\gamma m_0 \, \mathrm{d}\boldsymbol{v}/\mathrm{d}t = ze(\boldsymbol{v} \times \boldsymbol{B}) \tag{3}$$

Motion of an Electron in a Uniform, Static Magnetic field



We now split $\boldsymbol{v}$ into components parallel and perpendicular to the uniform magnetic field, $v_\parallel$ and $v_\perp$ respectively. The pitch angle θ of the particle's path is given by $\tan\theta = v_\perp / v_\parallel$, that is, θ is the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{B}$. Since $v_\parallel \times \boldsymbol{B} = 0$, $v_\parallel =$ constant. The acceleration is perpendicular to the magnetic field direction and to $v_\parallel$.

$$\gamma m_0 \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = zev_\perp B(\boldsymbol{i}_\perp \times \boldsymbol{i}_{\boldsymbol{B}}) = zevB(\boldsymbol{i}_{\boldsymbol{v}} \times \boldsymbol{i}_{\boldsymbol{B}}) \quad (4)$$

sinθ

where $\boldsymbol{i}_{\boldsymbol{v}}$ and $\boldsymbol{i}_{\boldsymbol{B}}$ are unit vectors in the directions of $\boldsymbol{v}$ and $\boldsymbol{B}$ respectively.

Gyrofrequencies

Thus, the motion of the particle consists of a constant velocity along the magnetic field direction and circular motion with radius r about it. This means that the particle moves in a *spiral path* with *constant pitch angle* θ. The radius r is known as the *gyroradius* of the particle. The angular frequency of the particle in its orbit $\omega_{\rm g}$ is known as the *angular cyclotron frequency* or *angular gyrofrequency* and is given by

$$\omega_{\rm g} = v_{\perp}/r = zeB/\gamma m_0 \tag{5}$$

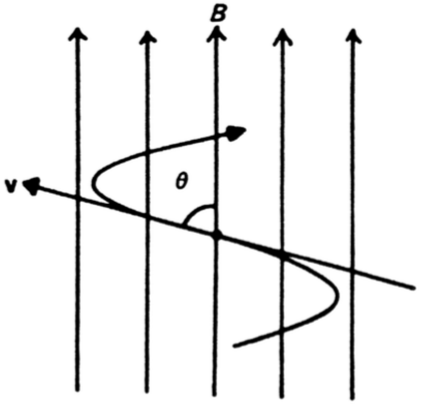
The corresponding *gyrofrequency* $\nu_{\rm g}$, that is, the number of times per second that the particle rotates about the magnetic field direction, is

$$\nu_{\rm g} = \omega_g/2\pi = zeB/2\pi\gamma m_0 \tag{6}$$

In the case of a non-relativistic particle, $\gamma = 1$ and hence $\nu_{\rm g} = zeB/2\pi m_0$.

A useful figure to remember is the non-relativistic gyrofrequency of an electron $\nu_{\rm g} = eB/2\pi m_{\rm e} = 28$ GHz T^{-1} where the magnetic field strength is measured in tesla; alternatively, $\nu_{\rm g} = 2.8$ MHz G^{-1} for those not yet converted from gauss (G) to teslas (T).

The Total Energy Loss Rate



Most of the essential tools needed in this analysis have already been derived. First of all, use the expression for the acceleration of the electron in its orbit and insert this into the expression for the radiation rate of a relativistic electron. The acceleration is always perpendicular to the velocity vector of the particle and to $\boldsymbol{B}$ and hence $a_{\parallel} = 0$. Therefore, the total radiation loss rate of the electron is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = \frac{\gamma^4 e^2}{6\pi\varepsilon_0 c^3}|a_\perp|^2 = \frac{\gamma^4 e^2}{6\pi\varepsilon_0 c^3}\frac{e^2 v^2 B^2 \sin^2\theta}{\gamma^2 m_e^2} \quad (7)$$

$$= \frac{e^4 B^2}{6\pi\varepsilon_0 c m_e^2}\frac{v^2}{c^2}\gamma^2 \sin^2\theta \quad (8)$$

The Average Energy Loss Rate

In the ultrarelativistic limit, $v \to c$, we may approximate this result by

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = 2\sigma_{\mathrm{T}} c U_{\mathrm{mag}} \gamma^2 \sin^2\theta \tag{14}$$

These results apply for electrons of a specific pitch angle θ. Particles of a particular energy E, or Lorentz factor γ, are often expected to have an isotropic distribution of pitch angles and therefore we can work out their average energy loss rate by averaging over such a distribution of pitch angles $p(\theta)\,\mathrm{d}\theta = \frac{1}{2}\sin\theta\,\mathrm{d}\theta$

$$\begin{aligned} -\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) &= 2\sigma_{\mathrm{T}} c U_{\mathrm{mag}} \gamma^2 \left(\frac{v}{c}\right)^2 \frac{1}{2}\int_0^{\pi} \sin^3\theta d\theta \\ &= \frac{4}{3}\sigma_{\mathrm{T}} c U_{\mathrm{mag}} \left(\frac{v}{c}\right)^2 \gamma^2 \qquad (15) \end{aligned}$$

There is a deeper sense in which (15) is the average loss rate for a particle of energy E. During its lifetime, it is likely that the high energy particle is randomly scattered in pitch angle and then (15) is the correct expression for its average energy loss rate.

Non-relativistic gyroradiation and cyclotron radiation

Consider first the simplest case of non-relativistic gyroradiation, in which case $v \ll c$ and hence $\gamma = 1$. Then, the expression for the loss rate of the electron is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right) = 2\sigma_T c U_{\mathrm{mag}} \left(\frac{v}{c}\right)^2 \sin^2\theta = \frac{2\sigma_T}{c} U_{\mathrm{mag}} v_\perp^2 \qquad (16)$$

and the radiation is emitted at the gyrofrequency of the electron $\nu_{\mathrm{g}} = eB/2\pi m_{\mathrm{e}}$.

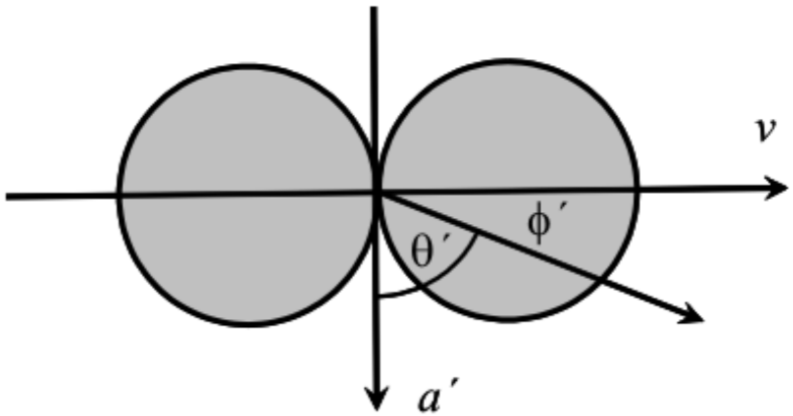
In the non-relativistic case, there are no beaming effects and the polarisation observed by the distant observer can be derived from the rules given above. When the magnetic field is perpendicular to the line of sight, *linearly polarised radiation* is observed because the acceleration vector is observed to perform simple harmonic motion in a plane perpendicular to the magnetic field by the distant observer. The electric field strength varies sinusoidally at the gyrofrequency. When the magnetic field is parallel to the line of sight, the acceleration vector is continually changing direction as the electron moves in a circular orbit about the magnetic field lines and therefore the radiation is observed to be 100% *circularly polarised*. Between these cases, the radiation is observed to be *elliptically polarised*, the ratio of axes of the polarisation ellipse being $\cos\theta$.

Spectrum of Synchrotron Radiation
Physical Arguments

One of the basic features of the radiation of relativistic particles in general is the fact that the radiation is *beamed* in the direction of motion of the particle. Part of this effect is associated with the relativistic aberration formulae between the frame of reference of the particle and the observer's frame of reference. There are, however, subtleties about what is observed by the distant observer because, in addition to aberration, we have to consider the time development of what is seen by the distant observer.

Let us consider first the simple case of a particle gyrating about the magnetic field at a pitch angle of 90°. The electron is accelerated towards its guiding centre, that is, radially inwards, and in its instantaneous rest frame it emits the usual dipole pattern with respect to the acceleration vector.

Beaming of the Emitted Radiation

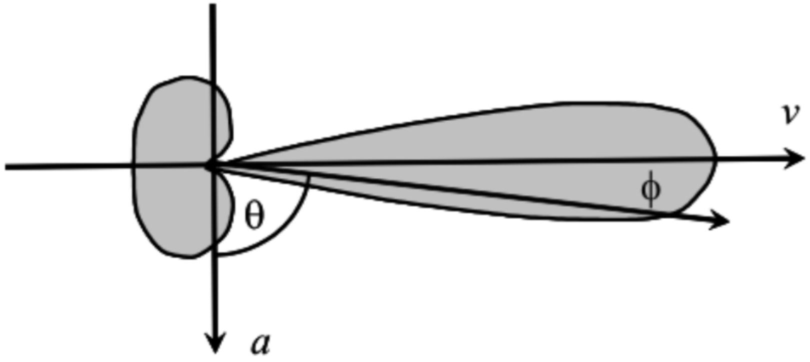


To centre of particle's orbit

We can therefore work out the radiation pattern in the laboratory frame of reference by applying the aberration formulae with the results illustrated schematically in the diagrams. The angular distribution of radiation with respect to the velocity vector in the frame S′ is $I_\nu \propto \sin^2\theta' = \cos^2\phi'$. We may think of this as being the probability distribution with which photons are emitted by the electron in its rest frame. The appropriate aberration formulae between the two frames are:

$$\sin\phi = \frac{1}{\gamma}\frac{\sin\phi'}{1 + (v/c)\cos\phi'} \qquad ; \qquad \cos\phi = \frac{\cos\phi' + v/c}{1 + (v/c)\cos\phi'} \tag{20}$$

Beaming of the Emitted Radiation



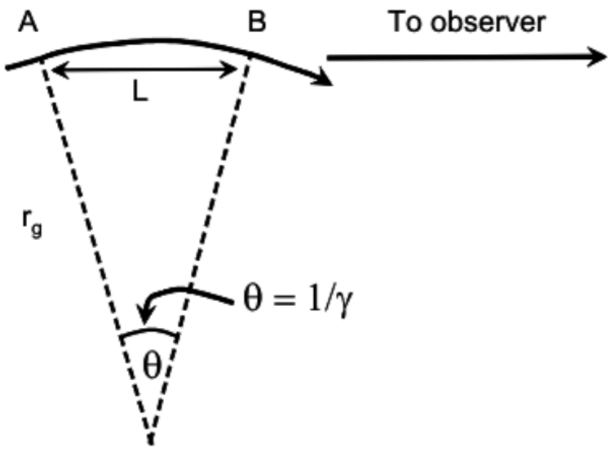
To centre of particle's orbit

Consider the angles $\phi' = \pm\pi/4$ in S', the angles at which the intensity of radiation falls to half its maximum value in the instantaneous rest frame. The corresponding angles ϕ in the laboratory frame of reference are

$$\sin\phi \approx \phi \approx 1/\gamma \tag{21}$$

The radiation emitted within $-\pi/4 < \phi' < \pi/4$ is beamed in the direction of motion of the electron within $-1/\gamma < \phi < 1/\gamma$. A large 'spike' of radiation is observed every time the electron's velocity vector lies within an angle of about $1/\gamma$ to the line of sight to the observer. The spectrum of the radiation is the Fourier transform of this pulse once the effects of time retardation and aberration are taken into account.

The Duration of the Observed Pulse



The observer sees significant radiation for only about $1/\gamma$ of the particle's orbit but the observed duration of the pulse is less than $1/\gamma$ times the period of the orbit because radiation emitted at the trailing edge of the pulse almost catches up with the radiation emitted at the leading edge. The observer is located at a distance R from the point A. The radiation from A reaches the observer at time R/c. The radiation is emitted from B at time L/v later and then travels a distance $(R - L)$ at the speed of light to the observer. The trailing edge of the pulse therefore arrives at a time $L/v + (R - L)/c$. The duration of the pulse as measured by the observer is therefore

$$\Delta t = \left[\frac{L}{v} + \frac{(R-L)}{c}\right] - \frac{R}{c} = \frac{L}{v}\left(1 - \frac{v}{c}\right) . \tag{22}$$

The Duration of the Observed Pulse

Notice that the observed duration of the pulse is much less than value L/v. Only if light propagated at an infinite velocity would the duration of the pulse be L/v. The factor $1-(v/c)$ is exactly the same factor which appears in the Liénard-Weichert potential and takes account of the fact that the source of radiation is not stationary but is moving towards the observer. We now rewrite the above expression noting that

$$\frac{L}{v} = \frac{r_{\mathrm{g}}\theta}{v} \approx \frac{1}{\gamma\omega_{\mathrm{r}}} = \frac{1}{\omega_{\mathrm{g}}} \tag{23}$$

where ω_{g} is the non-relativistic angular gyrofrequency and $\omega_r = \omega_{\mathrm{g}}/\gamma$. We can also rewrite $(1-v/c)$ as

$$\left(1-\frac{v}{c}\right) = \frac{[1-(v/c)]\,[1+(v/c)]}{[1+(v/c)]} = \frac{\left(1-v^2/c^2\right)}{1+(v/c)} \approx \frac{1}{2\gamma^2} \tag{24}$$

since $v \approx c$. Therefore, the observed duration of the pulse is roughly

$$\Delta t \approx \frac{1}{2\gamma^2\omega_{\mathrm{g}}} \tag{25}$$

The duration of the pulse in the laboratory frame of reference is roughly $1/\gamma^2$ times shorter than the non-relativistic gyroperiod $T_{\mathrm{g}} = 2\pi/\omega_{\mathrm{g}}$.

The Observed Frequency of Synchrotron Radiation

The maximum Fourier component of the spectral decomposition of the observed pulse of radiation is expected to correspond to a frequency $\nu \sim \Delta t^{-1}$, that is,

$$\nu \sim \Delta t^{-1} \sim \gamma^2 \nu_\mathrm{g} \tag{26}$$

where ν_g is the non-relativistic gyrofrequency.

In the above analysis, it has been assumed that the particle moves in a circle about the magnetic field lines, that is, the pitch angle θ is 90°. The same calculation can be performed for any pitch angle and then the result becomes

$$\nu \sim \gamma^2 \nu_\mathrm{g} \sin\theta \tag{27}$$

The reason for performing this simple exercise in detail is that the beaming of the radiation of ultrarelativistic particles is a very general property and does not depend upon the nature of the force causing the acceleration.

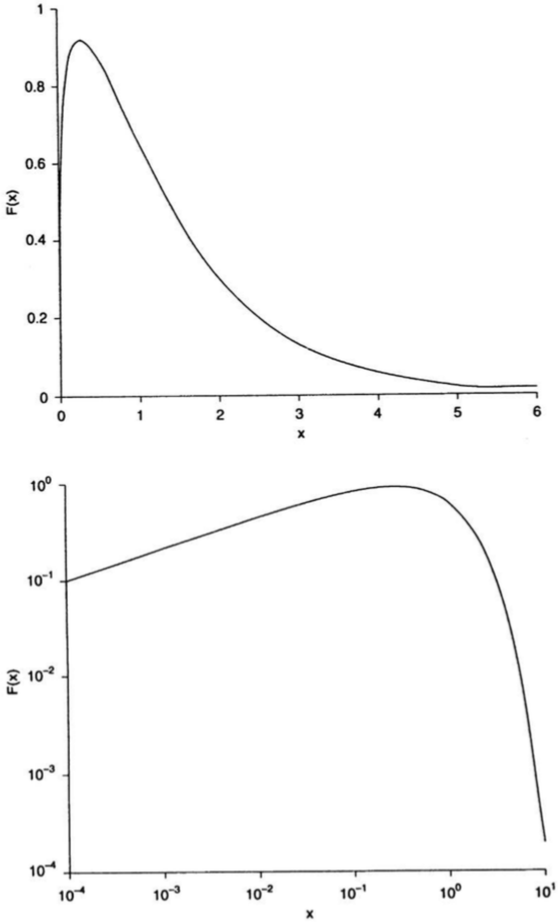
Synchrotron radiation - improved version

There is no particularly simple way of deriving the spectral distribution of synchrotron radiation and I do not find the analysis particularly appealing - see *HEA3* for the gruelling details. The analysis proceeds by the following steps:

- Write down the expression for the energy emitted per unit bandwidth for an arbitrarily moving electron;
- Select a suitable set of coordinates in which to work out the field components radiated by the electron spiralling in a magnetic field;
- Then battle away at the algebra to obtain the spectral distribution of the field components.

The Results

The emitted spectrum of a single electron, averaged over the particle's orbit is



$$j(\omega) = j_{\perp}(\omega) + j_{\parallel}(\omega) = \frac{\sqrt{3} e^3 B \sin\theta}{8\pi^2 \varepsilon_0 c m_{\mathrm{e}}} F(x) \quad (29)$$

where

$$x = \omega/\omega_{\mathrm{c}}, \quad \omega_{\mathrm{c}} = \frac{3}{2}\left(\frac{c}{v}\right)\gamma^3 \omega_{\mathrm{r}} \sin\theta \quad (30)$$

and

$$F(x) = x \int_x^{\infty} K_{5/3}(z)\,\mathrm{d}z. \quad (31)$$

ω_{c} is known as the critical angular frequency. $K_{5/3}(z)$ is a modified Bessel function of order $5/3$. The form of this spectrum in terms of angular frequency ω is shown in linear and logarithmic form in the diagrams. It has a broad maximum centred roughly at the frequency $\nu \approx \nu_{\mathrm{c}}$ with $\Delta\nu/\nu \sim 1$. The maximum of the emission spectrum has value $\nu_{\mathrm{max}} = 0.29\nu_{\mathrm{c}}$.

The Results (continued)

The high frequency emissivity of the electron is given by an expression of the form

$$j(\nu) \propto \nu^{1/2} \mathrm{e}^{-\nu/\nu_\mathrm{c}} \qquad (32)$$

which is dominated by the exponential cut-off at frequencies $\nu \gg \nu_\mathrm{c}$. This simply means that there is very little power at frequencies $\nu > \nu_\mathrm{c}$ which can be understood on the basis of the physical arguments developed earlier – there is very little structure in the observed polar diagram of the radiation emitted by the electron at angles $\theta \ll \gamma^{-1}$. At low frequencies, $\nu \ll \nu_\mathrm{c}$, the spectrum is given by $j(\nu) \propto \nu^{1/3}$

The ratio of the powers emitted in the polarisations parallel and perpendicular to the magnetic field direction is

$$\frac{I_\perp}{I_\parallel} = 7. \qquad (33)$$

To find the polarisation observed from a distribution of electrons at a particular observing frequency, however, we need to integrate over the energy spectrum of the emitting electrons.

The Synchrotron Radiation of a Power-law Distribution of Electron Energies

The emitted spectrum of electrons of energy E is quite sharply peaked near the critical frequency ν_{c} and is very much narrower than the breadth of the electron energy spectrum. Therefore, to a good approximation, it may be assumed that all the radiation of an electron of energy E is radiated at the critical frequency ν_{c} which we may approximate by

$$\nu \approx \nu_{\mathrm{c}} \approx \gamma^2 \nu_{\mathrm{g}} = \left(\frac{E}{m_{\mathrm{e}} c^2}\right)^2 \nu_{\mathrm{g}}; \qquad \nu_{\mathrm{g}} = \frac{eB}{2\pi m_{\mathrm{e}}}. \tag{34}$$

Therefore, the energy radiated in the frequency range ν to $\nu + \mathrm{d}\nu$ can be attributed to electrons with energies in the range E to $E + \mathrm{d}E$, which we assume to have power-law form $N(E) = \kappa E^{-p}$. We may therefore write

$$J(\nu)\,\mathrm{d}\nu = \left(-\frac{\mathrm{d}E}{\mathrm{d}t}\right) N(E)\,\mathrm{d}E. \tag{35}$$

(continued)

Now

$$E = \gamma m_e c^2 = \left(\frac{\nu}{\nu_g}\right)^{1/2} m_e c^2, \qquad dE = \frac{m_e c^2}{2\nu_g^{1/2}} \nu^{-1/2} d\nu, \tag{36}$$

and

$$-\left(\frac{dE}{dt}\right) = \frac{4}{3}\sigma_T c \left(\frac{E}{m_e c^2}\right)^2 \frac{B^2}{2\mu_0}. \tag{37}$$

Substituting these quantities into (35), the emissivity may be expressed in terms of κ, B, ν and fundamental constants.

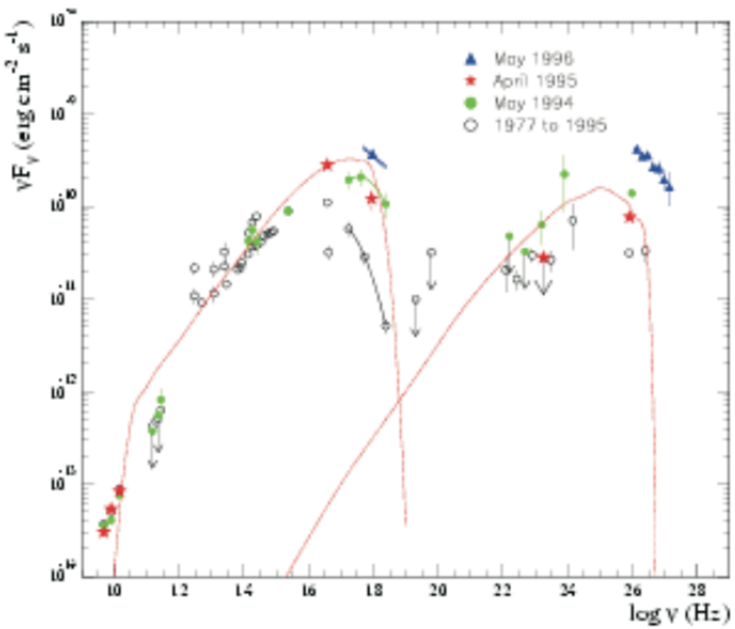
$$J(\nu) = (\text{constants})\ \kappa B^{(p+1)/2} \nu^{-(p-1)/2}. \tag{38}$$

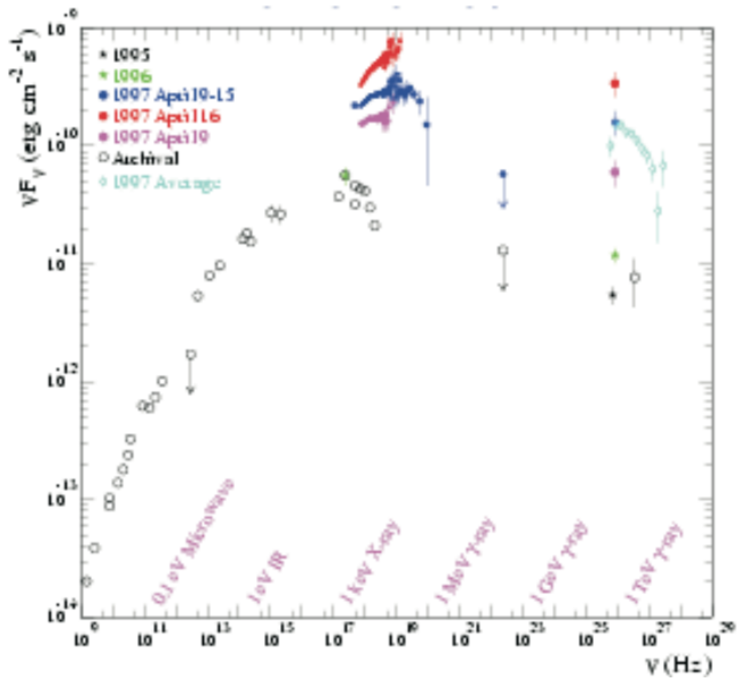
If the electron energy spectrum has power law index p, the spectral index of the synchrotron emission of these electrons, defined by $J(\nu) \propto \nu^{-\alpha}$, is $\alpha = (p-1)/2$. The spectral shape is determined by the shape of the electron energy spectrum rather than by the shape of the emission spectrum of a single particle. The quadratic nature of the relation between emitted frequency and the energy of the electron accounts for the difference in slopes of the emission spectrum and the electron energy spectrum.

Synchro-Compton Radiation and the Inverse Compton Catastrophe

Inverse Compton scattering is likely to be an important source of X-rays and γ-rays, for example, in the intense extragalactic γ-ray sources. Wherever there are large number densities of soft photons, the presence of ultrarelativistic electrons must result in the production of high energy photons, X-rays and γ-rays. The case of special interest in this chapter is that in which the same relativistic electrons which are the source of the soft photons are also responsible for scattering these photons to X-ray and γ-ray energies – this is the process known as *synchro-Compton Radiation*. One case of special importance is that in which the number density of low energy photons is so great that most of the energy of the electrons is lost by synchro-Compton radiation rather then by synchotron radiation. This line of reasoning leads to what is known as the *inverse Compton catastrophe*.

Ultra-High Energy γ-ray Sources





In the extreme γ-ray sources Markarian 421 and 501, it is very likely that some form of inverse Compton radiation is occurring, quite possible via the Synchro-Compton mechanism. These γ-ray sources are quite enormously luminous and variable. It is therefore likely that relativistic motions have to be involved to explain their luminosities and variability.

Back up

Non-relativistic and Thermal Bremsstrahlung

First, we evaluate the total energy loss rate by bremsstrahlung of a high energy but non-relativistic electron. We neglect the relativistic correction factors and hence obtain the low frequency radiation spectrum

$$I(\omega) = \frac{Z^2 e^6 N}{12\pi^3 \varepsilon_0^3 c^3 m_e^2} \frac{1}{v} \quad \ln \Lambda \tag{53}$$

where $\Lambda = b_{\text{max}}/b_{\text{min}}$. We have to make the correct choice of limiting collision parameters b_{max} and b_{min}.

For b_{max}, we note that we should only integrate out to those values of b for which $\omega b/v = 1$. For larger values of b, the radiation at frequency ω lies on the exponential tail of the spectrum and we obtain a negligible contribution to the intensity.

For b_{min}, at high velocities, $v \geq (Z/137)c$, the quantum restriction, $b_{\text{min}} \approx \hbar/2m_e v$, is applicable and can be derived from Heisenberg's uncertainty principle (see *HEA3*). This is the appropriate limit to describe, for example, the X-ray bremsstrahlung of hot intergalactic gas in clusters of galaxies. Thus, for high velocities, $\Lambda = 2m_e v^2/\hbar\omega$. There is, as usual, a cut-off at high frequencies $\omega \geq v/b$.

Thermal Bremsstrahlung

In order to work out the bremsstrahlung, or free-free emission, of a gas at temperature T, we integrate the expression (53) over a Maxwellian distribution of electron velocities

$$N_{\mathrm{e}}(v)\,\mathrm{d}v = 4\pi N_{\mathrm{e}} \left(\frac{m_{\mathrm{e}}}{2\pi kT}\right)^{3/2} v^2 \exp\left(\frac{m_{\mathrm{e}}v^2}{2kT}\right)\,\mathrm{d}v. \tag{55}$$

The algebra can become somewhat cumbersome at this stage. We can find the correct order-of-magnitude answer if we write $\frac{1}{2}m_{\mathrm{e}}v^2 = \frac{3}{2}kT$ in expression (53). Then, the emissivity of a plasma having electron density N_{e} becomes in the low frequency limit,

$$I(\omega) \approx \frac{Z^2 e^6 N N_{\mathrm{e}}}{12\sqrt{3}\pi^3\varepsilon_0^3 c^3 m_{\mathrm{e}}^2} \left(\frac{m_{\mathrm{e}}}{kT}\right)^{1/2} g(\omega, T) \tag{56}$$

where $g(\omega, T)$ is a *Gaunt factor*, corresponding to $\ln \Lambda$, but now integrated over velocity.

At high frequencies, the spectrum of thermal bremsstrahlung cuts off exponentially as $\exp(-\hbar\omega/kT)$, reflecting the population of electrons in the high energy tail of a Maxwellian distribution at energies $\hbar\omega \gg kT$.

Thermal Bremsstrahlung

The total energy loss rate of the plasma may be found by integrating the spectral emissivity over all frequencies. In practice, because of the exponential cut-off, we find the correct functional form by integrating (56) from 0 to $\omega = kT/\hbar$, that is,

$$-(\mathrm{d}E/\mathrm{d}t) = (\mathrm{constant})\, Z^2 T^{1/2} \bar{g} N N_\mathrm{e} \tag{57}$$

where $\bar{g}$ is a frequency averaged Gaunt factor. Detailed calculations give the following answers:

$$\kappa_\nu = \frac{1}{3\pi^2}\left(\frac{\pi}{6}\right)^{1/2} \frac{Z^2 \mathrm{e}^6}{\varepsilon_0^3 c^3 m_\mathrm{e}^2}\left(\frac{m_\mathrm{e}}{kT}\right)^{1/2} g(\nu, T) N N_\mathrm{e} \exp\left(-\frac{h\nu}{kT}\right) \tag{58}$$

$$= 6.8 \times 10^{-51} Z^2 T^{-1/2} N N_\mathrm{e}\, g(\nu, T) \exp(-h\nu/kT)\ \mathrm{W\,m^{-3}Hz^{-1}} \tag{59}$$

where the number densities of electrons N_e and of nuclei N are given in particles per cubic metre. At frequencies $\hbar\omega \ll kT$, the Gaunt factor has only a logarithmic dependence on frequency.

Thermal Bremsstrahlung

A suitable form for X-ray wavelengths is:

$$\text{X-ray} \quad g(\nu, T) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{kT}{h\nu}\right) , \tag{60}$$

The functional forms of the logarithmic term can be readily derived from the the above considerations.

The total loss rate of the plasma is

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{brems}} = 1.435 \times 10^{-40} Z^2 T^{1/2} \overline{g} N N_{\mathrm{e}} \quad \mathrm{W\,m^{-3}}. \tag{61}$$

Detailed calculations show that the frequency averaged Gaunt factor $\overline{g}$ lies in the range $1.1 - 1.5$ and a good approximation is $\overline{g} = 1.2$. A compilation of a large number of useful Gaunt factors for a wide range of physical conditions is given by Karzas and Latter (1961).

Non-relativistic Bremsstrahlung Losses

To find the energy loss rate of a single high energy electron, we integrate (52) over all frequencies. In practice, this means integrating from 0 to ω_{max} where ω_{max} corresponds to the cut-off, $b_{\mathrm{min}} \approx \hbar/2m_{\mathrm{e}}v$. This angular frequency is approximately

$$\omega_{\mathrm{max}} = 2\pi/\tau \sim 2\pi v/b_{\mathrm{min}} \approx 4\pi m_{\mathrm{e}} v^2/\hbar, \qquad (67)$$

that is, to order of magnitude $\hbar\omega \approx \frac{1}{2} m_{\mathrm{e}} v^2$. This is just the kinetic energy of the electron and is obviously the maximum amount of energy which can be lost in a single encounter with the nucleus. We should therefore integrate (52) from $\omega = 0$ to $\omega_{\mathrm{max}} \approx m_{\mathrm{e}} \nu v^2/2\hbar$ and thus,

$$-\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{brems}} \approx \int_0^{\omega_{\mathrm{max}}} \frac{Z^2 e^6 N}{12\pi^3 \epsilon_0^3 c^3 m_{\mathrm{e}}^2} \frac{1}{v} \ln\Lambda \, \mathrm{d}\omega \approx \frac{Z^2 e^6 N v}{24\pi^3 \epsilon_0^3 c^3 m_{\mathrm{e}} \hbar} \ln\Lambda \qquad (68)$$

Note that the total energy loss rate of the electron is proportional to v, that is, to the square root of the kinetic energy E: $-\mathrm{d}E/\mathrm{d}t \propto E^{1/2}$.

In practical applications of this formula, it is necessary to integrate over the energy distribution of the particles.

Curvature Radiation

Returning to an earlier part of the calculation, the observed frequency of the radiation can also be written

$$\nu \approx \gamma^2 \nu_{\rm g} = \gamma^3 \nu_{\rm r} = \frac{\gamma^3 v}{2\pi r_{\rm g}} \tag{28}$$

where $\nu_{\rm r}$ is the relativistic gyrofrequency and $r_{\rm g}$ is the radius of curvature of the particle's orbit. Notice that, in general, we may interpret $r_{\rm g}$ as the instantaneous radius of curvature of the particle's orbit and $v/r_{\rm g}$ is the angular frequency associated with it. This is a useful result because it enables us to work out the frequency at which most of the radiation is emitted, provided we know the radius of curvature of the particle's orbit. The frequency of the observed radiation is roughly γ^3 times the angular frequency v/r where r is the *instantaneous radius of curvature* of the particle in its orbit. This result is important in the study of *curvature radiation* which has important applications in the emission of radiation from the magnetic poles of pulsars.