Course on Foundations of Mathematics

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Introduction

In Lesson 1 we introduced the basic concept of a category and gave different examples of concrete categories (= objects are sets with a certain structure and morphisms are functions that preserve the structure) as well as of abstract categories where objects can be of any type.
We have seen that familiar concepts like sets, groups, topological spaces, lattices… all form categories. Keep in mind your favorite examples!

A category is a sort of "universe" with objects and morphisms.

The basic concept in a category is that of morphism (or map or arrow) indicated by $A \to B$, that means we are mainly interested in the way objects are related each other more than in the objects itself.

In category theory properties are consider in a "global" and not "local" way, that means properties depend on how objects and arrows behave with respect to the whole category universe.

Lesson 2

In this lesson 2 we will consider the following concepts:

1.Subcategories and their properties

2. Functors between categories

Notations:

The collection of all morphisms in ***A*** is denoted by *Mor* ***A***.

The morphisms between two objects *A* and *B*, ***A***(A,B), can be also denoted by $Mor(A,B)$.

The composite of morphisms $g \circ f$ can be also simply indicated by gf.

Subcategories

Given a category ***C***, a subcategory ***A*** of ***C*** indicated by $A \subseteq C$ is defined as follows:

Definition: A category ***A*** is a subcategory of ***C*** iff

Ob ***A*** $\subseteq$ *Ob* ***C***, *Mor* ***A*** $\subseteq$ *Mor* ***C***

and composition and identities in ***A*** behaves like in ***C.***

Subcategories: examples

Examples:

1. $\text{Set}_{\text{fin}} \subseteq \mathit{Set}$

Where in Set_{fin} objects are finite sets and morphisms are functions.

2. $\mathit{Set}_{inj} \subseteq \mathit{Set}$

Where in Set_{inj} objects are sets and morphisms are injective functions (composition of injective functions is still injective and identities are injective).

3. $\mathit{Set} \subseteq \mathit{Rel}$

Any function $f: A \to B$ represents a relation on *AxB* given by its graph.

Rel is not a concrete category.

4. $\text{Ab} \subseteq \mathit{Grp}$

Where *Ab* is the category of abelian groups and homomorphisms between them.

Examples:

5.*DistrLatt* $\subseteq$ *Latt*

Where *DistrLatt* is the category of distributive lattices and lattice functions (= preserve *sup* and *inf*).

6.*Bool* $\subseteq$ *DistrLatt* or also *Bool* $\subseteq$ *Latt*

Where *Bool* is the category with objects Boolean Algebras and morhisms functions that preserve *sup, inf, 0, 1*, and *complemen*t.

7.$\text{Top}_2 \subseteq$ *Top*

Where in Top_2 objects are Hausdorff topological spaces and morphisms are continuous functions.

Similarly, you can construct by yourself other examples by taking a concrete **category** ***C*** and then choose the subcategory ***A*** with objects, objects of ***C*** equipped with a specific property, and as morphism the same as in ***C*.**

Exercise:

In the following chains of inclusions of categories and subcategories verify when we get a full inclusion and when not. Motivate the answer.
If an inclusion is not full find a counterexample

Bool $\subseteq$ *DistribLatt with 0 and 1* $\subseteq$ *Distrib Latt* $\subseteq$ *Latt* $\subseteq$ *Ord* $\subseteq$ *Preord*

AbGrp $\subseteq$ *Grp* $\subseteq$ *Smgrwith1* $\subseteq$ *Smgr*

Definition: What is a Functor?

<u>Functors</u>

The notion of functor between two categories is the second base concept in category theory.

It refers to the way we can “move” form one categorical universe to another one.

A functor can be viewed as an arrow/morphism between two categories.

Many of the “natural” or “canonical” constructions in mathematics can be expressed as functors.

<u>Definition</u>. A functor F**:** ***A*** $\to$ ***B*** from the category ***A*** to the category ***B*** is given by a law that sends any object of **A** in an object of **B** and any morphism of **A** in a morphism of ***B***, in such a way that

- domains and codomains are preserved
- identities are preserved
- compositions are preserved.

That means:

Definition: What is a Functor?

Functors

F: $\boldsymbol{A} \rightarrow \boldsymbol{B}$ functor

When $f: A \rightarrow B$ is a morphism in $\boldsymbol{A}$, then
$F(f)$ is a morphism in **B** with domain $F(A)$ and codomain $F(B)$,

i.e $F(f) : F(A) \rightarrow F(B)$

Such that

$F(1_X) = 1_{F(X)}$

and $F(gf) = F(g)\, F(f)$,

where f and g are composable in $\boldsymbol{A}$.

Functors: examples

Examples

1) If **A** is a category then we can consider the identity functor id_A:***A*** $\to$ ***A*** which is the identity on both objects and morphisms

2) Common examples of functors are the so-called forgetful functors.

For instance:

a) There is a functor *U:Grp→ Set* defined as follows: if *G* is a group then *U(G)* is the underlying set of *G* (that is, its set of elements), and if *f:G→ H* is a group homomorphism then *U(f)* is the function *f* itself. So, *U* forgets the group structure of groups and forgets that group homomorphisms are homomorphisms.

b) Similarly, there is a functor *U:Ring→ Set* forgetting the ring structure on rings, and (for any field) there is a functor $U{:}Vect_k \to Set$ forgetting the vector space structure on vector spaces and the property of being linear for functions.

Functors: examples

Examples

c) Forgetful functors do not have to forget all the structure. For example, let *Ab* be the category of abelian groups. There is a functor *U:Ring→ Ab* that forgets the multiplicative structure, remembering just the underlying additive group. Or, let *Mon* be the category of monoids (semigroups with unit). There is a functor *U:Ring→ Mon* that forgets the additive structure, remembering just the underlying multiplicative monoid.

3) There is an inclusion functor E:Ab→ Grp defined by *U(A)=A* for any abelian group *A* and *U(f)=f* for any homomorphism *f* of abelian groups.
It forgets that abelian groups are abelian.

4) Free functors are in some sense dual to forgetful functors (as we will see in the next lessons), although they are less elementary.

Functors: examples

Free functors

Given any set *S*, one can build the free group *F(S*) on *S*. This is a group containing *S* as a subset and with no further properties other than those it is forced to have. Intuitively, the group *F(S*) is obtained from the set *S* by adding enough new elements such that it becomes a group, but without imposing any equations other than those forced by the definition of group. A little more precisely, the elements of *F(S)* are formal expressions or words such as

$x^{-4}yx^2zy^{-3}$ (where $x,y,z \in S$)

Two such words are seen as equal if one can be obtained from the other by the usual cancellation rules, so that, for example,

$x^3xy = x^4y = x^2y^{-1}yx^2y$

all represent the same element of *F(S*). To multiply two words, just write one followed by the other.

Functors: examples

For instance,

$(x^{-4}yx) \bullet (xzy^{-3}) = (x^{-4}yx^2zy^{-3})$

This construction assigns to each set S a group $F(S)$. F is a functor: any map of sets $f{:}S \to S'$ gives rise to a homomorphism of groups $F(f): F(S) \to F(S')$. For instance, take the map of sets $f{:}\{w,x,y,z\} \to \{u,v\}$ defined by

$f(w)=f(x)=f(y)=u$ *and* $f(z)=v$.

This gives rise to a homomorphism

$F(f){:}\ F(\{w,x,y,z\}) \to F(\{u,v\})$,

which maps, for example, $x^{-4}yx^2zy^{-3} \in F(\{w,x,y,z\})$ to $u^{-4}uu^2vu^{-3} = u^{-1}vu^{-3} \in F(\{u,v\})$.

(b) Similarly, we can construct the free commutative ring $F(S)$ on a set S, giving a functor F from *Set* to the category *CRing* of commutative rings. In fact, $F(S)$ is something familiar, namely, the ring of polynomials over Z with variables $x_s (s \in S)$. For example, if S is a two-element set then $F(S) = Z[x,y]$.

Functors: examples

c) We can also construct the free vector space on a set. Fix a field *k*. The free functor *F:Set*→ $Vect_k$ is defined on objects by taking *F(S)* as the set of all formal k-linear combinations of elements of *S;* it will *be* a vector space with basis *S*. Hence, an element of *F(S)* is an expression $\sum\lambda_s s$ with $s \in S$ and where each λ_s is a scalar and there are only finitely many values of *s* such that $\lambda s \neq 0$. Elements of *F (S)* can be canonically added, and there is also a canonical scalar multiplication on *F(S*).

Any function *f: S* → *S'* will be send by *F* in a corresponding linear function *F(f): F(S)* →*F(S')* such that $F(f)\,(\sum\lambda_s s) = \sum\lambda_s f(s)$.

5. Functors in algebraic topology.

Historically, some of the first examples of functors arose in algebraic topology. There, the strategy is – roughly speaking - to learn about a space by extracting data from it in some clever way, assembling that data into an algebraic structure, then studying the algebraic structure instead of the original space. Algebraic topology therefore involves many functors from categories of topological spaces to categories of algebras.

Functors: Example

(a) Let $Top*$ be the category of topological spaces equipped with a base point, together with the continuous base point-preserving maps. There is a functor $\pi_1 : Top* \to Grp$ assigning to each space X with base point x the fundamental group $\pi_1(X,x)$ of X at x.

(b)For each $n \in \mathbf{N}$, there is a functor $H_n : Top \to Ab$ assigning to a space its nth homology group.

Functor operations.

For all categories **A**, there is an identity functor $Id_A: \mathbf{A} \to \mathbf{A}$, given by the rule $Id_A(X) = X$ and $Id_A(f) = f$ for all objects X and arrows f in A.

If $F: \mathbf{A} \to \mathbf{B}$ *and* $G: \mathbf{B} \to \mathbf{C}$, then the composite functor $GF: \mathbf{A} \to \mathbf{C}$ *exists*, with composition defined component-wise.

Functors

Definition.

We say that a functor *F:* ***A*** $\to$ ***B*** is faithful if the maps on morphisms *F(X,Y):* ***A****(X,Y)* $\to$ ***B****(FX, FY)* (that send any *f,* morphism in *A,* in *F(f),* morphism in *B,*) are injective for all *X,Y* $\in$***A***.

We say that F: **A** $\to$ **B** is full if the maps *F(X,Y):* ***A****(X,Y)* $\to$ ***B****(FX, FY)* are all surjective.

Finally, we say that *F* is fully faithful if it is both full and faithful, ie. all the *F(X,Y):* ***A****(X,Y)* $\to$ ***B****(FX, FY)* are bijective.

Exercises:

1. Verify in the above examples of functors when they are full or faifhul.

2. Show that functors preserve isomorphisms.

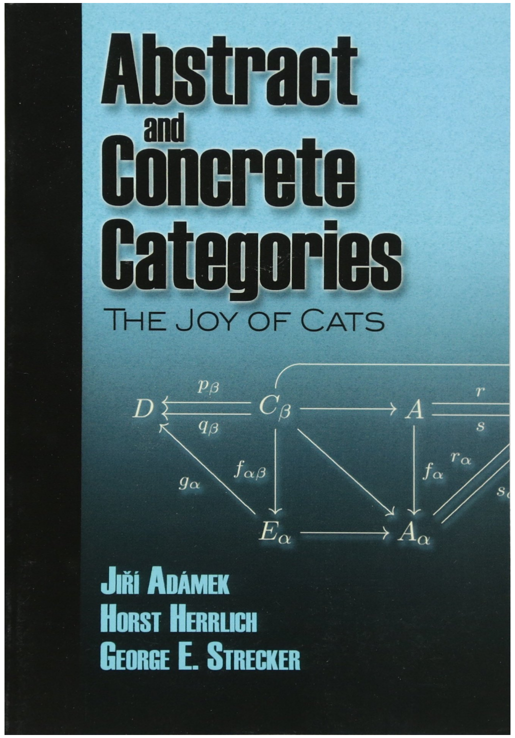
3. Prove that, every monotone function is a functor, when the underlying partial orders are viewed as categories.

4. Every monoid homomorphism is a functor, when the underlying monoids are viewed as categories

5. The following law *Z:Grp→Grp* associating to a group its center Z(G) can define a functor?

6. Construct the free functor from *Set* to Smgr.

THANK YOU FOR YOUR ATTENTION



Categorists have developed a symbolism that allows one quickly to visualize quite complicated acts by means of diagrams