LESSON 5.

1. Proof of Normalization Lemma

Even if it is known as Normalization "Lemma", this is a deep theorem in algebra, with many applications, not merely a lemma to prove the Nullstellensatz. Later we will see its interesting geometric interpretation.

It takes its name from Emmy Noether, who in 1926 proved it under the hypothesis that K was infinite. The case where K is a finite field was demonstrated by Oscar Zariski in 1943. To prove the Normalization Lemma, we will first see a couple of results about integral elements over a ring. Then we will see a proof over an infinite field, rather similar to the original one. It is less technical than any proof of the general case. For other proofs see [Atiyah-MacDonald] or [Lang, Algebra, 2nd ed.].

Let $A \subseteq B$ be rings, where A is a subring of B. In this case we also say that B is an A-algebra. Note that B has a natural structure of A-module. If B is finitely generated **as A-module**, then B is called a **finite A-algebra**. This means that there exist elements $b_1, \ldots, b_r \in B$ such that $B = b_1A + b_2A + \ldots + b_rA$, i.e. any element of B is a linear combination with coefficients in A of the generators $b_1, \ldots, b_r$: if $b \in B$, then there is an expression $b = a_1b_1 + \cdots + a_rb_r$, with $a_1, \ldots, a_r \in A$.

If B is finitely generated **as a ring containing** A, then B is called a **finitely generated A-algebra**. In this case there exists a finite number of elements of B, $b_1, \ldots, b_r$, such that $B = A[b_1, \ldots, b_r]$; B is the minimal ring containing A and the elements $b_1, \ldots, b_r$. For any element of B there is an expression as polynomial with coefficients in A in the elements $b_1, \ldots, b_r$. Another way to express that B is a finitely generated A-algebra is saying that B is (isomorphic to) a quotient of a polynomial ring in a finite number of variables with coefficients in A. Indeed, if $B = A[b_1, \ldots, b_r]$, we can define a surjective ring homomorphism φ sending any polynomial $f(x_1, \ldots, x_r) \in A[x_1, \ldots, x_r]$ to $f(b_1, \ldots, b_r)$. So, by the homomorphism theorem, $B \simeq A[x_1, \ldots, x_r]/\ker\varphi$.

Theorem 1.1. *Let $b \in B$, let $A[b] \subseteq B$ be the A-algebra generated by b.*

The following are equivalent:

1) b is integral over A;

2) $A[b]$ is a finite A-algebra;

3) there exists a subring $C \subset B$, with $A[b] \subseteq C$, such that C is a finite A-algebra.

Proof.

1) $\Rightarrow$ 2) By assumption there is a relation $b^n + a_1 b^{n-1} + \cdots + a_n = 0$, with $a_1, \ldots, a_n \in A$. Therefore, for any $r \geq 0$, $b^{n+r} = -(a_1 b^{n+r-1} + \cdots + a_n b^r)$. By induction it follows that all positive powers of b belong to the A-module generated by $1, b, \ldots, b^{n-1}$.

2) $\Rightarrow$ 3) It is enough to take $C = A[b]$.

3) $\Rightarrow$ 1) Let $c_1, \ldots, c_r$ be generators of C as A-module: $C = c_1 A + \cdots + c_r A$. Then, for any $i = 1, \ldots, r$, bc_i is a linear combination of $c_1, \ldots, c_r$ with coefficients in A. So there exists an $r \times r$ matrix $M = (m_{ij})_{i,j=1\ldots,r}$ with entries in A such that

$$bc_i = \sum_{j=1}^{r} m_{ij} c_j, \tag{1}$$

i.e. $(bE_r - M)\underline{c} = 0$, where $\underline{c} = (c_1 \ldots c_r)$ and E_r is the identity matrix. Multiplying equations (1) at the left by the adjoint matrix ${}^{ad}(bE_r - M)$, we get $\det(bE_r - M)c_i = 0$ for any i. Since $c_1, \ldots, c_r$ generate C, there is an expression $1 = c_1\alpha_1 + \cdots + c_r\alpha_r$. Therefore $\det(bE_r - M) = \det(bE_r - M) \cdot 1 = \det(bE_r - M)c_1\alpha_1 + \cdots + \det(bE_r - M)c_r\alpha_r = 0$. The expansion of $\det(bE_r - M)$ gives a relation of integral dependence of b over A. $\square$

Remark. Equation (1) says that b is an eigenvalue of the matrix M. The conclusion is that b is a root of the characteristic polynomial of M. But, since we work over a ring not over a field, we cannot reach directly the conclusion. In fact we have to use the assumption that $c_1, \ldots, c_r$ generate C as A-module.

We will need also the following easy property, known as "**Transitivity of finiteness**": Suppose that N is a finitely generated B-module. Then N is also an A-module, by restriction of the scalars. Assume also that B is finitely generated as an A-module. Then N is finitely generated as an A-module. Indeed if $y_1, \ldots, y_n$ generate N over B and $x_1, \ldots, x_m$ generate B as A-module, then the mn products $x_i y_j$ generate N over A.

Corollary 1.2. *Let $A \subseteq B$.*

1. Let $b_1, \ldots, b_n \in B$ be integral over A. Then $A[b_1, \ldots, b_n]$ is a finite A-module.

2. Transitivity of integral dependence: Let $A \subset B \subset C$. If B is integral extension of A and C is integral extension of B, then C is integral extension of A.

Proof. 1. By induction on n. The case $n = 1$ is part of Theorem 1.1. Assume $n > 1$, let $A_r = A[b_1, \ldots, b_r]$; then by inductive hypothesis A_{n-1} is a finitely generated A-module. $A_n = A_{n-1}[b_n]$ is a finitely generated A_{n-1}-module by the case $n = 1$, since b_n is integral over A_{n-1}. Then the thesis follows by the transitivity of finiteness.

2. Let $c \in C$, then we have an equation $c^n + b_1 c^{n-1} + \cdots + b_n = 0$, with $b_i \in B$ for any index i. The ring $B' = A[b_1, \ldots, b_n]$ is a finitely generated A-module by part 1., and $B'[c]$ is

a finitely generated B'-module, since c is integral over B'. Hence $B'[c]$ is a finite A-module, by transitivity of finiteness, and therefore c is integral over A by Theorem 1.1 (iii). □

We are now ready to prove

Theorem 1.3. Normalisation Lemma. *Let $A = K[y_1, \ldots, y_n]$ be a finitely generated K–algebra and an integral domain. Let $r := tr.d.\ Q(A)/K = tr.d.\ K(y_1, \ldots, y_n)/K$. Then there exist elements $z_1, \ldots, z_r \in A$, algebraically independent over K, such that A is integral over the K-algebra $B = K[z_1, \ldots, z_r]$.*

Proof. The proof is by induction on n.

If $n = 1$, then $A = K[y]$. If y is transcendental over K, then $r = 1$, and $A = B$. If $r = 0$, then y is algebraic over K, A is an algebraic extension of finite degree of K, and $B = K$.

Let $n \geq 2$ and assume the theorem is true for K-algebras with $n-1$ generators. Let $\varphi : K[x_1, \ldots, x_n] \to A$ be the surjective homomorphism sending any polynomial $f(x_1, \ldots, x_n)$ to $f(y_1, \ldots, y_n)$. If φ is an isomorphism, then $r = n$ and $B = A$. So we assume that $ker\ \varphi \neq (0)$ and $r < n$: there exists a non-zero polynomial f such that $f(y_1, \ldots, y_n) = 0$. Possibly renaming the variables, we can assume that x_n appears explicitly in f.

If f is monic of degree d with respect to x_n, then $A = K[y_1, \ldots, y_n]$ is a finite module over $K[y_1, \ldots, y_{n-1}]$, generated by $1, y_n, \ldots, y_n^{d-1}$. By Theorem 1.1, every element of A is integral over $K[y_1, \ldots, y_{n-1}]$. By inductive assumption, there exists $B = K[z_1, \ldots, z_r]$ with $z_1, \ldots, z_r$ algebraically independent over K, such that $K[y_1, \ldots, y_{n-1}]$ is integral over B. By Corollary 1.2, 2., also A is integral over B.

It remains the case where in the kernel of φ there is no monic polynomial in x_n. We claim we can "change coordinates" **linearly** in $K[x_1, \ldots, x_n]$ in such a way that the polynomial f becomes monic. That will mean there is another surjection $K[x_1, \ldots, x_n] \to A$ such that some element of the kernel is monic in x_n.

We consider the linear change of coordinates $x_i \to x_i + a_i x_n$, for $1 \leq i \leq n-1$ and $x_n \to x_n$, where the a_i's are elements of K. Write f as sum of its homogeneous components $f = f_d +$ lower degree terms, where $d = \deg f$. Under this transformation, $f \to f(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n)$. We claim it is possible to choose the coefficients a_i so that this new polynomial has non zero coefficient of x_n^d. Just expand, and get $f(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n) = f_d(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n) +$ lower degree terms. Then expand the top degree term and get $f_d(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n) = f_d(a_1, \ldots, a_{n-1}, 1)x_n^d +$ lower degree terms in x_n. Adding gives

$$f(x_1 + a_1 x_n, \ldots, x_{n-1} + a_{n-1}x_n, x_n) = f_d(a_1, \ldots, a_{n-1}, 1)x_n^d + \text{lower degree terms in } x_n.$$

Thus we only have to choose the a_i's so that $f_d(a_1, \ldots, a_{n-1}, 1) \neq 0$. Since f_d is a non-zero homogeneous polynomial of degree $d \geq 1$, $f_d(a_1, \ldots, a_{n-1}, 1)$ is a non-zero polynomial of degree less than or equal to d in $a_1, \ldots, a_{n-1}$. Since the field K is infinite, we are done thanks to Exercise (1) in Lesson 3. $\square$

Remarks. This proof has been adapted from MathOverflow, a "question and answer site for professional mathematicians": https://mathoverflow.net/questions/92354/noether-normalization The same proof can be found in the book [M. Reid, Undergraduate Algebraic Geometry]. The original article of Emmy Noether is unfortunately in German: *Der Endlichkeitssatz der Invarianten endlicher Gruppen der Charakteristik p*, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, 1926.

A nice article on Normalization Lemma, by Judith Sally, can be found in the book "Emmy Noether in Bryn Mawr", published in the occasion of her 100th birthday.

Emmy Noether (1882-1935) is the founder of modern algebra; her story is very interesting and in some aspects symbolic of the difficulties encountered by women mathematicians. As quoted in Wikipedia,

"In a letter to The New York Times, Albert Einstein wrote:

In the judgment of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began. In the realm of algebra, in which the most gifted mathematicians have been busy for centuries, she discovered methods which have proved of enormous importance in the development of the present-day younger generation of mathematicians.

On 2 January 1935, a few months before her death, mathematician Norbert Wiener wrote that

Miss Noether is ... the greatest woman mathematician who has ever lived; and the greatest woman scientist of any sort now living, and a scholar at least on the plane of Madame Curie.
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See also http://www.enciclopediadelledonne.it/biografie/emmy-noether/