## §8. Derivatives

**8.1. Derivative of a polynomial.** The derivative of a polynomial function, as of any function, is defined by limiting processes. The formula used for differentiating such functions, however, involves no such processes in any explicit fashion. We shall now see that a purely formal application of this formula will give us many of the properties of derivatives.

For any element

$$f = a_0 + a_1x + \cdots + a_nx^n$$

of $D[x]$ we define the *derivative* of $f$ with respect to $x$ to be

$$f' = a_1 + 2a_2x + \cdots + na_nx^{n-1}.$$

The (partial) derivative of $f(x_1,\cdots,x_r)$ with respect to $x_i$ may then be defined by regarding $f(x_1,\cdots,x_r)$ as a polynomial over $D[x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_r]$. The following properties of derivatives will be useful:

(i) $(f+g)' = f' + g'$.

(ii) If $a$ is a constant, $a' = 0$ and $(af)' = af'$.

(iii) $(fg)' = f'g + fg'$; $(f^n)' = nf^{n-1}f'$.

(iv) $f(g_1(x),\cdots,g_r(x))' = \Sigma_i f_i\,(g_1(x),\cdots,g_r(x))g'_i(x)$,

where $f_i(x_1,\cdots,x_r)$ is the derivative of $f$ with respect to $x_i$, $i = 1,\cdots,r$.

(i) and (ii) follow at once from the definition, and the first part of (iii) can be easily checked; the last part of (iii) is obtained by repeated application of the first part. (iv) may be proved first in the case in which $f$ consists of a single term, by use of (iii), and then in the general case by use of (i) and (ii).

Derivatives have certain undesirable properties for polynomials over a domain of characteristic $p \neq 0$. This is due to the fact that for such a domain any polynomial of the type $a_0 + a_1x^p + a_2x^{2p} + \cdots$ has zero for its derivative. Hence throughout the rest of this section we shall assume that $D$ has characteristic zero. We also assume that $D$ has unique factorization.

Our most important use of derivatives will be in the investigation of multiple factors of polynomials. A fundamental theorem in this connection is the following.

Theorem 8.1. *If $g$ is irreducible, and not a constant, $g^2 \mid f$ if and only if $g \mid f$ and $g \mid f'$.*

Proof. If $g \mid f$ then $f = gh$, and so $f' = gh' + g'h$. If, also, $g \mid f'$, then $g \mid g'h$. Since $\deg g' < \deg g$, $g \nmid g'$, and so $g \mid h$, since $g$ is irreducible. Therefore, $g^2 \mid f$. Conversely, if $f = g^2k$ then $f' = 2gg'k + g^2k'$, and so $g \mid f$ and $g \mid f'$.

As an important special case we obtain from Theorem 7.2,

THEOREM 8.2. *$(x - a)^2 \mid f(x)$ if and only if $f(a) = f'(a) = 0$.*

**8.2. Taylor's Theorem.** By induction, we define the $n$th derivative of $f(x)$ to be $f^{(n)}(x) = f^{(n-1)}(x)'$. Similarly, the higher partial derivatives can be defined. The partial derivative of $f(x_1, \cdots, x_n)$ with respect to $x_i$ may be designated by $\partial f/\partial x_i$, $f_{x_i}$, or $f_i$, with obvious extensions to higher derivatives. The relation $f_{ij} = f_{ji}$ is easily verified, and from this it follows that all the higher partial derivatives are independent of the order of differentiation.

We shall now prove Taylor's Theorem.

THEOREM 8.3. *For any polynomial $f(x)$ of degree n and any constant a,*

$$f(x) = f(a) + f'(a)(x - a) + \frac{1}{2!} f''(a)(x - a)^2 + \cdots + \frac{1}{n!} f^{(n)}(a)(x - a)^n.$$

PROOF. Let

$$f(x + a) = a_0 + a_1 x + \cdots + a_n x^n. \tag{8.1}$$

Differentiating successively with respect to $x$, we obtain, using (iv),

$$f'(x + a) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1},$$

$$f''(x + a) = 2a_2 + 2 \cdot 3 a_3 x + \cdots + n(n - 1) a_n x^{n-2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f^{(n)}(x + a) = n!\, a_n.$$

Putting $x = 0$ in each of these equations, we have

$$a_0 = f(a), \quad a_1 = f'(a), \quad a_2 = \frac{1}{2!} f''(a), \cdots, a_n = \frac{1}{n!} f^{(n)}(a).$$

Substituting in (8.1), and replacing $x$ by $x - a$, we have the desired result.

This theorem is easily extended to polynomials in several variables.

THEOREM 8.4. *Let $f(x_1 \cdots, x_r) \in D[x_1, \cdots, x_r]$, let $a_1, \cdots, a_r \in D$, and designate by $f_{ij} \ldots$ the result of substituting $a_1, \cdots, a_r$ for $x_1, \cdots, x_r$ in the partial derivative of f with respect to $x_i, x_j, \cdots$. Then*

$$f(x_1, \cdots, x_r) = f(a_1, \cdots, a_r) + \Sigma_i f_i(x_i - a_i) + \frac{1}{2!} \Sigma_{i,j} f_{ij}(x_i - a_i)(x_j - a_j) + \cdots$$

PROOF. By the previous theorem expand

$F(t) = f(a_1 + (x_1 - a_1)t, \cdots, a_r + (x_r - a_r)t)$ in powers of $t$. Using (iv) we obtain

$$F(t) = f(a_1, \cdots, a_r) + \Sigma_i f_i(x_i - a_i)t + \frac{1}{2!}\Sigma_{i,j} f_{ij}(x_i - a_i)(x_j - a_j)t^2 + \cdots.$$

The desired result is obtained by putting $t = 1$.

**8.3. Exercises.** 1. Properties (i),$\cdots$,(iv) will hold for rational functions if and only if we define $(f/g)'$ to be $(f'g - fg')/g^2$.

2. Extend Theorems 8.1 and 8.2 from squares to $n$th powers.

3. Prove Theorem 8.2 and its generalization for domains of any characteristic.