Introduction to probability

Stochastic Modelling and Simulation

Exercise 1.1

Let X be a random variable distributed according to a Bernoulli distribution with parameter p. Show that $\mathbb{E}[X] = p$ and $Var[X] = p(1-p)$

Exercise 1.2

Ten persons came into contact with a person infected with tuberculosis. The probability of being infected after contacting a person with tuberculosis is 0.1.

i) What is the probability that nobody is infected?

ii) What is the probability that at least 2 persons are infected?

iii) What is the expected number of infected persons?

Exercise 2.1

Let X and Y be two random variables. Prove the following:

i) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$, when X and Y are independent.

ii) $\mathbb{E}[X + Y] = \mathbb{E}[X] + \cdot\mathbb{E}[Y]$.

iii) $\mathrm{Var}(X+Y) = \mathrm{Var}(X)+\mathrm{Var}(Y)$ if X, and Y are independent. How can you express the covariance among X and Y in case they are not independent r.v.?

iv) *Optional:* $\mathbb{E}[\mathbb{E}[Y \mid X]] = \mathbb{E}[Y]$ for jointly distributed random variables.

Exercise 2.2

Two fair dice are thrown. Let X be the random variable that denotes the number of spots shown on the first die and Y the number of spots that show on the second die. It follows that X and Y are independent and identically distributed. Compute $\mathbb{E}[X^2]$ and $\mathbb{E}[XY]$ and observe that they are not the same.

Exercise 3

Let X be an exponentially distributed random variable and let A and B be two constants such that $A \geq 0$ and $B \geq 0$. Prove that:

i) $Pr(X > A + B) = Pr(X > A)Pr(X > B)$ and that

ii) $Pr(X > A + B \mid X > A) = Pr(X > B)$.

Exercise 4.1

The time spent waiting for a bus is normally distributed with mean equal to 10 minutes and standard deviation equal to 10 minutes. Find:

i) the probability of waiting less than 12 minutes and

ii) the probability of waiting more than 15 minutes.

Exercise 4.2

A normally distributed random variable has mean $\mu = 4$ and $Pr(2 \leq X \leq 6) = 0.68$. Find the standard deviation of X.

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