# SEISMOLOGY

Master Degree Programme in Physics - UNITS
Physics of the Earth and of the Environment

# WAVE EQUATION (Strings)

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# What is a wave?

**Small** perturbations of a **stable** equilibrium point → **Linear restoring force** → **Harmonic Oscillation**

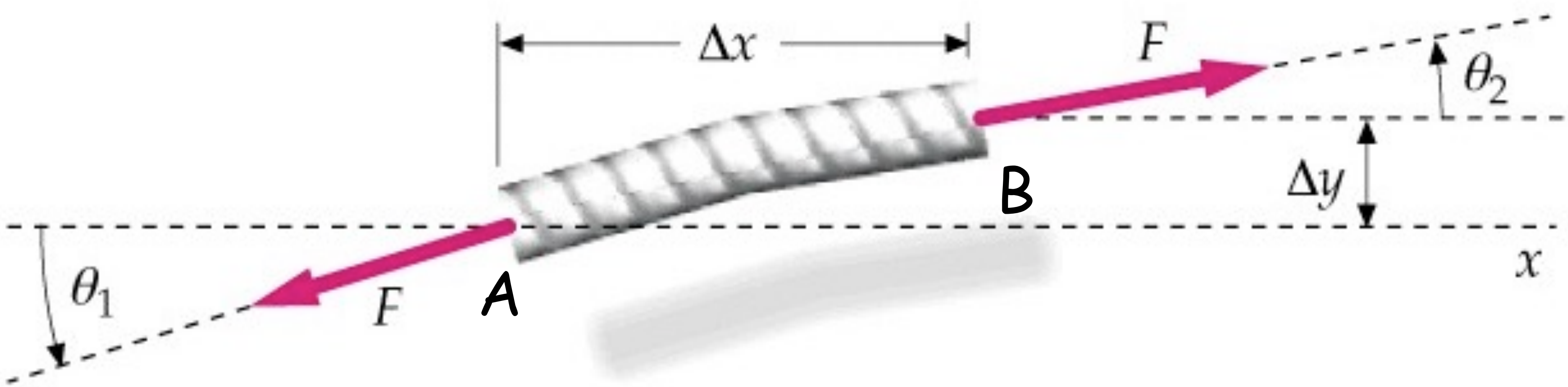
**Coupling** of harmonic oscillators → the disturbances can **propagate**, superpose and stand

**WAVE: organized propagating imbalance**, satisfying differential equations of motion

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

General form of LWE

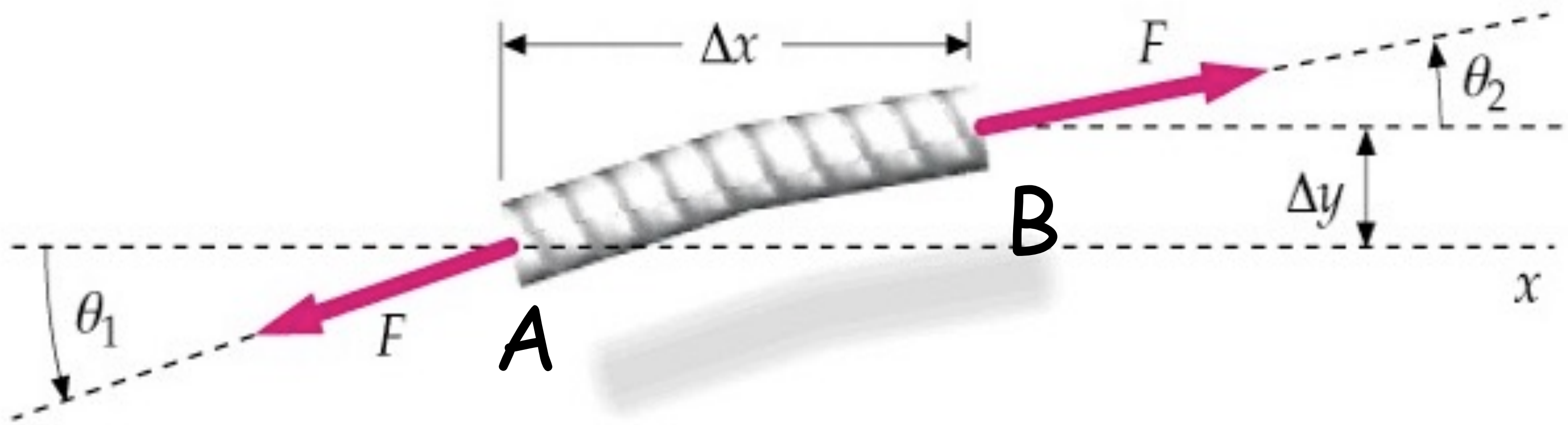
# Derivation of the wave equation




Consider a small segment of string of length $\Delta x$ and tension F

The ends of the string make **small** angles $\theta_1$ and $\theta_2$ with the x-axis.

The vertical displacement $\Delta y$ is very **small** compared to the length of the string




Resolving forces vertically

$$\sum F_y = F\sin\theta_2 - F\sin\theta_1$$

$$= F(\sin\theta_2 - \sin\theta_1)$$

From small angle approximation
$\sin\theta \sim \tan\theta$

$$\sum F_y \approx F(\tan\theta_2 - \tan\theta_1)$$

The tangent of angle A (B) = pendence of the curve in A (B)

given by $\dfrac{\partial y}{\partial x}$

$$\therefore \quad \sum F_y \approx F\left(\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right)$$

We now apply N2 to segment

$$\sum F_y = ma = \mu\,\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right)$$

$$\mu\,\Delta x\left(\frac{\partial^2 y}{\partial t^2}\right) = F\left(\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A\right)$$

$$\frac{\mu}{F}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\left[(\partial y/\partial x)_B - (\partial y/\partial x)_A\right]}{\Delta x}$$

$$\frac{\mu}{F}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{[(\partial y/\partial x)_B - (\partial y/\partial x)_A]}{\Delta x}$$

The derivative of a function is defined as

$$\left(\frac{\partial f}{\partial x}\right) = \lim_{\Delta x \to 0} \frac{[f(x+\Delta x) - f(x)]}{\Delta x}$$

If we associate f(x+Δx) with $(\partial y/\partial x)_B$ and f(x) with $(\partial y/\partial x)_A$

as Δx -> 0

$$\frac{\mu}{F}\left(\frac{\partial^2 y}{\partial t^2}\right) = \frac{\partial^2 y}{\partial x^2}$$

This is the linear wave equation for waves on a string

# Solution of the wave equation

Consider a wavefunction of the form $y(x,t) = A\sin(kx-\omega t)$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(kx-\omega t) \qquad \frac{\partial^2 y}{\partial x^2} = -k^2 A\sin(kx-\omega t)$$

If we substitute these into the linear wave equation

$$\frac{\mu}{F}\left(-\omega^2 A\sin(kx-\omega t)\right) = -k^2 A\sin(kx-\omega t)$$

$$\frac{\mu}{F}\omega^2 = k^2$$

Using $v = \omega/k$ , $v^2 = \omega^2/k^2 = F/\mu$

$$v = \sqrt{F/\mu}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

General form of LWE

# The linear wave equation

We introduce the concept of a wavefunction to represent waves travelling on a string.

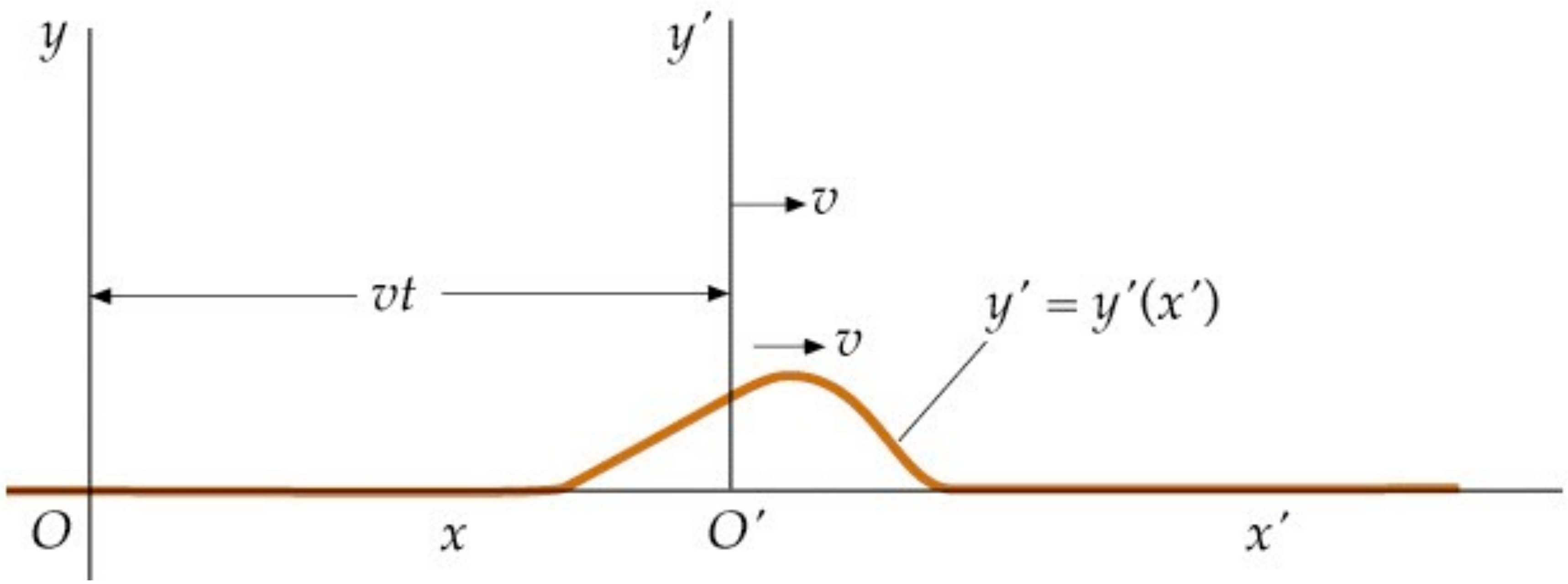
All wavefunctions y(x,t) represent solutions of the

**LINEAR WAVE EQUATION**

The wave equation provides a complete description of the wave motion and from it we can derive the wave velocity

# The Wavefunction

This is a mathematical description of a travelling wave




Consider a transverse wave on a string moving along the x-axis with constant speed v.

Transverse displacement of string is y, maximum = $y_m$

At some time t later the pulse is vt further down the string, but shape of pulse is unchanged

The shape of the pulse can be described by $y = f(x)$

If the shape doesn't change as a function of time then we can represent the displacement y for all later times measured in a stationary frame with the origin at 0 as

$y = f(x - vt)$ for a wave moving to right

$y = f(x + vt)$ for a wave moving to left

The displacement y is sometimes referred to as the

**WAVEFUNCTION**

and is usually written as $y(x,t)$

# D'Alembert's solution

D'Alembert (1747) "Recherches sur la courbe que forme une corde tendue mise en vibration" (Researches on the curve that a tense cord forms [when] set into vibration), Histoire de l'académie royale des sciences et belles lettres de Berlin, vol. 3, pages 214-219.

D'Alembert (1750) "Addition au mémoire sur la courbe que forme une corde tenduë mise en vibration," Histoire de l'académie royale des sciences et belles lettres de Berlin, vol. 6, pages 355-360.

$$y(x,t) \to y(\xi,\eta) \text{ with } \xi = x - vt,\ \eta = x + vt$$

$$y_x = \frac{\partial y}{\partial x} = y_\xi \xi_x + y_\eta \eta_x = y_\xi + y_\eta; \quad y_{xx} = \frac{\partial}{\partial x}\left(y_x\right) = y_{\xi\xi} + 2y_{\xi\eta} + y_{\eta\eta}, \quad y_{tt} = v^2\left(y_{\xi\xi} - 2y_{\xi\eta} + y_{\eta\eta}\right)$$

$$\Rightarrow y_{\xi\eta} = \frac{\partial^2 y}{\partial\xi\,\partial\eta} = \frac{\partial}{\partial\xi}\left(\frac{\partial y}{\partial\eta}\right) = 0$$

$$y = h(\xi) + g(\eta) \;\Rightarrow\; y(x,t) = h(x - vt) + g(x + vt)$$

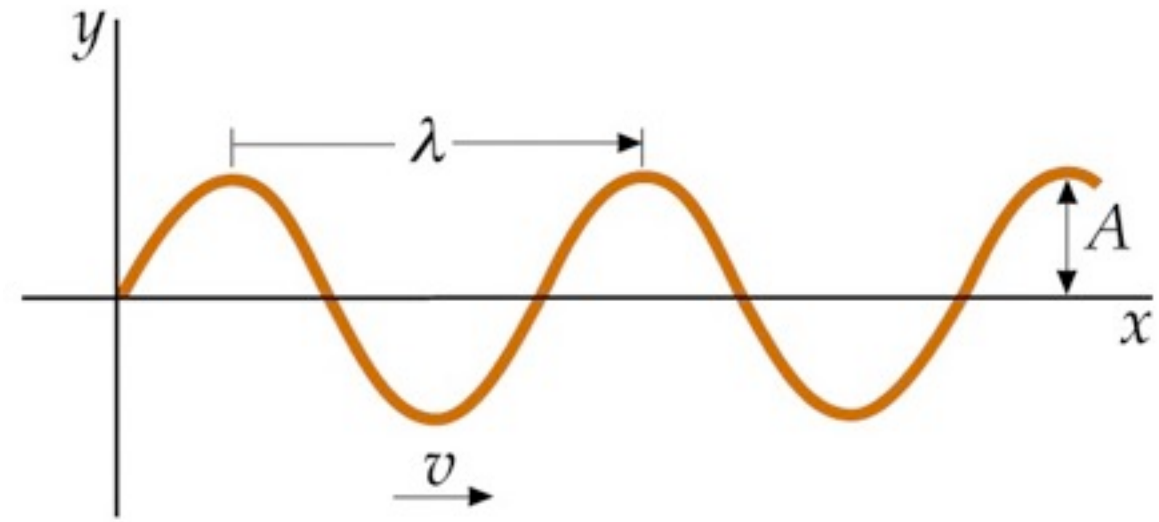
and if the initial conditions are $y(x,0)=f(x)$ and initial velocity=0

$$y(x,t) = \frac{1}{2}\left[f(x - vt) + f(x + vt)\right]$$

# Harmonic Waves

A harmonic wave is sinusoidal in shape, and has a displacement $y$ at time t=0

$$y = A\sin\left(\frac{2\pi}{\lambda}x\right)$$




A is the amplitude of the wave and λ is the wavelength (the distance between two crests)

if the wave is moving to the right with speed (or phase velocity) v, the wavefunction at some t is given by

$$y = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

Time taken to travel one wavelength is the period T

Phase velocity, wavelength and period are related by

$$v = \frac{\lambda}{T} \quad \text{or} \quad \lambda = vT$$

$$\therefore \quad y = A\sin\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]$$

The wavefunction shows the periodic nature of y:

at any time t y has the same value at x, x+λ, x+2λ.........

and at any x y has the same value at times t, t+T, t+2T......

It is convenient to express the harmonic wavefunction by defining the **wavenumber k**, and the **angular frequency** $\omega$

$$\text{where} \quad k = \frac{2\pi}{\lambda} \quad \text{and} \quad \omega = \frac{2\pi}{T}$$

$$\therefore \quad y = A\sin(kx - \omega t)$$

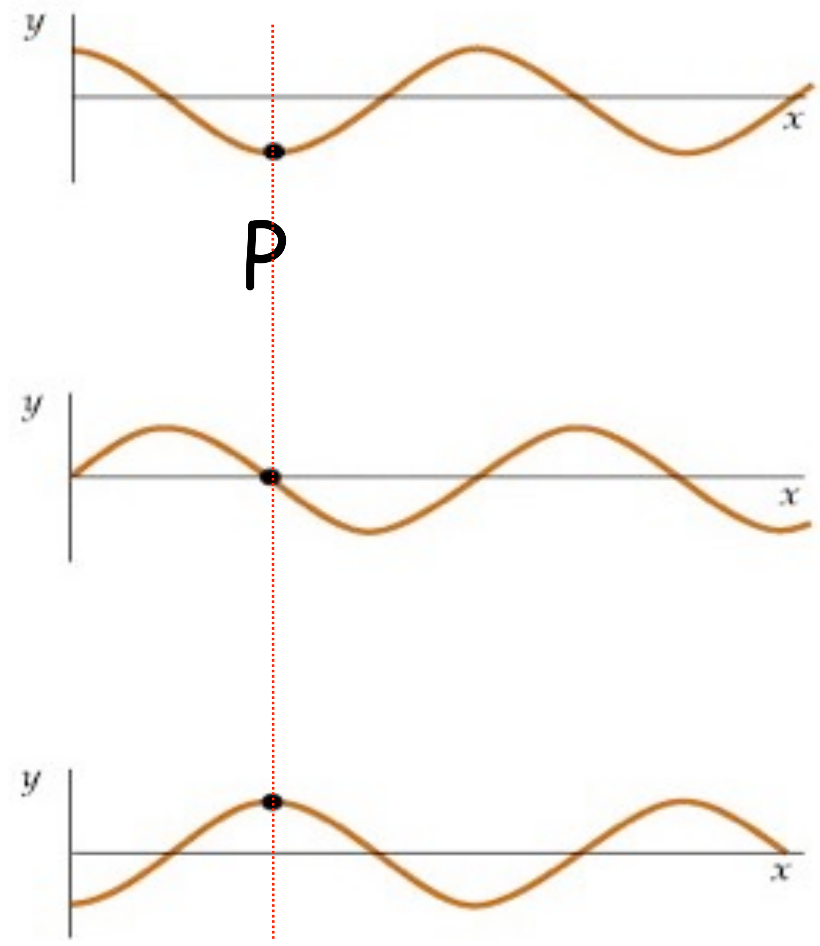
This assumes that the displacement is zero at x=0 and t=0. If this is not the case we can use a more general form

$$y = A\sin(kx - \omega t - \phi)$$

where $\phi$ is the **phase constant** and is determined from initial conditions

The wavefunction can be used to describe the motion of any point P.

If $y = A\sin(kx - \omega t)$




Transverse velocity $v_y$

$$v_y = \left.\frac{dy}{dt}\right|_{x=\text{constant}}$$

$$= \frac{\partial y}{\partial t}$$

$$= -\omega A\cos(kx - \omega t)$$

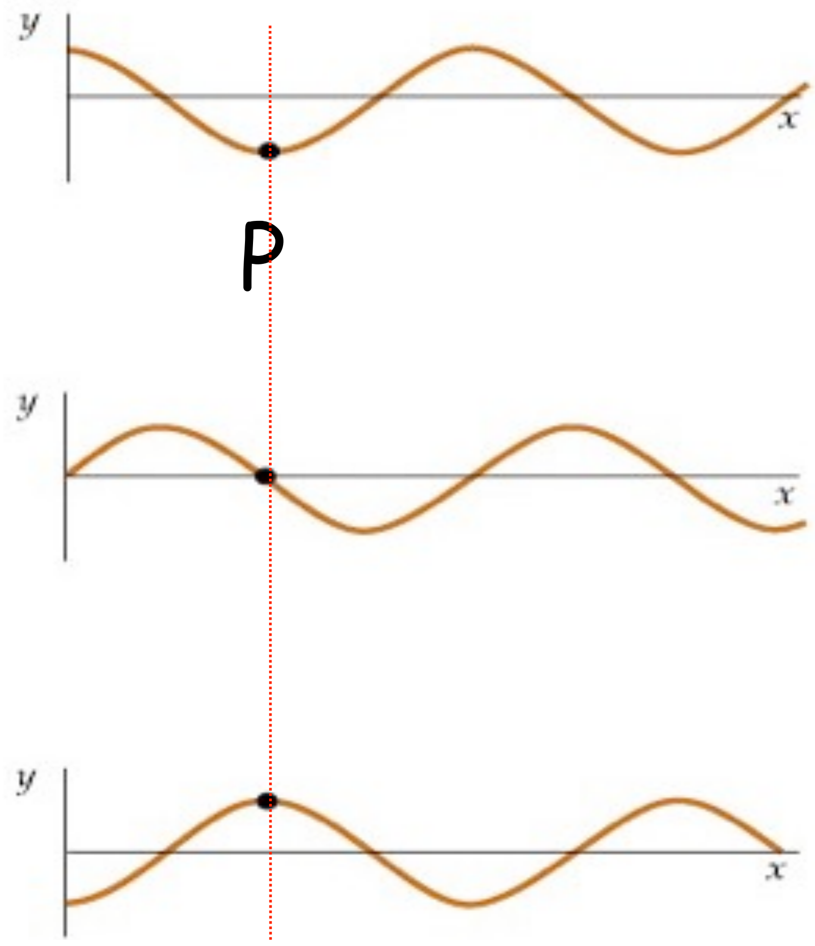
which has a maximum value $(v_y)_{max} = \omega A$ when y = 0

Transverse acceleration $a_y$

$$a_y = \left.\frac{dv_y}{dt}\right|_{x=\text{constant}}$$

$$= \frac{\partial v_y}{\partial t}$$

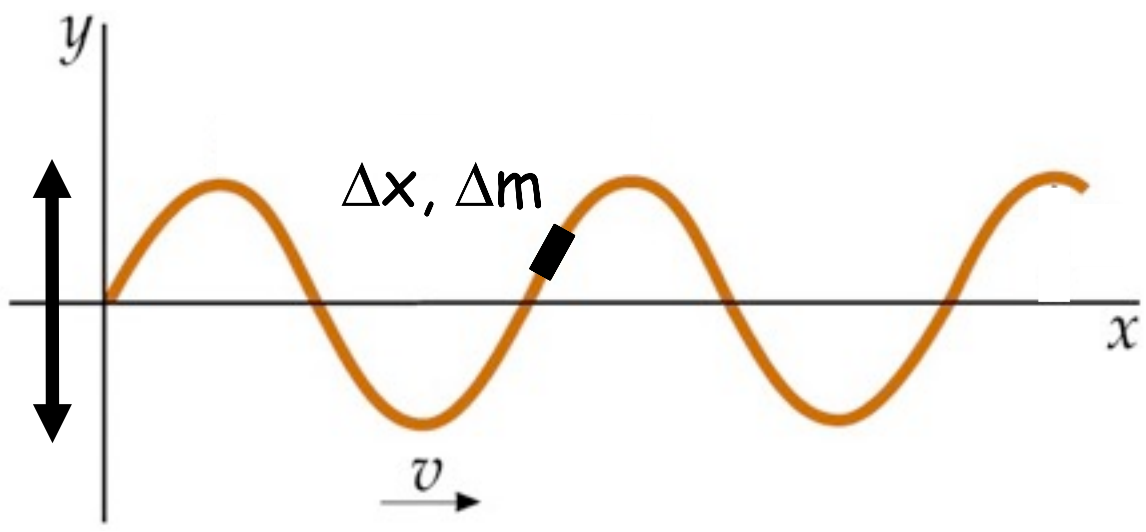
$$= -\omega^2 A \sin(kx - \omega t)$$




which has a maximum value $(a_y)_{max} = \omega^2 A$ when $y = -A$

NB: x-coordinates of P are constant

# Energy of waves on a string




Consider a harmonic wave travelling on a string.

Source of energy is an external agent on the left of the wave which does work in producing oscillations.

Consider a small segment, length $\Delta x$ and mass $\Delta m$.

The segment moves vertically with SHM, frequency $\omega$ and amplitude A.

Generally $E = \frac{1}{2}kA^2 = \frac{1}{2}m\omega^2 A^2$ (where k is the "force constant" of the restoring force)

# Energy of waves on a string

$$E = \frac{1}{2} m\omega^2 A^2$$

If we apply this to our small segment, the total energy of the element is

$$\Delta E = \frac{1}{2}(\Delta m)\,\omega^2 A^2$$

If $\mu$ is the mass per unit length, then the element $\Delta x$ has mass $\Delta m = \mu\ \Delta x$

$$\Delta E = \frac{1}{2}(\mu\,\Delta x)\,\omega^2 A^2$$

If the wave is travelling from left to right, the energy $\Delta E$ arises from the work done on element $\Delta m_i$ by the element $\Delta m_{i-1}$ (to the left).

# Energy of waves on a string

Similarly $\Delta m_i$ does work on element $\Delta m_{i+1}$ (to the right) ie. energy is transmitted to the right.

The rate at which energy is transmitted along the string is the power and is given by dE/dt.

If $\Delta x$ -> 0 then

$$\text{Power} = \frac{dE}{dt} = \frac{1}{2}(\mu \frac{dx}{dt})\omega^2 A^2$$

but dx/dt = speed

$$\therefore \quad \text{Power} = \frac{1}{2}\mu\, \omega^2 A^2 v$$

# Energy of waves on a string

$$\text{Power} = \frac{1}{2}\mu\,\omega^2 A^2 v$$

Power transmitted on a harmonic wave is proportional to

(a) the wave speed v

(b) the square of the angular frequency $\omega$

(c) the square of the amplitude $A$

All harmonic waves have the following general properties:

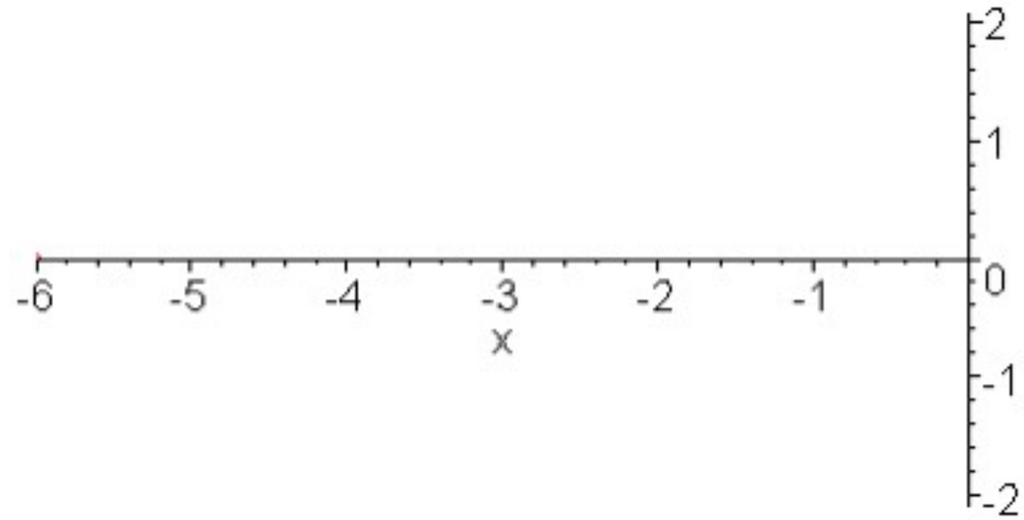
**The power transmitted by any harmonic wave is proportional to the square of the frequency and to the square of the amplitude.**

# Standing waves

If a string is clamped at both ends, waves will be reflected from the fixed ends and a standing wave will be set up.

The incident and reflected waves will combine according to the principle of superposition

Essential in music and quantum theory !



# Wavefunction for a standing wave

Consider two sinusoidal waves in the same medium with the same amplitude, frequency and wavelength but travelling in opposite directions

$$y_1 = A_o \sin(kx - \omega t) \qquad\qquad y_2 = A_o \sin(kx + \omega t)$$

$$y = A_o \left[\sin(kx - \omega t) + \sin(kx + \omega t)\right]$$

Using the identity $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$$y = 2A_o \sin(kx)\cos(\omega t)$$

This is the wavefunction of a standing wave

$$y = 2A_0 \sin(kx) \cos(\omega t)$$

Amplitude = $2A_0\sin(kx)$

Angular frequency = $\omega$

Every particle on the string vibrates in SHM with the same frequency.

The amplitude of a given particle depends on $x$

Compare this to travelling harmonic wave where all particles oscillate with the same amplitude and at the same frequency

# Antinodes

$$y = 2A_0 \sin(kx)\cos(\omega t)$$

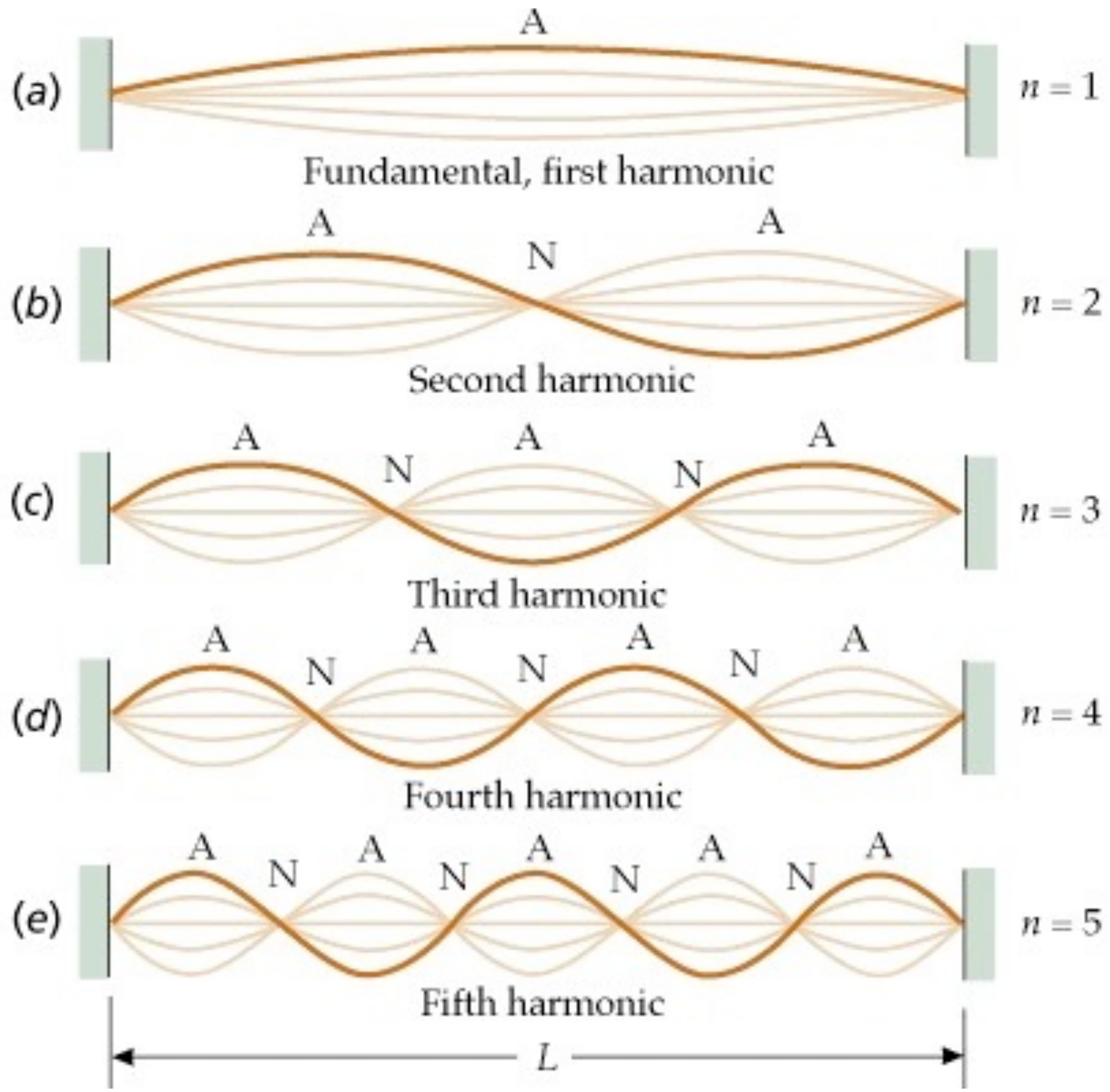
At any x maximum amplitude ($2A_o$) occurs when sin(kx) = 1

$$\text{or when} \quad kx = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots\ldots$$

but $k = 2\pi / \lambda$ and positions of maximum amplitude occur at

$$x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4} \ldots\ldots = \frac{n\lambda}{4} \quad \text{with } n = 1, 3, 5, \ldots$$

Positions of maximum amplitude are **ANTINODES** and are separated by a distance of $\lambda/2$.




Tipler Fig 16-11

# Nodes

$$y = 2A_0 \sin(kx)\cos(\omega t)$$

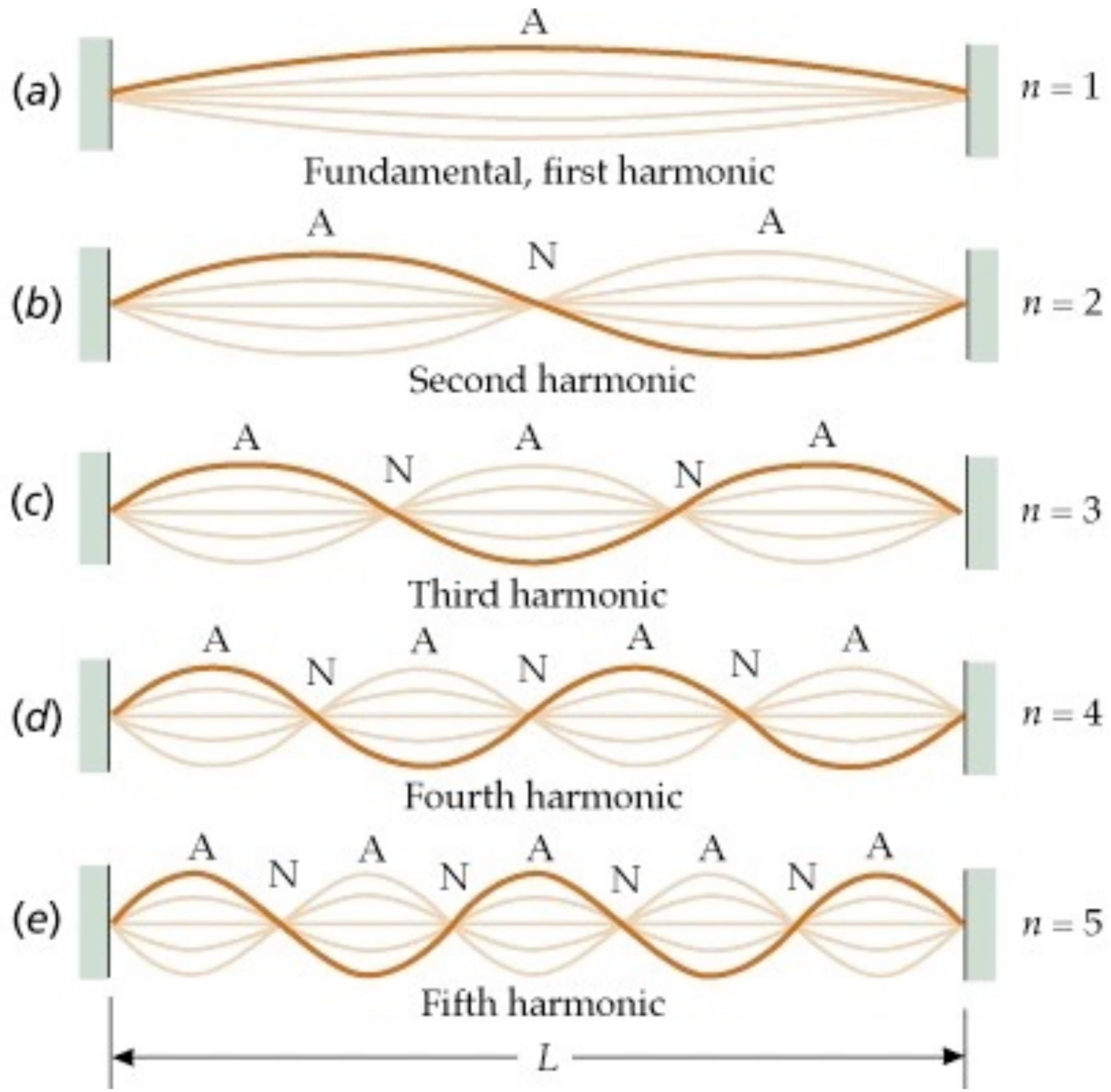
Similarly zero amplitude occurs when sin(kx) = 0

or when $kx = \pi,\ 2\pi,\ 3\pi.......$

$$x = \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}....... = \frac{n\lambda}{2} \quad \text{with } n = 1, 2, 3,....$$

Positions of zero amplitude are **NODES** and are also separated by a distance of $\lambda/2$.

The distance between a node and an antinode is $\lambda/4$

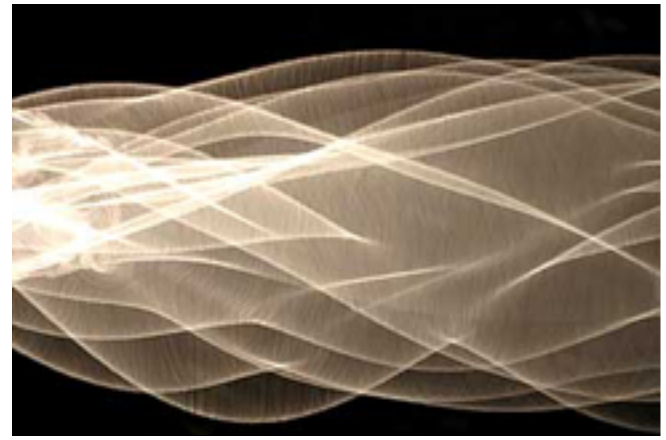



Tipler Fig 16-11

# Separation of variables

- A starting point to study differential equations is to guess solutions of a certain form (ansatz). Dealing with linear PDEs, the superposition principle principle guarantees that linear combinations of separated solutions will also satisfy both the equation and the homogeneous boundary conditions.
    - Separation of variables: a PDE of n variables $\Rightarrow$ n ODEs
    - Solving the ODEs by BCs to get normal modes (solutions satisfying PDE and BCs).
- The proper choice of linear combination will allow for the initial conditions to be satisfied
    - Determining exact solution (expansion coefficients of modes) by ICs

# Separation of variables: string



$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

and if it has separable solutions:

$$y(x,t) = X(x)T(t)$$

$$\frac{d^2X(x)}{dx^2} + k^2X(x) = 0 \qquad T''(t) + c^2k^2T(t) = 0$$

$$X(x) = A\cos(kx) + B\sin(kx) \qquad T(t) = C\cos(\omega t) + D\sin(\omega t)$$

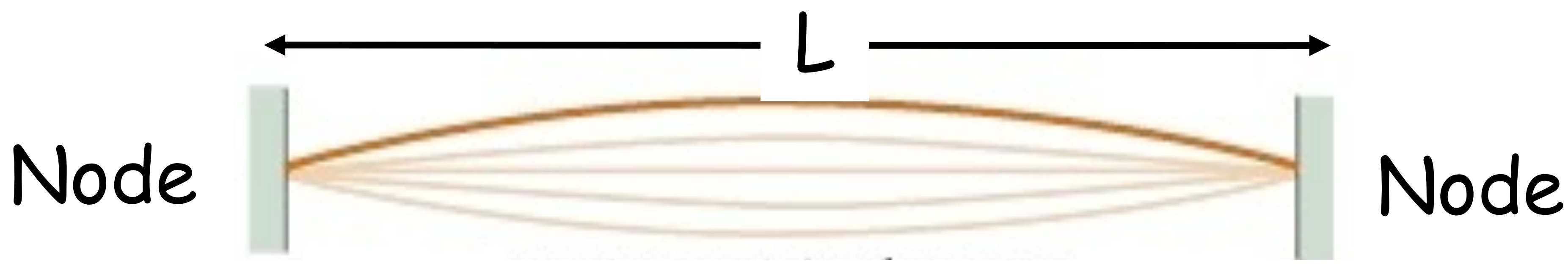
$\omega = ck$

To be determined by **initial** and **boundary** conditions

Consider a string of length L and fixed at both ends

The string has a number of natural patterns of vibration called **NORMAL MODES**

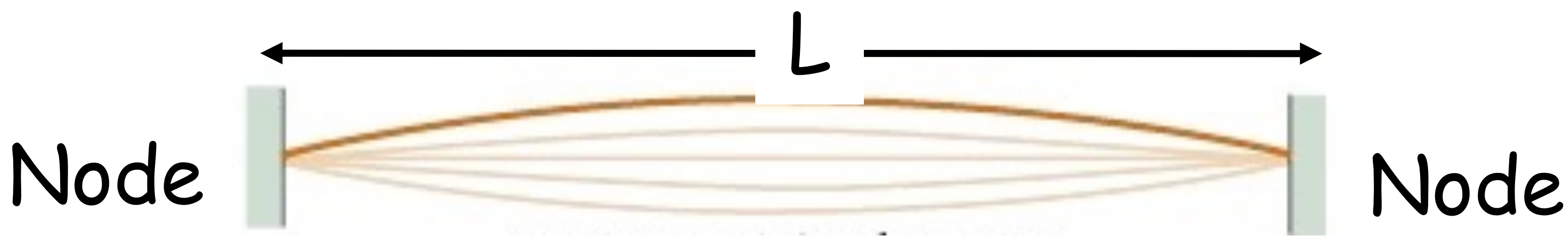
Each normal mode has a characteristic frequency which we can easily calculate




When the string is displaced at its mid point the centre of the string becomes an antinode.

# Standing waves in a string fixed at both ends

For first normal mode $L = \lambda_1 / 2$




The next normal mode occurs when the length of the string $L$ = one wavelength, ie $L = \lambda_2$

The third normal mode occurs when $L = 3\lambda_3 /2$

Generally normal modes occur when $L = n\lambda_n /2$

ie $\lambda_n = \frac{2L}{n}$ where $n = 1,2,3 \ldots\ldots$

# Standing waves in a string fixed at both ends

The natural frequencies associated with these modes can be derived from $f = v/\lambda$

$$f = \frac{v}{\lambda} = \frac{n}{2L}v \quad \text{with } n = 1,2,3.....$$

For a string of mass/unit length $\mu$, under tension F we can replace v by $(F/\mu)^{\frac{1}{2}}$

$$f = \frac{n}{2L}\sqrt{\frac{F}{\mu}} \quad \text{with } n = 1,2,3.....$$

The lowest frequency (**fundamental**) corresponds to n = 1

$$\text{ie} \quad f = \frac{1}{2L}v \qquad \text{or} \quad f = \frac{1}{2L}\sqrt{\frac{F}{\mu}}$$

# Musical Interpretation

The frequencies of modes with n = 2, 3, … (**harmonics**) are integral multiples of the fundamental frequency, $2f_1$, $3f_1$……

These higher natural frequencies together with the fundamental form a **harmonic series**.

The fundamental $f_1$ is the first harmonic, $f_2 = 2f_1$ is the second harmonic, $f_n = nf_1$ is the nth harmonic

In music the allowed frequencies are called **overtones** where the second harmonic is the first overtone, the third harmonic the second overtone etc.

We can obtain these expressions from the wavefunctions

Consider wavefunction of a standing wave:

$$y(x,t) = 2A_0 \sin(kx)\cos(\omega t)$$

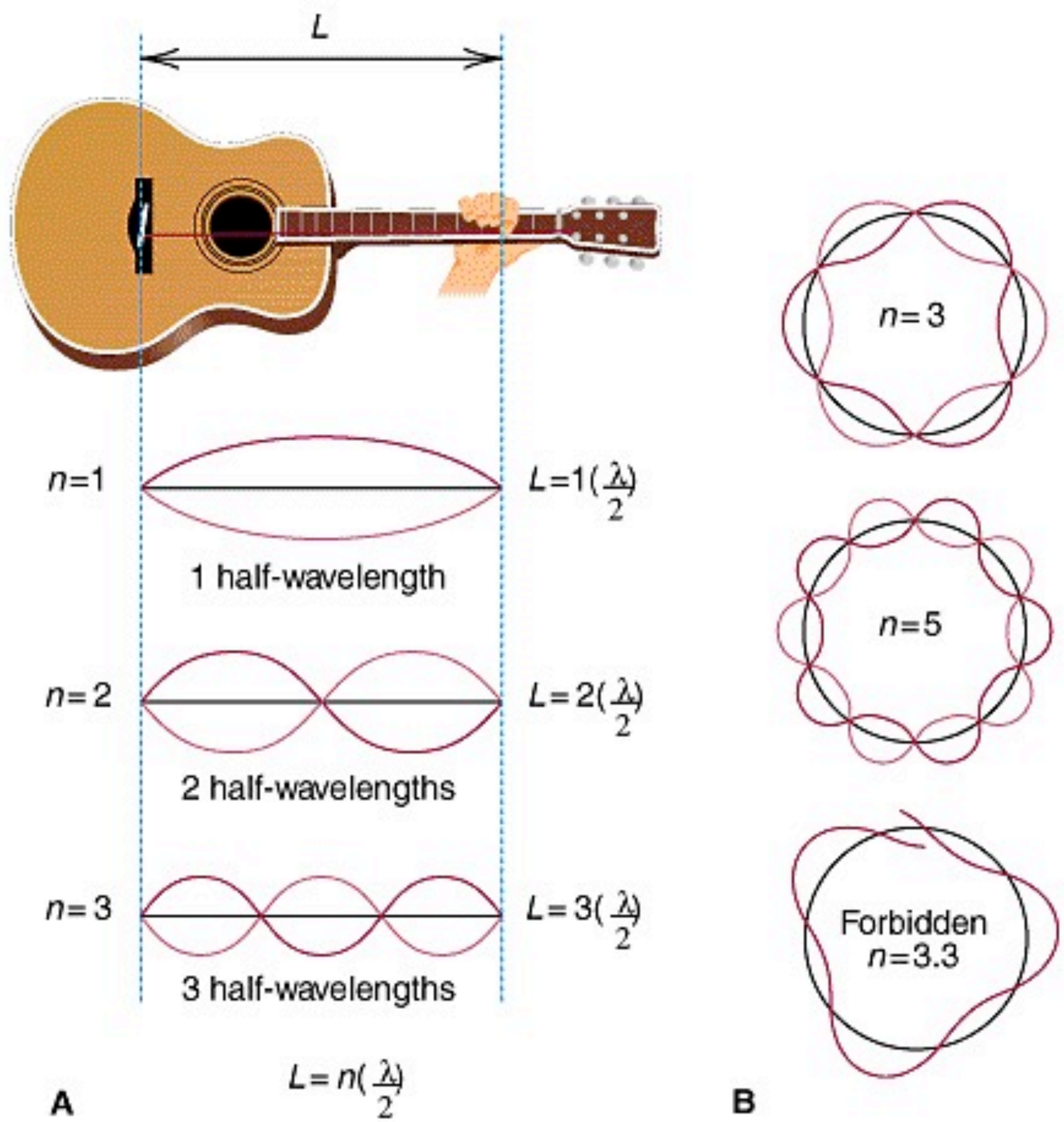
String is fixed at both ends $\therefore$ $y(x,t) = 0$ at $x = 0$ and $L$

$y(0,t)=0$ when $x = 0$ as $\sin(kx) = 0$ at $x = 0$

$y(L,t) = 0$ when $\sin(kL) = 0$ ie $k_n L = n\pi$ $n=1,2,3....$

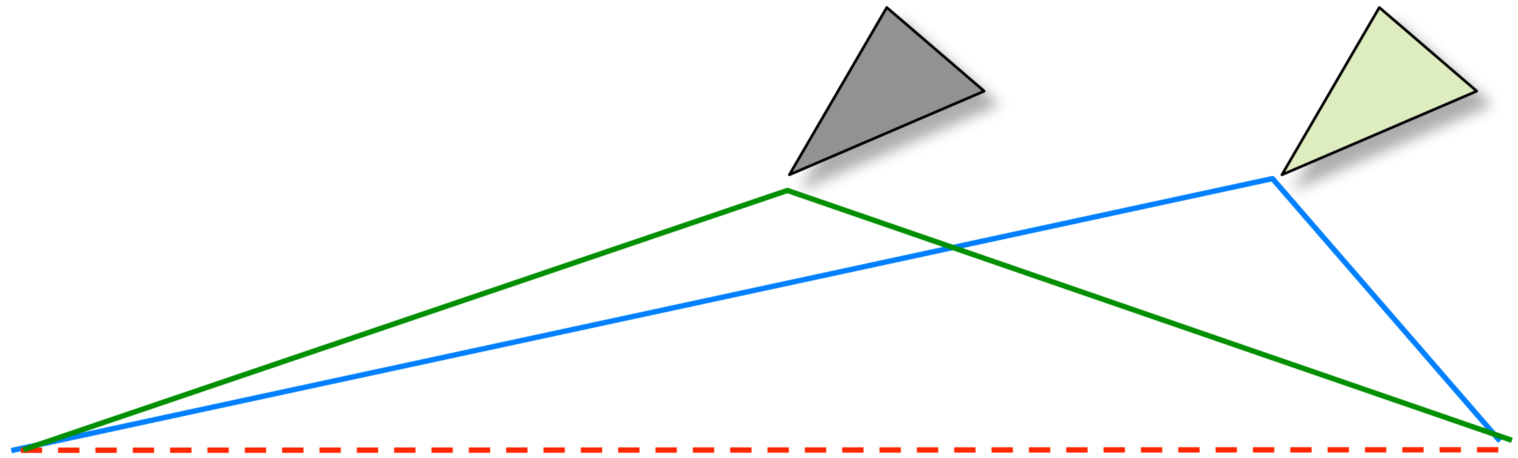
but $k_n = 2\pi / \lambda$ $\therefore$ $(2\pi / \lambda_n)L = n\pi$ or $\boxed{\lambda_n = 2L/n}$

# Guitars and Quantum mechanics



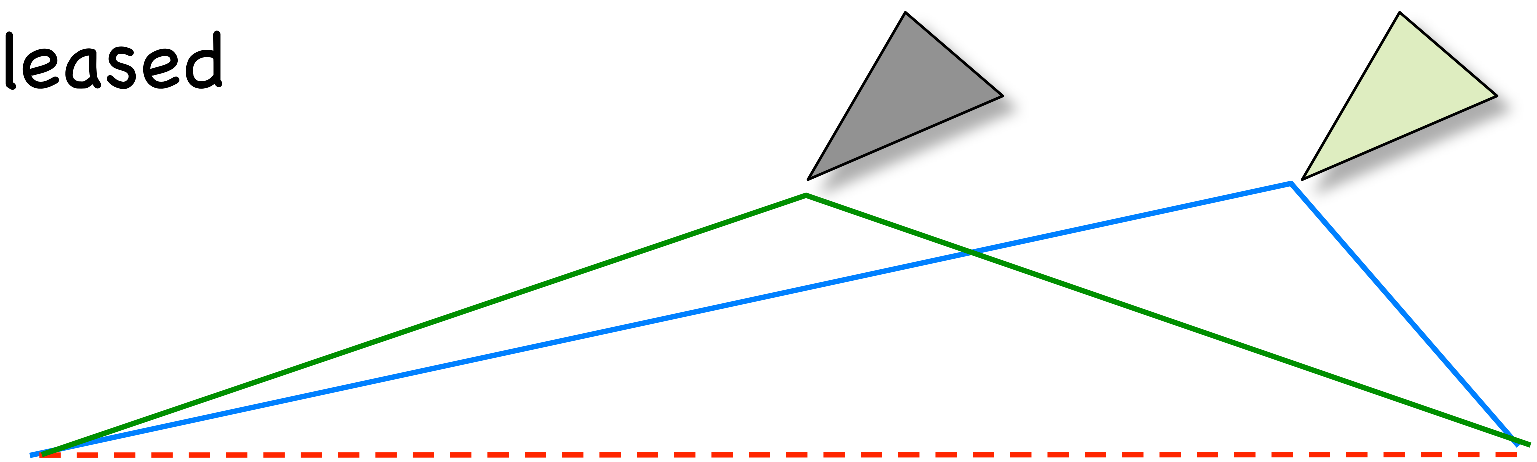
# Plucked string

- Can one predict the amplitude of each mode (overtone/harmonic?) following plucking?



- Using the procedure to measure the Fourier coefficients it is possible to predict the amplitude of each harmonic tone.

- You know the shape just before it is plucked.
- You know that each mode moves at its own frequency
- The shape when released
- We rewrite this as



$$f(x,t=0)$$

$$f(x,t=0)=\sum_{m=0}^{\infty} A_m \sin\frac{2\pi mx}{2L}$$

# Plucked string

Each harmonic has its own frequency of oscillation, the m-th harmonic moves at a frequency $f_m = mf_0$ or m times that of the fundamental mode.

$$f(x, t=0) = \sum_{m=0}^{\infty} A_m \sin\frac{2\pi mx}{2L} \quad \text{initial condition}$$

$$f(x,t) = \sum_{m=0}^{\infty} A_m \sin\frac{2\pi mx}{2L} \cos 2\pi m f_0 t$$

http://www.falstad.com/loadedstring/

# Modal summation on a string

Recall modes on a string:

$$u(x, t) = \sum_{n=0}^{\infty} A_n \, U_n(x, \omega_n) \cos(\omega_n t)$$

This is the sum of standing waves or *eigenfunctions,* $U_n(x, \omega_n)$, each of which is weighted by the amplitude $A_n$ and vibrates at its *eigenfrequency* $\omega_n$.

The eigenfunctions and eigenfrequencies are constants due to the physical properties of the string.

The amplitudes depend on the position and nature of the source that excited the motion.

The eigenfunctions were constrained by the boundary conditions, so that

$$U_n(x, \omega_n) = \sin(n\pi x/L) = \sin(\omega_n x/\mathrm{v}) \qquad \omega_n = n\pi \mathrm{v}/L = 2\pi \mathrm{v}/\lambda$$

$$u(x, t) = \sum_{n=0}^{\infty} \sin(n\pi x_s/L)\, F(\omega_n)\, \sin(n\pi x/L)\, \cos(\omega_n t)$$

The source, at $x_s = 8$, is described by

$$F(\omega_n) = \exp[-(\omega_n \tau)^2/4]$$

with $\tau = 0.2$.