Lesson 2.
Projective algebraic sets and Zariski topology on the projective space.

We want to define now the projective algebraic sets, or projective varieties, in $\mathbb{P}^n$.

The idea is the same as in the affine space: a projective variety is the set of solutions of a system of polynomial equations. The difference is that a point in the projective space does not have a well defined set of coordinates: homogeneous coordinates are defined only up to proportionality. So it may happens that, given a polynomial F and a point $P \in \mathbb{P}^n$ with homogeneous coordinates $[x_0, \ldots, x_n]$, the n-tuple $x_0, \ldots, x_n$ is a zero of F, but other proportional n-tuples of the form $[\lambda x_0, \ldots, \lambda x_n]$ are not.

To give a good definition, we have to consider only homogeneous polynomials, because for them the problem does not exist. Otherwise, to say that a point $p \in \mathbb{P}^n$ is a zero of a polynomial F, we must require that it annihilates F for any choice of its homogeneous coordinates.

Let's now formalize what I have anticipated.

Let $K[x_0, x_1, \ldots, x_n]$ be the polynomial ring in $n+1$ variables. Fix a polynomial $G(x_0, x_1, \ldots, x_n) \in K[x_0, x_1, \ldots, x_n]$ and a point $P[a_0, a_1, \ldots, a_n] \in \mathbb{P}^n$: then, in general,

$$G(a_0, \ldots, a_n) \neq G(\lambda a_0, \ldots, \lambda a_n),$$

so the value of G at P is not defined.

2.7. Example. Let $G = x_1 + x_0 x_1 + x_2^2$, $P[0,1,2] = [0,2,4] \in \mathbb{P}^2_{\mathbb{R}}$. So $G(0,1,2) = 1 + 4 \neq G(0,2,4) = 2 + 16$. But if $Q = [1,0,0] = [\lambda, 0, 0]$, then $G(1,0,0) = G(\lambda, 0, 0) = 0$ for all λ.

2.8. Definition. Let $G \in K[x_0, x_1, \ldots, x_n]$: G is *homogeneous of degree d*, or G is *a form* of degree d, if G is a linear combination of monomials of degree d.

2.9. Lemma. *If G is homogeneous of degree d, $G \in K[x_0, x_1, \ldots, x_n]$, and t is a new variable, then $G(tx_0, \ldots, tx_n) = t^d G(x_0, \ldots, x_n)$.*

Proof. It is enough to prove the equality for monomials, i.e. for

$$G = a x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n} \text{ with } i_0 + i_1 + \ldots + i_n = d :$$

$$G(tx_0, \ldots, tx_n) = a(tx_0)^{i_0}(tx_1)^{i_1} \ldots (tx_n)^{i_n} = a t^{i_0+i_1+\cdots+i_n} x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n} =$$
$$= t^d G(x_0, \ldots, x_n).$$

$\square$

2.10. Definition. Let G be a homogeneous polynomial of $K[x_0, x_1, \ldots, x_n]$. A point $P[a_0, \ldots, a_n] \in \mathbb{P}^n$ is a *zero of G* if $G(a_0, \ldots, a_n) = 0$. In this case we write $G(P) = 0$.

Note that by Lemma 2.9 if $G(a_0, \ldots, a_n) = 0$, then

$$G(\lambda a_0, \ldots, \lambda a_n) = \lambda^{\deg G} G(a_0, \ldots, a_n) = 0$$

for every choice of $\lambda \in K^*$. (Remind: K^* denotes $K \setminus \{0\}$.)

2.11. Definition. A subset Z of $\mathbb{P}^n$ is a *projective algebraic set*, or a *projective variety*, if Z is the set of common zeroes of a set of homogeneous polynomials of $K[x_0, x_1, \ldots, x_n]$.

If $T \subset K[x_0, x_1, \ldots, x_n]$ is any subset formed by homogeneous polynomials, then the corresponding algebraic set will be denoted by $V_P(T)$.

We want now to give an interpretation of projective varieties as sets of zeroes of ideals, as we did in the affine case, see Proposition 2.3. But of course the ideal generated by a family of homogeneous polynomials contains also polynomials that are not homogeneous.

Let $\alpha = \langle T \rangle$ be the ideal generated by the polynomials of T, all assumed to be homogeneous. For any $F \in \alpha$, there is en expression $F = \sum_i H_i F_i$, $F_i \in T$.

So if $P \in V_P(T)$, and $P[a_0, \ldots, a_n]$, then

$$F(a_0, \ldots, a_n) = \sum H_i(a_0, \ldots, a_n) F_i(a_0, \ldots, a_n) = 0$$

for any choice of coordinates of P, regardless if F is homogeneous or not. We say that P is a *projective zero* of F.

We want to formalize this situation in the context of the *graded rings*, of which the polynomial rings are a prototype. In particular in a graded ring there will be a situation similar to the following one:

If F is a polynomial, then F can be written *in a unique way* as a sum of homogeneous polynomials, called the homogeneous components of F: $F = F_0 + F_1 + \ldots + F_d$, where, for any index i, either the degree of F_i is equal to i, or $F_i = 0$.

We give the following definition:

2.12. Definition. Let A be a ring (always assumed to be commutative with unit). A is called *a graded ring over* $\mathbb{Z}$ if there exists a family of additive subgroups of A $\{A_i\}_{i \in \mathbb{Z}}$ such that:

(i) $A = \oplus_{i \in \mathbb{Z}} A_i$; and

(ii) $A_i A_j \subset A_{i+j}$ for all pair of indices.

The elements of A_i are called *homogeneous of degree i* and A_i the homogeneous component of A of degree i. Condition (i) regards the additive structure of A; it means that any element a of A has a unique finite expression $a = \Sigma_{i\in\mathbb{Z}} a_i$, finite sum of homogeneous elements. Condition (ii) regards the multiplicative structure: a product of homogeneous elements is homogeneous of degree the sum of the degrees. Notice that 0 belongs to all homogeneous components of A.

The standard example of graded ring is the polynomial ring with coefficients in a ring R. R is the homogeneous component of degree 0, the variables have all degree 1. In this case the homogeneous components of negative degrees are all zero.

2.13 Proposition - Definition. *Let $I \subset A$ be an ideal of a graded ring. I is called* **homogeneous** *if the following equivalent conditions are fulfilled:*

(i) I is generated by homogeneous elements (this means: there is a system of generators formed by homogeneous elements);

(ii) $I = \oplus_{k\in\mathbb{Z}}(I \cap A_k)$, i.e. if $F = \Sigma_{k\in\mathbb{Z}} F_k \in I$, then all homogeneous components F_k of F belong to I.

Proof of the equivalence.

"(ii)⇒(i)": given a system of generators of I, write each of them as sum of its homogeneous components: $F_i = \Sigma_{k\in\mathbb{Z}} F_{ik}$. Then a set of homogeneous generators of I is formed by all the elements F_{ik}.

"(i)⇒(ii)": let I be generated by a family of homogeneous elements $\{G_\alpha\}$, with $\deg G_\alpha = d_\alpha$. If $F \in I$, then F is a combination of the elements G_α with suitable coefficients H_α; write each H_α as sum of its homogeneous components: $H_\alpha = \Sigma H_{\alpha k}$. Note that the product $H_{\alpha k} G_\alpha$ is homogeneous of degree $k + d_\alpha$. By the unicity of the expression of F as sum of homogeneous elements, it follows that all of them are combinations of the generators $\{G_\alpha\}$ and therefore they belong to I. □

Let $I \subset K[x_0, x_1, \dots, x_n]$ be a homogeneous ideal. Note that, by the noetherianity, I admits a finite set of homogeneous generators.

Let $P[a_0, \dots, a_n] \in \mathbb{P}^n$. If $F \in I$, $F = F_0 + \dots + F_d$, then $F_0 \in I, \dots, F_d \in I$. We say that P *is a zero of* I if P is a projective zero of any polynomial of I or, equivalently, of any homogeneous polynomial of I. This also means that P is a zero of any homogeneous polynomial of a set generating I. The set of zeroes of I will be denoted $V_P(I)$: all projective algebraic subsets of $\mathbb{P}^n$ are of this form.

As in the affine case, the projective algebraic subsets of $\mathbb{P}^n$ satisfy the axioms of the closed sets of a topology, called the Zariski topology of $\mathbb{P}^n$. This time the empty set can be expressed as $V_P(1)$, as well as $V_P(x_0, \dots, x_n)$: indeed the n- tuple $[0, ..., 0]$ is not a point of $\mathbb{P}^n$. As for the other axioms of closed sets, the idea is always the same: the equations of the intersection of a family of algebraic sets are the union of all the equations, while the union of two algebraic sets X and Y is defined by all the possible products of two equations, one of X and the other of Y.

From the point of view of ideals, it is useful to make the following remark, whose proof follows form 2.13. Let I, J be homogeneous ideals of $K[x_0, x_1, \dots, x_n]$. Then $I + J$, IJ and $I \cap J$ are homogeneous ideals.

Indeed: both I and J are generated by homogeneous polynomials, $I + J$ is generated by the union of all of them, IJ is generated by products of two of them, one in I and the other in J, so in both cases by homogeneous polynomials.

For $I \cap J$ it is enough to use 2.13 (ii).

Note that also all subsets of $\mathbb{A}^n$ and $\mathbb{P}^n$ have a structure of topological space, with the induced topology, which is still called the Zariski topology.

Exercises to §2.

1. Let $F \in K[x_1, \dots, x_n]$ be a non–constant polynomial. The set $\mathbb{A}^n \setminus V(F)$ will be denoted $\mathbb{A}^n_F$. Prove that $\{\mathbb{A}^n_F | F \in K[x_1, \dots, x_n] \setminus K\}$ is a topology basis for the Zariski topology.

2. Let $B \subset \mathbb{R}^n$ be a ball. Prove that B is not Zariski closed.

3*. Prove that the map $\phi : \mathbb{A}^1 \to \mathbb{A}^3$ defined by $t \to (t, t^2, t^3)$ is a homeomorphism between $\mathbb{A}^1$ and its image, for the Zariski topology.

4. Let $X \subset \mathbb{A}^2_{\mathbb{R}}$ be the graph of the map $\mathbb{R} \to \mathbb{R}$ such that $x \to \sin x$. Is X closed in the Zariski topology? (hint: intersect X with a line....)