Course on Foundations of Mathematics

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Lesson 1

- **Motivations for a course on Category Thoery**

Category theory represents a powerful mathematical language that can concisely express and abstract the relationships between various mathematical ideas; infact it can describe precisely many similar phenomena that occur in different mathematical fields.

Categories abound in mathematics and in related fields such as computer science. Such entities as sets, vector spaces, groups, topological spaces, Banach spaces, manifolds, ordered sets, automata, languages, etc., all naturally give rise to categories.

Also, Category theory provides a vehicle that allows to transport problems from one area of mathematics to another area, where solutions are some times easier. For example, algebraic topology can be described as an investigation of topological problems by algebraic methods.

Motivations

- **<u>Motivations for a course on Category Thoery</u>**

"Category theory takes a bird's eye view of mathematics. From high in the sky, details become invisible, but we can spot patterns that were impossible to detect from ground level. How is the lowest common multiple of two numbers like the direct sum of two vector spaces? What do discrete topological spaces, free groups, and fields of fractions have in common? We will discover answers to these and many similar questions, seeing patterns in mathematics that you may never have seen before.

The most important concept in this book is that of *universal property*. The further you go in mathematics, especially pure mathematics, the more universal properties you will meet. We will spend most of our time studying different manifestations of this concept".

Source Tom Leinster: Basic Category theory

Motivations

To give an example of a construction with similar properties that occurs in completely different mathematical fields and that can be described by a universal property, you can think at the concept of "product" for sets, groups, topological spaces, vector spaces…

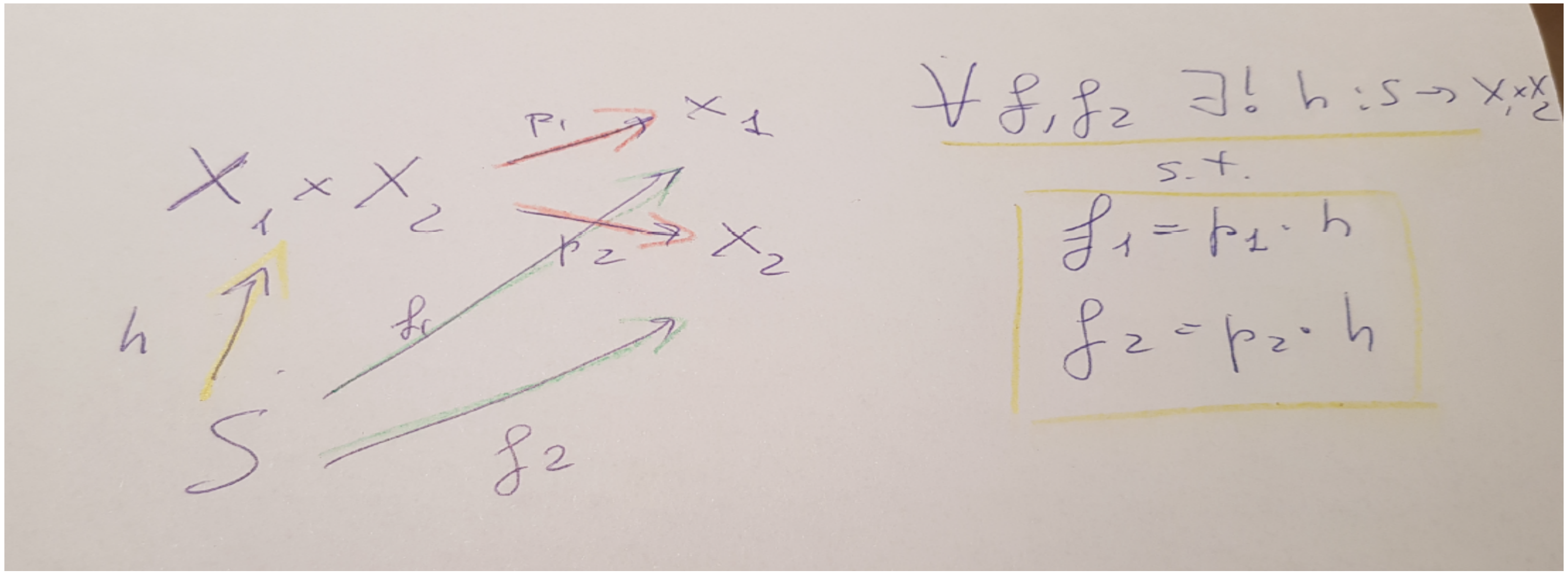
- A cartesian product of sets *A* and *B* indicated by *AxB* is defined by its elements: $AxB = \{(a,b) \text{ with } a \in A \text{ and } b \in B\}$
- The group product *GxH* of two groups id done as in the case of sets with an operation defined pointwise: $(g,h) \circ (g',h') = (g \circ g', h \circ h')$
- The topological product of two topological spaces X and Y is obtained by defining the classical product topology on the cartesian product XxY. You can consider that the product topology is the coarsest topology (i.e. the topology with the fewest open sets for which all the projections are continuous).

<u>How can we find a unique definition that capture all the three examples ?</u>

The idea of product

Category theory allows us to say precisely what it means to "act like" a product object.

We can define a product of two objects in a general contex via the following "**universal property**": an object X is the product of X_1 and X_2 denoted $X_1{\times}X_2$, if and only if there are two projections p_1 and p_2 such that for any pair of morphisms f_1 and f_2 from an arbitrary object S to X_1 and X_2 respectively there exists a "unique morphism" h from S to $X_1{\times}X_2$ such that the following diagram commutes:



The idea of product

You can check, that in case lile sets, groups or topological spaces this definition does indeed agrees with the classical definition of product. The beautiful thing about such an abstract definition is that it captures the “idea” of what a product is, and we get a unique definition that applies to many different contexts. Universal property will be at the core of our course.

Brief Historical Sketch

The notions of categories, functors, natural transformations, limits and colimits appeared almost out of nowhere in a paper by Eilenberg & Mac Lane (1945) entitled “General Theory of Natural Equivalences.” We say “almost,” because their earlier paper (1942) already contains specific functors and natural transformations at work, limited to groups. A desire to clarify and abstract their 1942 results led Eilenberg & Mac Lane to devise category theory. The central notion at the time, as their title indicates, was that of natural transformation. In order to give a general definition of the latter, they defined functor, borrowing the term from Carnap, and in order to define functor, they borrowed the word ‘category’ from the philosophy of Aristotle, Kant, and C. S. Peirce, but redefining it mathematically.

Source; Stanford Encyclopedia of Philosophy

Brief Historical Sketch

After their 1945 paper, it was not clear that the concepts of category theory would amount to more than a convenient language; this indeed was the status quo for about fifteen years. Category theory was employed in this manner by Eilenberg & Steenrod (1952), in an influential book on the foundations of algebraic topology, and by Cartan & Eilenberg (1956), in a ground breaking book on homological algebra.

These books allowed new generations of mathematicians to learn algebraic topology and homological algebra directly in the categorical language, and to master the method of diagrams. Indeed, without the method of diagram chasing, many results in these two books seem inconceivable, or would have required a considerably more intricate presentation.

Brief Historical Sketch

The situation changed radically with Grothendieck's (1957) landmark paper entitled "Sur quelques points d'algèbre homologique", in which the author employed categories intrinsically to define and construct more general theories which he then applied to specific fields, e.g., to algebraic geometry. Kan (1958) showed that a new idea of "adjoint functors" could capture fundamental concepts and costructions in other areas (in his case, homotopy theory).

At this point, category theory became more than a convenient language, by virtue of many interesting developments.

Brief Historical Sketch

Thanks in large part to the efforts of Freyd and Lawvere, in the 70' category theory gradually became an autonomous field of research. And indeed, it did grow rapidly as a discipline, but also in its applications, mainly in its source contexts, namely algebraic topology and homological algebra, but also in algebraic geometry and, after the appearance of Lawvere's Ph.D. thesis, in universal algebra. This thesis also constitutes a landmark in this history of the field, for in it Lawvere proposed the category of categories as a foundation for category theory, set theory and, thus, the whole of mathematics, as well as using categories for the study of logic.

Brief Historical Sketch

From the 1980s to the present, category theory has found new applications. In theoretical computer science, category theory is now firmly rooted, and contributes, among other things, to the development of new logical systems and to the semantics of programming. Its applications to mathematics are becoming more diverse, even touching on theoretical physics, which employs higher-dimensional category theory.

To conclude: Category theory is now a common tool in the mathematician's toolbox; It is also clear that category theory organizes and unifies much of mathematics.

Definition: What is a Category?

Definition: A category $\boldsymbol{C}$ consists of:

- a collection ob($\boldsymbol{C}$) of objects;

- for each $A, B \in$ ob($\boldsymbol{C}$), a collection $\boldsymbol{C}(A, B)$ of morphisms

such that:

- for each $A, B, C \in$ ob($\boldsymbol{C}$), there is a function
  $\boldsymbol{C}$(A,B) x $\boldsymbol{C}$(B,C) $\rightarrow$ $\boldsymbol{C}$(AC), called composition and indicated with the symbol $\circ$

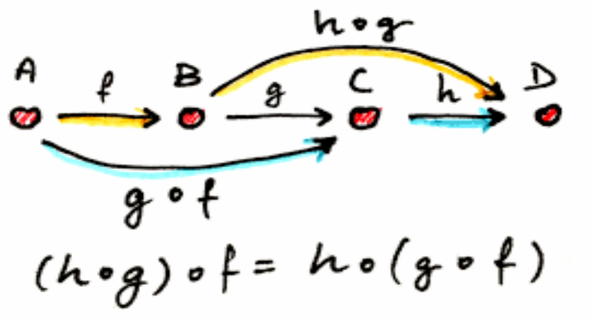
satisfying the following two axioms:

- for each $A \in$ ob(C), there exists a morphism 1_A of $C(A, A)$, called the identity on A, such that for each $f \in C(A, B)$, we have $f \circ 1_A = f = 1_B \circ f$.
- associativity: for each $f \in C(A, B)$, $g \in C(B,C)$ and $h \in C(C, D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$;

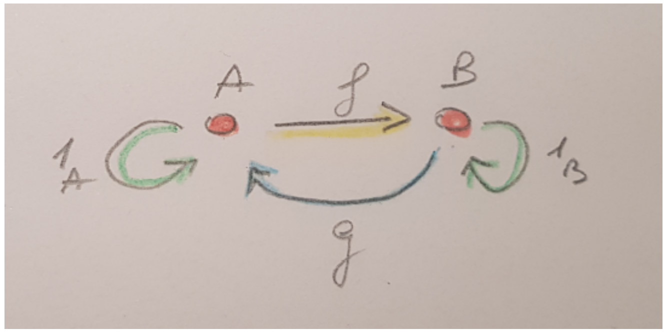
Morphisms

A morphism $f \in$ ***C*** (A, B) can be indicated by an arrow, f : A $\rightarrow$ B where A is called the domain of f and B the codomain of f.

Associativity:



Definition: A morphism f is an **isomorphism** iff there exists an inverse morphism $g: B \rightarrow A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.



Exercises

Exercise1: Any identity morphism is an isomorphism.

Exercise 2: Composition of isomorphisms is an isomorphism.

Exercise 3 : Show that a morphism in a category can have at most one inverse. That is, given a morphism $f : A \to B$, show that there is at most one morphism $g: B \to A$ such that $g \circ f = 1_A$ and $f \circ g = 1_B$.

Exercise 4: which are the morphisms that are isomorphisms in the examples mentioned in the following slides?

Examples of categories

Concrete categories: categories of mathematical structures

(a) **Set** the category described as follows. Its objects are sets. Given sets *A* and *B*, a morphism from *A* to *B* is a function from *A* to *B.*

Composition in the category is ordinary composition of functions, and the identity maps are the identity functions. This is a basic model you can always keep in mind to have some intuition about the category concept.

(b) **Grp** is the category of groups, whose objects are groups and whose maps are group homomorphisms.

Axioms are easily verified since the composition of homomorphisms is again an homomorphism and identity functions are homomophisms.

Examples of categories

Similarly

(c) **Ring** is the category of rings and ring homomorphisms.

(d) For each field *k*, there is a category ***Vect****$_k$* of vector spaces over k and linear maps between them.

(e) ***Top*** is the category of topological spaces and continuous maps.

(f) ***Latt*** is the category with objects lattices and morphisms lattice morphisms (functions that preserve sup and inf).

Examples of categories

The examples of categories mentioned so far are important, but could give a false impression. In each of them in fact, the objects of the category are sets with structure (such as a group structure, a topology, or, in the case of Set, no structure at all). The morphisms are the functions preserving the structure. And in each of them, there is a clear sense of what the elements of a given object are. However, not all categories are like this. In general, the objects of a category are not necessarily ‘sets equipped with extra stuff’. Thus, in a general category, it does not make sense to talk about the ‘elements’ of an object. Similarly, in a general category, the morphisms need not necessarily be mappings or functions in the usual sense.

The value of category theory stays in this power of abstraction and in its strong capacity of generalization.

Abstraction

A category is a system of related objects. The objects do not live in isolation: there is some notion of map between objects, binding them together.
In a category morphisms (or arrows or maps) are more important than objects!

Objects itself correspond to the identity morphisms.

Looking mainly at morphisms means that we must consider the way how objects relate each other not the objects in itself (an object does not always has elements).

Working in a category means to conside a global approach instead than a local one. Properties of objects depend on how they are in relation with other objects of the same category.
These concepts will become clear during the course.

Examples of categories: Abstract Categories

Preorder Sets as categories

A **preorder** is a reflexive and transitive binary relation. A preordered set *(S, ≤)* is a set S together with a preorder ≤ on it.

Examples:

- *S* = N and ≤ has its usual meaning;
- *S* is the set of subsets of {1,...,10} and ≤ is the inclusion ;
- *S* = Z and a ≤ b means that *a* divides *b*.

A **preordered set *(S, ≤)* can be regarded as a category S** in which objects are objects of the set ***S*** and, for each *A, B* $\in$ ***S*** , there is at most one morphism *A* → *B*. This is the case when *A ≤ B;* otherwise ***S****(A,B)* is empty.

S is clearly a category since composition of morphisms,associativity and identity axioms are automatic consequences of the transitive and reflexive properties of ≤.

Examples of categories: Abstract Categories

Category Mat of matrices: objects all natural numbers; **Mat** (m, n) is the set of all real $m \times n$ matrices (a morphism is a matrix), $id_n : n \to n$ is the unit diagonal $n \times n$ matrix, and composition of matrices is defined by the usual multiplication of matrices.

Category Rel of relations: objects are sets and morphisms are relations between sets. Composition is the classical composition of relations and identity the identity relation.

Examples of categories: Abstract Categories

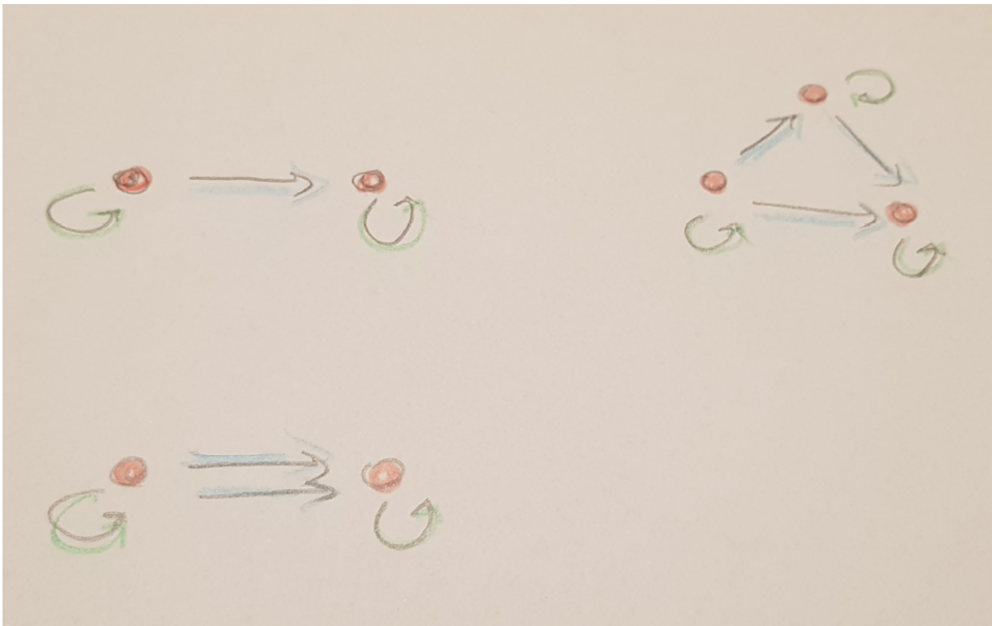
Example of a category with a unique object and many morphisms.

Any **monoid** (an algebraic structure with a single associative binary operation and an identity object) forms a category with a single object *x*. The morphisms from *x* to *x* are precisely the elements of the monoid, the identity morphism of *x* is the identity of the monoid, and the categorical composition of morphisms is given by the monoid operation. .

Similarly any gropup can be seen as a category with a single object in which every morphism is *invertible*, that is, for every morphism *f* there is a morphism *g* that is both left and right inverse to *f* under composition.

Examples of categories: Abstract Categories

We can also consider examples of **finite categories** with a finite number of objects and of arrows; we can draw these examples as graphs



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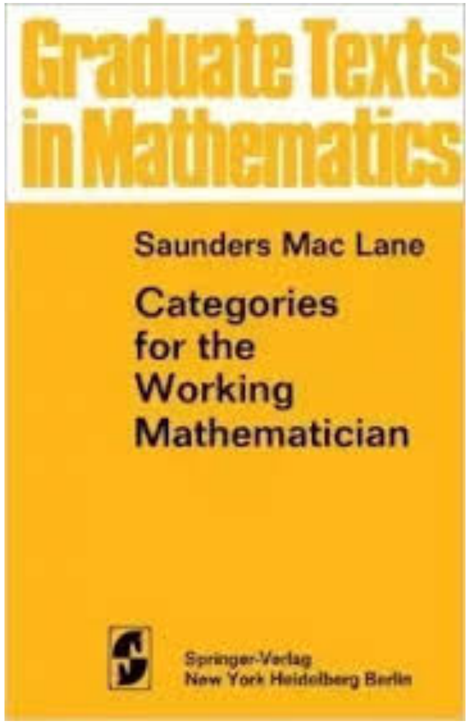

Eilenberg and Mc Lane