LESSON 6.

1. The projective closure of an affine algebraic set.

In this section we will interpret the affine space $\mathbb{A}^n$ as identified with the open subet $U_0 \subset \mathbb{P}^n$. As we have seen in Lesson 3, 1.6, this is possible via the homeomorphisms, inverse each other, $\varphi_0 : U_0 \to \mathbb{A}^n$ and $j_0 : \mathbb{A}^n \to U_0$. Similar considerations hold for any index $i = 0, \dots, n$.

Given an affine variety $X \subset \mathbb{A}^n = U_0 \subset \mathbb{P}^n$, in this way it becomes a subet of $\mathbb{P}^n$ and it makes sense to consider its closure in the Zariski topology of the projective space.

Definition 1.1. The **projective closure** of X, $\overline{X}$, is the closure of X in the Zariski topology of $\mathbb{P}^n$.

Since the map φ_0 is a homeomorphism, we have: $\overline{X} \cap \mathbb{A}^n = X$ because X is closed in $\mathbb{A}^n$. The points of $\overline{X} \cap H_0$, where H_0 is the hyperplane at infinity $V_P(x_0)$, are called the "points at infinity" of X in the fixed embedding.

Remark. Note that, if K is an infinite field, then the projective closure of $\mathbb{A}^n$ is $\mathbb{P}^n$, i.e. the affine space is dense in the projective space.

Indeed, let F be a homogeneous polynomial of degree d vanishing along $\mathbb{A}^n = U_0$. We can write $F = F_0 x_0^d + F_1 x_0^{d-1} + \cdots + F_d$, where F_i is a homogeneous polynomial of degree i in $x_1, \dots, x_n$ for any i. By assumption, for every $P(a_1, \dots, a_n) \in \mathbb{A}^n$, $P \in V_P(F)$, i.e. $F(1, a_1, \dots, a_n) = 0 = {}^aF(a_1, \dots, a_n)$. So ${}^aF \in I(\mathbb{A}^n)$. We claim that $I(\mathbb{A}^n) = (0)$: if $n = 1$, this follows from the principle of identity of polynomials, because K is infinite. If $n \geq 2$, assume that $F(a_1, ..., a_n) = 0$ for all $(a_1, ..., a_n) \in K^n$ and consider $F(a_1, ..., a_{n-1}, x)$: either it has positive degree in x for some choice of $(a_1, ..., a_n)$, but then it has finitely many zeroes against the assumption; or it is always constant in x, so F belongs to $K[x_1, ..., x_{n-1}]$ and we can conclude by induction. So the claim is proved. We get therefore that $F_0 = F_1 = \ldots = F_d = 0$ and $F = 0$.

We want to find the relation between the equations of $X \subset \mathbb{A}^n$ and those of its projective closure $\overline{X} \subset \mathbb{P}^n$.

Proposition 1.2. *Let $X \subset \mathbb{A}^n$ be an affine variety, $\overline{X}$ be its projective closure. Then*

$$I_h(\overline{X}) = {}^hI(X) := \langle {}^hF | F \in I(X) \rangle.$$

Proof. Let $F \in I_h(\overline{X})$ be a homogeneous polynomial. If $P(a_1, \dots, a_n) \in X$, then $[1, a_1, \dots, a_n] \in \overline{X}$, so $F(1, a_1, \dots, a_n) = 0 = {}^aF(a_1, \dots, a_n)$. Hence ${}^aF \in X$. There exists $k \geq 0$ such that $F = (x_0^k)^h({}^aF)$ (see proof of Proposition 1.3, Lesson 3), so $F \in {}^hI(X)$. Hence $I_h(\overline{X}) \subset {}^hI(X)$.

Conversely, if $G \in I(X)$ and $P(a_1, \dots, a_n) \in X$, then $G(a_1, \dots, a_n) = 0 = {}^hG(1, a_1, \dots, a_n)$, so ${}^hG \in I_h(X)$ (here X is seen as a subset of $\mathbb{P}^n$). So ${}^hI(X) \subset I_h(X)$. Since $I_h(X) = I_h(\overline{X})$ (see Exercise 1), we have the claim. □

In particular, if X is a hypersurface and $I(X) = \langle F \rangle$, then $I_h(\overline{X}) = \langle {}^hF \rangle$.

Next example, that will occupy the rest of this Lesson, will show that, **in general**, from $I(X) = \langle F_1, \dots, F_r \rangle$, it does not follow ${}^hI(X) = \langle {}^hF_1, \dots, {}^hF_r \rangle$. Only in the last thirty years, thanks to the development of symbolic algebra and in particular of the theory of Gröbner bases, the problem of characterizing the systems of generators of $I(X)$, whose homogeneization generates ${}^hI(X)$, has been solved.

The example of the skew cubic is of fundamental importance in algebraic geometry, because of the many geometrical phenomena that appear, and are developed in different classes of varieties of which the skew cubic is the first case.

Example 1.3 (The skew cubic)**.**

The affine skew cubic is the following closed subset X of $\mathbb{A}^3$: $X = V(y - x^2, z - x^3)$ (we use variables x, y, z). X is the image of the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ such that $\varphi(t) = (t, t^2, t^3)$. Note that $\varphi : \mathbb{A}^1 \to X$ is a homeomorphism (see Exercise 3, Lesson 2). Let α be the ideal $\langle y - x^2, y - x^3 \rangle$. Note that $X = V(\alpha)$. We claim that $\alpha = I(X) = \{F \in K[x, y, z] \mid F(x, x^2, x^3) = 0$ for any $x \in K$. Proceeding as in Lesson 4, Example 1.2, we consider the development of any polynomial $G \in K[x, y, z]$ in Taylor series around (x, x^2, x^3), and we get the claim. We observe also that α is a prime ideal; to see this, we consider the ring homomorphism $K[x, y, z] \to K[x]$ such that $F(x, y, z) \to F(x, x^2, x^3)$: it is surjective and its kernel is α, therefore the quotient ring $K[x, y, z]/\alpha$ is isomorphic to $K[x]$, which is an integral domain. Therefore α is prime.

Let $\overline{X}$ be the projective closure of X in $\mathbb{P}^3$. First we will study $\overline{X}$ geometrically, then we will determine its homogeneous ideal. We claim that it is the image of the map $\psi : \mathbb{P}^1 \to \mathbb{P}^3$ such that $\psi([\lambda, \mu]) = [\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3]$. We identify $\mathbb{A}^1$ with the open subset of $\mathbb{P}^1$ defined by $\lambda \neq 0$ i.e. U_0, and $\mathbb{A}^3$ with the open subset of $\mathbb{P}^3$ defined by $x_0 \neq 0$ (U_0 again). Note that $\psi|_{\mathbb{A}^1} = \varphi$, because $\psi([1, t]) = [1, t, t^2, t^3] =$ via the identification of $\mathbb{A}^3$ with $U_0 = (t, t^2, t^3) = \varphi(t)$. Moreover $\psi([0, 1]) = [0, 0, 0, 1]$. So $\psi(\mathbb{P}^1) = X \cup \{[0, 0, 0, 1]\}$.

Let G be a homogeneous polynomial of $K[x_0, x_1, x_2, x_3]$ such that $X \subset V_P(G)$. Then $G(1, t, t^2, t^3) = 0 \ \forall t \in K$, so $G(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3) = 0 \ \forall \mu \in K, \forall \lambda \in K^*$. Since K is infinite, then $G(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3)$ is the zero polynomial in λ and μ, so $G(0, 0, 0, 1) = 0$ and $V_P(G) \supset \psi(\mathbb{P}^1)$, therefore $\overline{X} \supset \psi(\mathbb{P}^1)$.

Conversely, we prove that $\psi(\mathbb{P}^1)$ is Zariski closed, more precisely

$$\psi(\mathbb{P}^1) = V_P(F_0, F_1, F_2) \text{ where } F_0 := x_1x_3 - x_2^2, F_1 := x_1x_2 - x_0x_3, F_2 := x_0x_2 - x_1^2.$$

One inclusion is clear: every point of $\mathbb{P}^3$ of coordinates $[\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3]$ satisfies the three quadratic equations $F_0 = F_1 = F_2 = 0$. Conversely, let $F_i(y_0, \dots, y_3) = 0 \ \forall i = 1, \dots, 3$, i.e. $y_1y_3 = y_2^2$, $y_1y_2 = y_0y_3$, $y_0y_2 = y_1^2$. We observe that either $y_0 \neq 0$ or $y_3 \neq 0$, otherwise also $y_1 = y_2 = 0$.

Assume $y_0 \neq 0$, then, using the three equations, we get

$$[y_0, y_1, y_2, y_3] = [y_0^3, y_0^2y_1, y_0^2y_2, y_0^2y_3] = [y_0^3, y_0^2y_1, y_0y_1^2, y_0y_1y_2] = [y_0^3, y_0^2y_1, y_0y_1^2, y_1^3] = \psi([y_0, y_1]).$$

Similarly, if $y_3 \neq 0$, $[y_0, y_1, y_2, y_3] = \psi([y_2, y_3])$. So $\psi(\mathbb{P}^1) = \overline{X}$.

The three polynomials F_0, F_1, F_2 are the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

with entries in $K[x_0, x_1, x_2, x_3]$. Let $F = y - x^2$, $G = z - x^3$ be the two generators of $I(X)$; ${}^hF = x_0x_2 - x_1^2$, ${}^hG = x_0^2x_3 - x_1^3$, hence $V_P({}^hF, {}^hG) = V_P(x_0x_2 - x_1^2, x_0^2x_3 - x_1^3) \neq \overline{X}$, because $V_P({}^hF, {}^hG)$ contains the whole line "at infinity" $V_P(x_0, x_1)$, which is not contained in $\overline{X}$.

We shall prove now the non-trivial fact:

Proposition 1.4. $I_h(\overline{X}) = \langle F_0, F_1, F_2 \rangle$.

Proof. For all integer number $d \geq 0$, let $I_h(\overline{X})_d := I_h(\overline{X}) \cap K[x_0, x_1, x_2, x_3]_d$: it is a K-vector space of dimension $\leq \binom{d+3}{3}$. We define a K-linear map ρ_d having $I_h(\overline{X})_d$ as kernel:

$$\rho_d : K[x_0, x_1, x_2, x_3]_d \to K[\lambda, \mu]_{3d}$$

such that $\rho_d(F) = F(\lambda^3, \lambda^2\mu, \lambda\mu^2, \mu^3)$. Since ρ_d is clearly surjective, we compute

$$\dim I_h(\overline{X})_d = \binom{d+3}{3} - (3d + 1) = (d^3 + 6d^2 - 7d)/6.$$

For $d \geq 2$, we define now a second K-linear map

$$\varphi_d : K[x_0, x_1, x_2, x_3]_{d-2}^{\oplus 3} \to I_h(\overline{X})_d$$

such that $\varphi_d(G_0, G_1, G_2) = G_0F_0 + G_1F_1 + G_2F_2$. Our aim is to prove that φ_d is surjective. The elements of its kernel are called the *syzygies of degree* d among the polynomials F_0, F_1, F_2.

Two obvious syzygies of degree 3 are constructed by developing, according to the Laplace rule, the determinant of the matrix obtained repeating one of the rows of M, for example

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

It gives $x_0F_0 + x_1F_1 + x_2F_2 = 0$, so (x_0, x_1, x_2) is a syzygy of degree 3. Similarly (x_1, x_2, x_3).

We put $H_1 = (x_0, x_1, x_2)$ and $H_2 = (x_1, x_2, x_3)$, they both belong to $\ker \varphi_3$. Note that H_1 and H_2 give rise to syzygies of all degrees ≥ 3, in fact we can construct a third linear map

$$\psi_d : K[x_0, x_1, x_2, x_3]^{\oplus 3}_{d-3} \to \ker \varphi_d$$

putting $\psi_d(A, B) = H_1A + H_2B = (x_0, x_1, x_2)A + (x_1, x_2, x_3)B = (x_0A + x_1B, x_1A + x_2B, x_2A + x_3B)$.

Claim. ψ_d *is an isomorphism.*

Assuming the claim, we are able to compute $\dim \ker \varphi_d = 2\binom{d}{3}$, therefore

$$\dim Im\ \varphi_d = 3\binom{d+1}{3} - 2\binom{d}{3}$$

which coincides with the dimension of $I_h(\overline{X})_d$ previously computed. This proves that φ_d is surjective for all d and concludes the proof of the Proposition.

Proof of the Claim. Let (G_0, G_1, G_2) belong to $\ker \varphi_d$. This means that the following matrix N with entries in $K[x_0, x_1, x_2, x_3]$ is non-invertible:

$$N := \begin{pmatrix} G_0 & G_1 & G_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Therefore, the rows of N are linearly dependent over the quotient field of the polynomial ring $K(x_0, \dots, x_3)$. Since the last two rows are linearly independent, there exist reduced rational functions $\frac{a_1}{a_0}, \frac{b_1}{b_0} \in K(x_0, x_1, x_2, x_3)$, such that

$$G_0 = \frac{a_1}{a_0}x_0 + \frac{b_1}{b_0}x_1 = \frac{a_1b_0x_0 + a_0b_1x_1}{a_0b_0}$$

and similarly

$$G_1 = \frac{a_1b_0x_1 + a_0b_1x_2}{a_0b_0}, G_2 = \frac{a_1b_0x_2 + a_0b_1x_3}{a_0b_0}$$

The G_i's are polynomials, therefore the denominator a_0b_0 divides the numerator in each of the three expressions on the right hand side. Moreover, if p is a prime factor of a_0, then p

divides the three products b_0x_0, b_0x_1, b_0x_2, hence p divides b_0. We can repeat the reasoning for a prime divisor of b_0, so obtaining that $a_0 = b_0$ (up to invertible constants). We get:

$$G_0 = \frac{a_1x_0 + b_1x_1}{b_0}, G_1 = \frac{a_1x_1 + b_1x_2}{b_0}, G_2 = \frac{a_1x_2 + b_1x_3}{b_0},$$

therefore b_0 divides the numerators

$$c_0 := a_1x_0 + b_1x_1, c_1 := a_1x_1 + b_1x_2, c_2 := a_1x_2 + b_1x_3.$$

Hence b_0 divides also $x_1c_0 - x_0c_1 = b_1(x_1^2 - x_0x_1) = -b_1F_2$, and similarly $x_2c_0 - x_0c_2 = b_1F_1$, $x_2c_1 - x_1c_2 = -b_1F_0$. But F_0, F_1, F_2 are irreducible and coprime, so we conclude that $b_0 \mid b_1$. But b_0 and b_1 are coprime, so finally we get $b_0 = a_0 = 1$. $\square$

As an important by-product of the proof of Proposition 1.4 we have the **minimal free resolution** of the R-module $I_h(\overline{X})$, where $R = K[x_0, x_1, x_2, x_3]$:

$$0 \to R^{\oplus 2} \xrightarrow{\psi} R^{\oplus 3} \xrightarrow{\varphi} I_h(\overline{X}) \to 0$$

where ψ is represented by the transposed of the matrix M and φ by the triple of polynomials (F_0, F_1, F_2).

Exercises 1.5. 1*. Let $X \subset \mathbb{A}^n$ be a closed subset, $\overline{X}$ be its projective closure in $\mathbb{P}^n$. Prove that $I_h(X) = I_h(\overline{X})$.

2. Find a system of generators of the ideal of the affine skew cubic X, such that, if you homogeneize them, you get a system of generators for $I_h(\overline{X})$.