MECCANICA RAZIONALE

– LEZIONE DEL 24 MARZO 2020

– PRIMA PARTE

Le equazioni cardinali della statica.

$\underline{R}^e = \underline{0}$ forze esterne attive & reattive

$\underline{M}^e(O) = \underline{0}$

→ corpo rigido ($\longleftrightarrow$ PLV)

solo necessarie per più corpi rigidi

→ applicare l'ECS al sistema intero (più rigidi articolati & vincolati) & a tutti i sottosistemi

strategia : equilibrio $\longleftarrow$ PLV

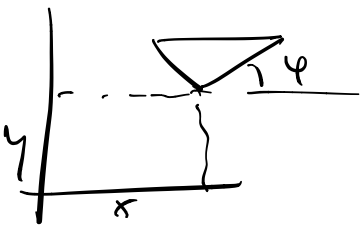
$\left(\frac{dV}{dq} = 0 \quad , \quad \frac{d^2V}{dq^2}\Big|_{eq} \lessgtr \right)$

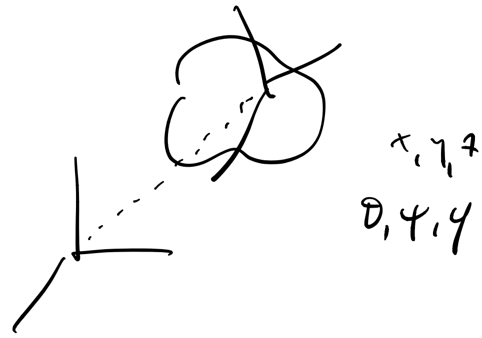
determinare le reazioni vincolari (all'eq) $\longleftrightarrow$ ECS

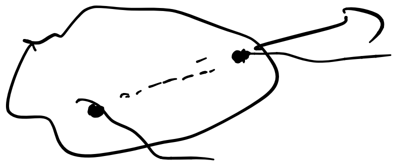
SFORZI INTERNI AD UN RIGIDO

Corpo rigido

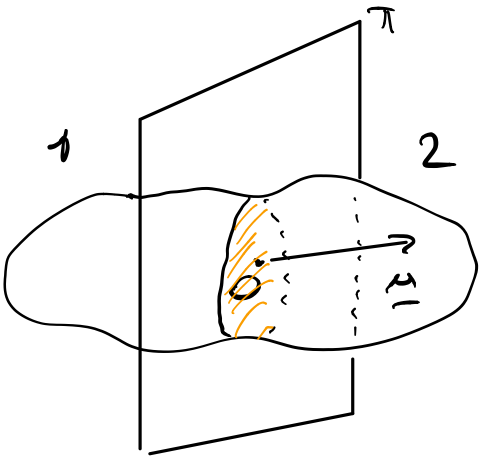
rigido piano







forze interne: mantengono il corpo rigido



Sezioniamo con un piano π

$O \in \pi$ $\quad \underline{n}$ normale al piano, orientato $1 \longrightarrow 2$

Consideriamo sistema all'eq e guardiamo la parte 1. Eq di 1 = eq rispetto alle forze esterne su 1, e alle forze interne di 2 su 1.

$$\underline{R}_1^{(e)} + \underline{R}^{(i)} = \underline{0}$$

E.C.S.

risultante forze esterne su 1

risultante forze interne di 2 su 1

$$\underline{M}^{(e)}_1(O) + \underline{M}^{(i)}(O) = \underline{0}$$

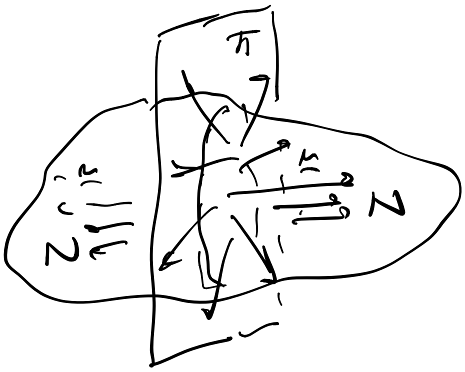
momenti esterni su O — momenti di 2 agenti su 1

Diamo alcune definizioni

$$\underline{R}^{(i)} = N\underline{n} + \underline{T} \Rightarrow N = \underline{R}^{(i)}\cdot\underline{n} \quad \text{(scalare)}$$

SFORZO NORMALE

($N>0$ trazione, $N<0$ compressione)



$$\underline{T} = \underline{R}^{(i)} - N\underline{n}$$

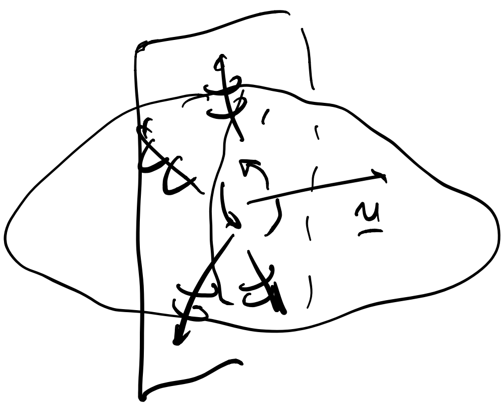
SFORZO DI TAGLIO

(vettore)

$$\underline{M}^{(i)}(O) = M_T\,\underline{n} + \underline{M}_f(O) \Rightarrow M_T(O) = \underline{M}^{(i)}(O)\cdot\underline{n}$$

MOMENTO TORCENTE

(scalare)

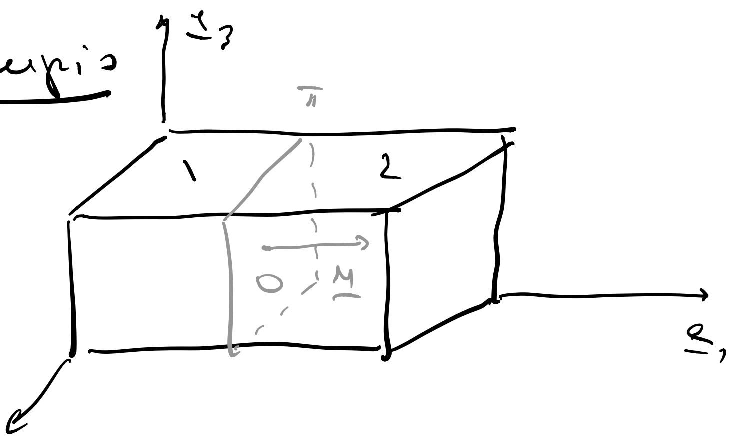


$$\underline{M}_f(O) = \underline{M}^{i}(O) - M_T\,\underline{n}$$

MOMENTO FLETTENTE

(vettore)

Esempio



Supponiamo

$$\underline{R}^e_1 = f\,\underline{e}_1 + 2f\,\underline{e}_2$$

$$\underline{M}^e_1(O) = \mu\,\underline{e}_1 + 4\mu\,\underline{e}_3$$

Calcoliamo $\underbrace{\underline{R}^{int}}$

$$\underline{R}^e_1 + N\underline{u} + \underline{T} = \underline{0} \quad \Rightarrow \quad f\underline{e}_1 + 2f\underline{e}_2 + N\underline{e}_1 + \underline{T} = \underline{0}$$

$$\underline{u} = \underline{e}_1$$

da cui $\quad N = -f \quad , \quad \underline{T} = -2f\,\underline{e}_2$

$$\underline{M}^e_1(O) + M_T\,\underline{u} + \underline{M}_f = \underline{0} \quad \Rightarrow$$

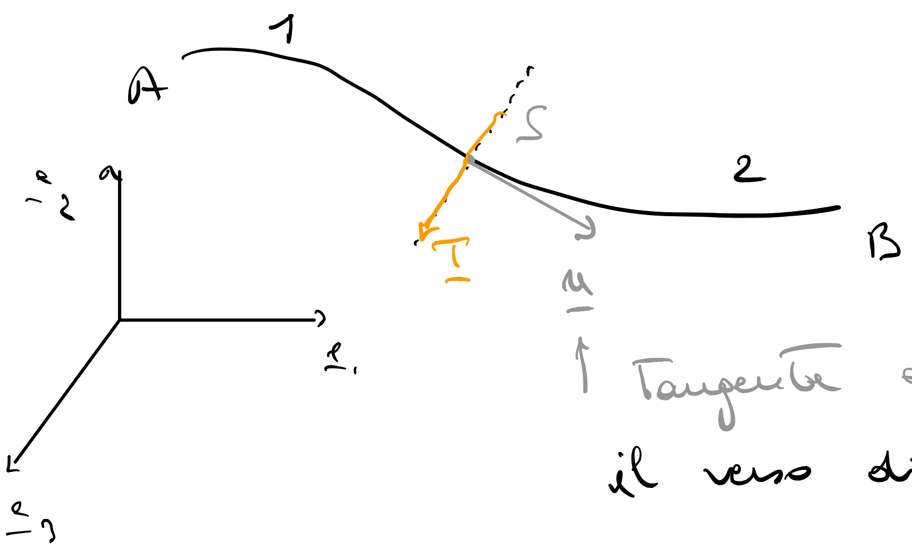
$$\mu\,\underline{e}_1 + 4\mu\,\underline{e}_3 + M_T\,\underline{e}_1 + \underline{M}_f = \underline{0}$$

$$M_T = -\mu \quad , \quad \underline{M}_f = -4\mu\,\underline{e}_3$$

Archi rigidi piani

Rigidi unidimensionali piani

Sezione = ortogonale alla curva e determinata da un punto



Tangente geometrica in S

il verso di $\underline{u}$ da $1 \to 2$

$\underline{x} = \underline{x}(s) \qquad s \in [0, L) \qquad L$ lunghezza curva

$\underline{x}_A = \underline{x}(0) \qquad s = 0 \qquad$ Parte 1 : AS precede S

$\underline{x}_B = \underline{x}(L) \qquad s = L \qquad$ Parte 2 : SB segue S

Dalla geometria $\underline{n} = \frac{d}{ds} \underline{x}(s)$

Sul piano:

- $\underline{R}^{(i)} \in$ piano $\underline{e}_1, \underline{e}_2$ $\quad \underline{n}$ Tangente alla curva

  $\Rightarrow \underline{T}$ è ortogonale a $\underline{n}$ e sul piano

- Tutte le forze (interne e esterne) sono sul piano $\Rightarrow \underline{M}^{(i)}_{(S)}$ è ortogonale al piano

  In particolare è ortogonale a $\underline{n}$

  $\Rightarrow M_T$ (= componente lungo $\underline{n}$)

  $\quad \parallel$

  $\quad 0$ per sistemi piani

  L'unica componente $\neq 0$ è $\underline{M}_f$

  che è diretto lungo $\underline{e}_3$

Quindi sul piano

$$\underline{R}^{(e)}_1 + N\underline{n} + \underline{T} = 0 \qquad \text{2 incognite}$$

$$\underline{M}^e_1(S) + \underline{M}_f(S) = 0 \qquad \text{1 incognita}$$

$\uparrow$ vettore lungo $\underline{e}_3$

Vogliamo calcolarle per ogni sezione S

$\Rightarrow$ vogliamo gli sforzi interni in ogni punto dell'arco, per sapere se regge

$\rightarrow$ 3 funzioni $N(s)$, $\underline{T}(s)$, $\underline{M}_f(s)$

Per capire come determinarle, prima occorre conoscere la distribuzione delle forze esterne

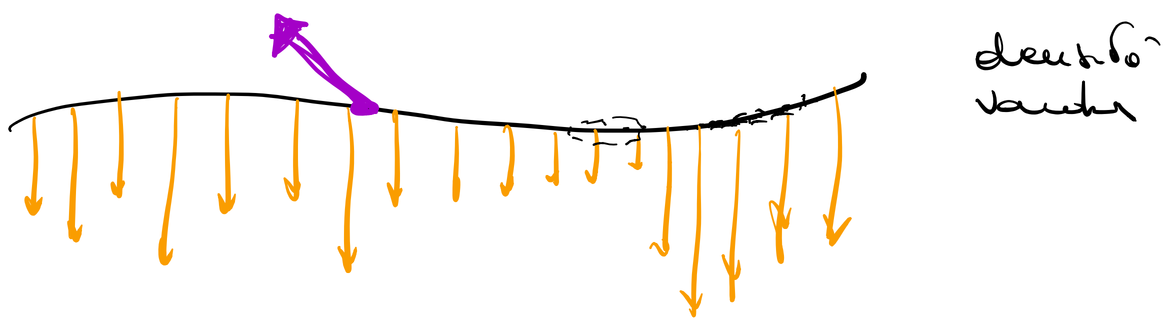
- forze concentrate (applicate in un punto)
- forze distribuite (la forza peso)

$\downarrow$

forza specifica : $\boxed{\underline{f}(s)}$

[peso $\underline{f}(s) = \rho(s)\,\underline{g}$]

$\uparrow$ densità

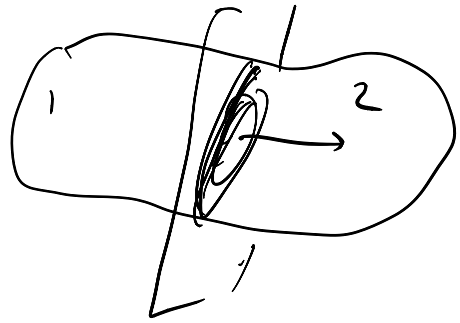


MECCANICA RAZIONALE

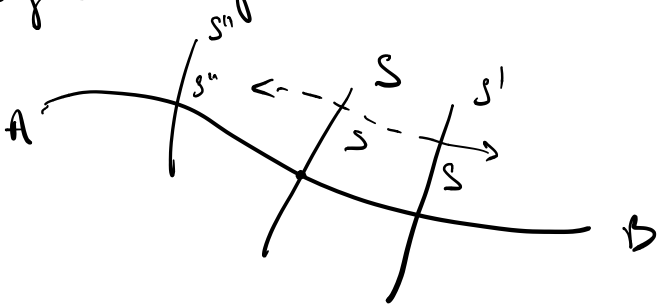
- LEZIONE DEL 24 MARZO 2020

- SECONDA PARTE

ECS :



Rigidi piani



$$\left.\begin{array}{l}\underline{R}^{(i)}(s)\\ \underline{M}^{(i)}(s)\end{array}\right\} \quad \begin{array}{l}N(s)\\ T(s)\\ \underline{M}_f(s)\end{array}$$

Teorema (Relazioni differenziali per gli sforzi interni)

Supponiamo che

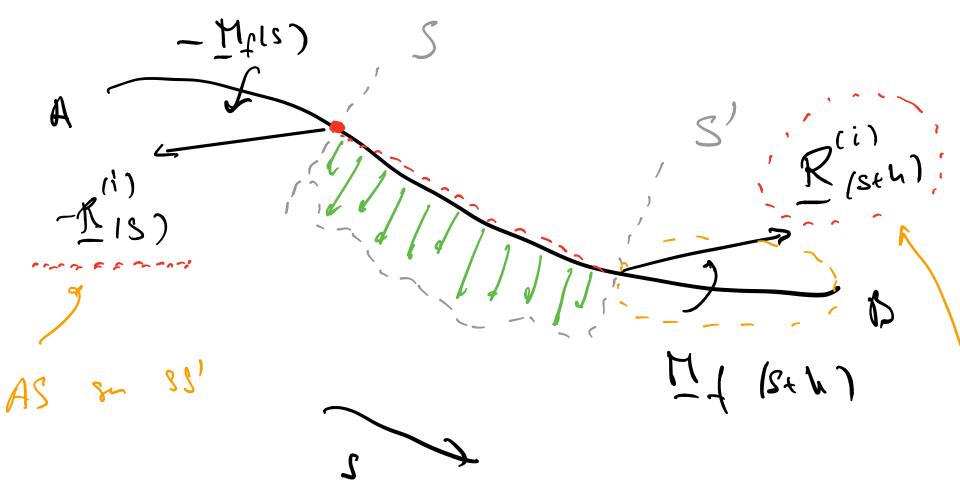
- no forze concentrate (in un intorno della sezione)
- forze specifiche sono regolari
- arco regolare

Allora gli sforzi interni sono regolari e

$$\frac{d}{ds}\left(N(s)\,\underline{u}(s) + \underline{T}(s) \right) = -\underline{f}(s)$$

$$\frac{d}{ds}\,\underline{M}_f(s) = \underline{T}(s) \wedge \underline{u}(s)$$

<u>Dimostrazione</u> Consideriamo



consideriamo l'arco tra S e S'

$S \longrightarrow s$

$S' \longrightarrow s+h$

$h > 0$

$\underline{R}^{(i)} = N\,\underline{u} + \underline{T}$

$\underline{M}_f$

Forze che agiscono su SS' :

- forze esterne, con forza specifica $\underline{f}(\xi)$
  $\xi \in [s, s+h]$

$$\rightarrow \quad \underline{R}^{SS'} = \int_s^{s+h} \underline{f}(\xi)\,d\xi$$

$$\rightarrow \quad \underline{M}^{SS'}(s) = \int_s^{s+h} \left(\underline{x}(\xi) - \underline{x}(s)\right) \wedge \underline{f}(\xi)\,d\xi$$

$$\sum_{\xi} \left(\underline{x}_\xi - \underline{x}_s\right) \wedge \underline{f}_\xi$$

polo

- sforzi interni di S'B su SS'

$\underline{R}^{i}(s+h)$

$\underline{M}_f(s+h)$

$s+h$ corrisponde a S'

- sforzi interni di AS su SS'

  $-\underline{R}^{(i)}(s)$

  $-\underline{M}_f(s)$

Bilancio :

$$\begin{cases} \underline{R}^{SS'} + \underline{R}^{(i)}(s+h) - \underline{R}^{(i)}(s) = \underline{0} \\ \underline{M}^{SS'} + \underline{M}_f(s+h) - \underline{M}_f(s) + \\ \quad + (\underline{x}(s+h) - \underline{x}(s)) \wedge \underline{R}^{(i)}(s+h) = \underline{0} \end{cases}$$

- $$\frac{\underline{R}^{(i)}(s+h) - \underline{R}^{(i)}(s)}{h} = -\frac{1}{h}\underline{R}^{SS'}$$

- $$\frac{\underline{M}_f(s+h) - \underline{M}_f(s)}{h} = -\frac{1}{h}\underline{M}^{SS'}(s) + \underline{R}^{(i)}(s+h) \wedge \frac{\underline{x}(s+h) - \underline{x}(s)}{h}$$

Prendiamo il limite $h \to 0$

Forza specifica sia regolare: la forza deve essere limitata $\| \underline{f} \| < F$

- $$\underline{R}^{ss'} = \int_s^{s+h} \underline{f}(\xi)\, d\xi \xrightarrow[h \to 0]{} 0$$

[pensiamo di avere g tale che $f = \frac{dg}{ds}$

$$\int_s^{s+h} \frac{dg}{ds} = -g(s) + g(s+h) \xrightarrow[h \to 0]{} 0$$]

- $$\frac{\underline{R}^{ss'}}{h} = \frac{1}{h} \int_s^{s+h} \underline{f}(\xi)\, d\xi \xrightarrow[h \to 0]{} \underline{f}(s)$$

[$$\frac{-g(s) + g(s+h)}{h} \xrightarrow[h \to 0]{} \frac{dg}{ds} = f$$]

$$\frac{\underline{R}^{(i)}_{(s+h)} - \underline{R}^{(i)}_{(s)}}{h} = - \frac{\underline{R}^{ss'}}{h} \Rightarrow \frac{d}{ds} \underline{R}^{(i)}_{(s)} = - f(s)$$

- $$\left\| \frac{\underline{M}^{ss'}_{(s)}}{h} \right\| = \frac{1}{h} \left\| \int_s^{s+h} (\underline{x}(\xi) - x(s)) \wedge \underline{f}(\xi)\, d\xi \right\|$$

$$\le \frac{1}{h} \int_s^{s+h} \underbrace{\| \underline{x}(\xi) - \underline{x}(s) \|}_{\le h} \underbrace{\| \underline{f}(\xi) \|}_{\le F} d\xi$$

$$\le F h \xrightarrow[h\to 0]{} 0 \qquad \le \frac{1}{h}\, h F \underbrace{\int_s^{s+h} d\xi}_{h} = \frac{h h F}{h} \to 0$$

- $$\frac{\underline{x}(s+h) - \underline{x}(s)}{h} \xrightarrow[h\to 0]{} \frac{d\underline{x}(s)}{ds} = \underline{u}(s)$$

$$\leadsto \frac{d}{ds}\underline{M}_f(s) = \underline{R}^{(i)}_{(s)} \wedge \underline{u}(s)$$

$$\underline{R}^{(i)}_{(s)} = N\underline{u} + \underline{T}(s)$$

$$= \underline{T}(s) \wedge \underline{u}(s)$$

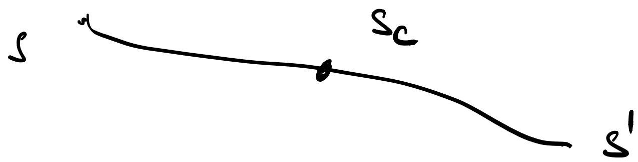
(ripetiamo il ragionamento con $h < 0$)

$$\begin{cases} \dfrac{d}{ds}(N\underline{u} + \underline{T}) = -\underline{f}(s) \\[2ex] \dfrac{d}{ds}\underline{M}_f(s) = \underline{T}(s) \wedge \underline{u}(s) \end{cases}$$

Assumiamo di avere carichi concentrati

$\underline{F}_c$ applicata in s_c

$\underline{R}^{(i)}, \underline{M}_f$ regolari per $s \neq s_c$



Aggiungiamo il contributo $\underline{F}_c$

- nell' eq $\underline{R}^{ss'}$ $\rightarrow$ aggiungo $\underline{F}_c$

- $\underline{M}^{ss'}_{(1)}$ aggiungiamo il momento di $\underline{F}_c$ rispetto ad S_1 cioè

$$(\underline{x}(s_c) - \underline{x}(s)) \wedge \underline{F}_c$$

Conseguenze

- $$\underline{R}^{ss'} \xrightarrow[h \to 0]{} \underline{F}_c$$

→ significa $\underline{R}^{(i)}$ ha un salto in s_c:

$$\underline{F}_c + [\underline{R}^{(i)}]_{s=s_c} = \underline{0}$$

$$[\underline{R}^{i}]_{s=s_c} = \lim_{s \to s_c^+} \underline{R}_s^{(i)} - \lim_{s \to s_c^-} \underline{R}^{(i)}(s)$$

- $$\underline{M}^{ss'} \xrightarrow[h \to 0]{} \underline{0}$$: il momento flettente è continuo in $s = s_c$

Infatti:

$$\frac{d}{ds} \underline{M}_f(s) = \underline{T}(s) \wedge \underline{u}$$

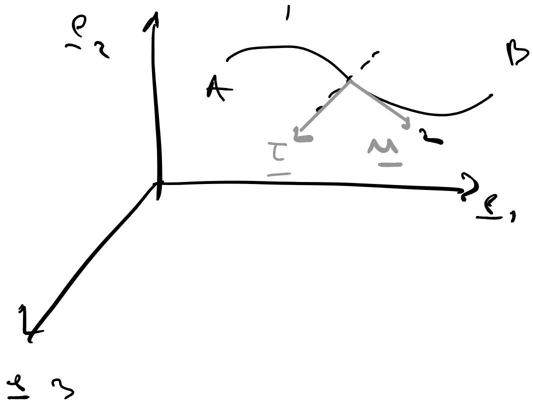
se T è discontinuo allora $\frac{d}{ds}\underline{M}_f$ è discontinua

MECCANICA RAZIONALE

— LEZIONE DEL 24 MARZO 2020

— TERZA PARTE

Esempio



$$\underline{n} = \frac{d\underline{x}(s)}{ds}$$

$$\underline{\tau} = \underline{n} \wedge \underline{e}_3 \qquad \left(\tilde{\underline{\tau}} = \underline{e}_3 \wedge \underline{n} \right)$$

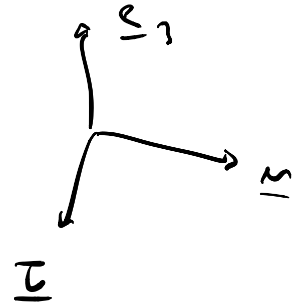
$$\underline{T}(s) = T(s)\,\underline{\tau}$$

$$\underline{M}_f(s) = M_f(s)\,\underline{e}_3$$

Sforzi interni:

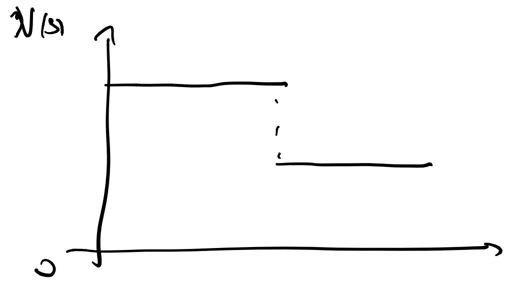
$N = N(s)$
$T = T(s)$
$M_f = M_f(s)$

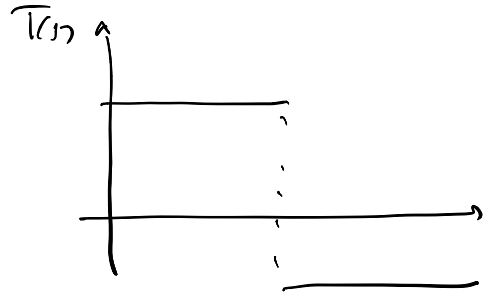
3 funzioni scalari

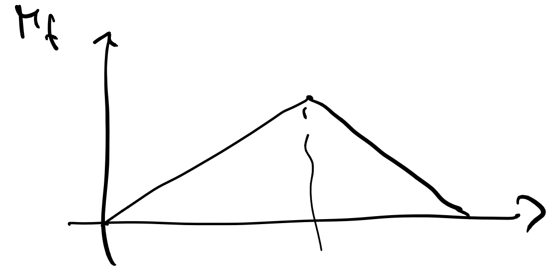


→ diagramma degli sforzi interni

grafico delle funzioni $N(s)$, $T(s)$, $M_f(s)$

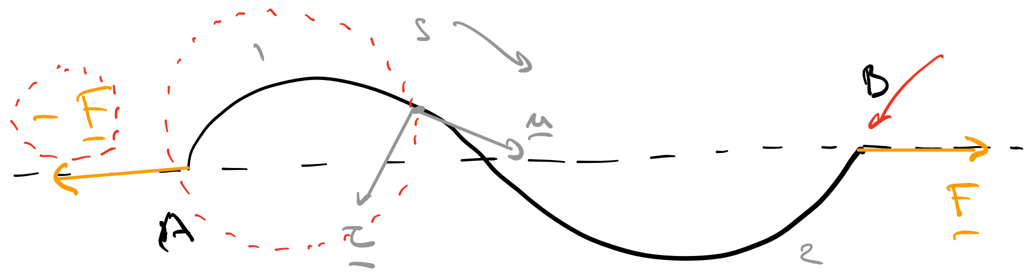






Archi scarichi

Arco scarico = è soggetto solo a due forze applicate alle estremità dell'arco (coppia di braccio nullo)



$\underline{\tau} = \underline{m} \wedge \underline{e}_3$

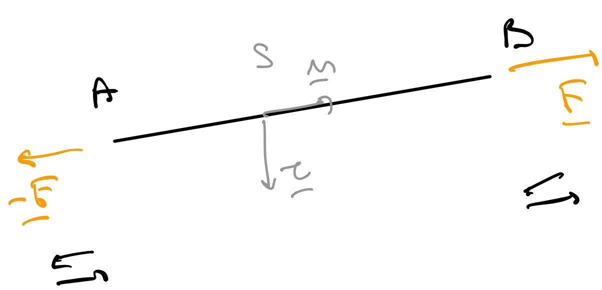
$\underline{T} = T\,\underline{\tau}$

Equilibrio della parte 1

$$\begin{cases} -\underline{F} + N\underline{m} + T\underline{\tau} = \underline{0} \\ (\underline{x}_A - \underline{x}_s) \wedge (-\underline{F}) + M_f\,\underline{e}_3 = \underline{0} \end{cases}$$

Arco curvo : la direzione di $\underline{m}$ e $\underline{\tau}$ cambia con s

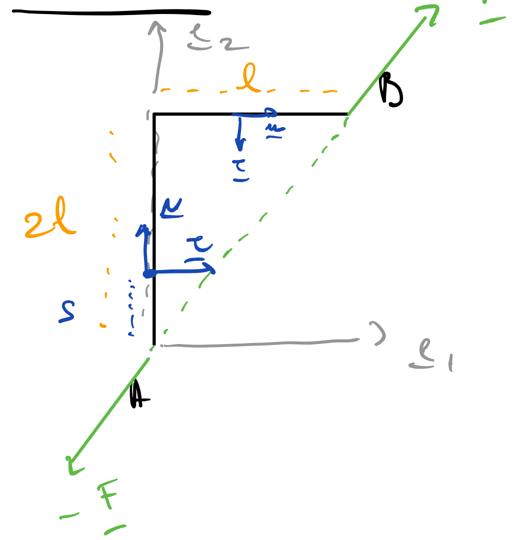
$\underline{F} \perp \underline{\tau}$



$\Rightarrow T = 0 \,,\; M_f = 0$

$$N \begin{cases} > 0 & \text{trazione} \\ < 0 & \text{compressione} \end{cases}$$

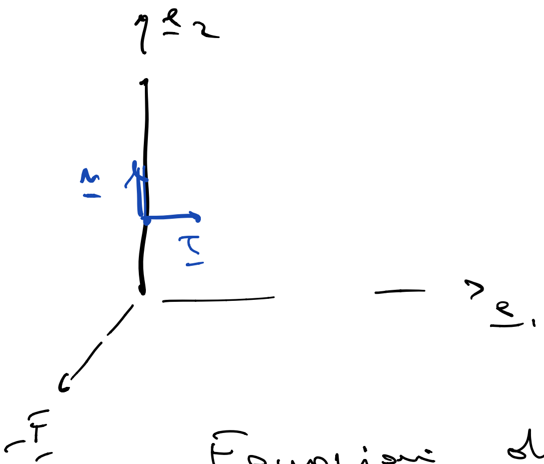
Esercizio



Vogliamo calcolare gli sforzi interni ad AB per $\underline{F} = \text{vers}(\underline{x}_B - \underline{x}_A)\, F$, $F>0$

Dividiamo il problema in due: $0 < s < 2\ell$, $2\ell < s < 3\ell$

Primo caso: $0 < s < 2\ell$



$\underline{n} = \underline{e}_2$

$\underline{\tau} = \underline{e}_1$

Equazioni di bilancio:

$$-\underline{F} + N\underbrace{\underline{e}_2}_{\underline{n}} + T\underbrace{\underline{e}_1}_{\underline{\tau}} = 0$$

$$\underline{F} = F\,\text{vers}(\underline{x}_B - \underline{x}_A) = F\,\underbrace{\frac{1}{\sqrt{5}}(\underline{e}_1 + 2\underline{e}_2)}_{\text{versore}}$$

$$F\,\frac{\underline{e}_1 + 2\underline{e}_2}{\sqrt{5}} + N\underline{e}_2 + T\underline{e}_1 = 0$$

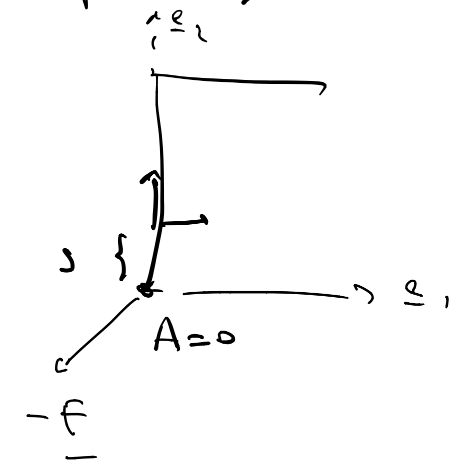
$$N = \frac{2}{\sqrt{5}}F, \quad T = \frac{1}{\sqrt{5}}F$$

$$M_f \underline{e}_3 + (\underline{x}_A - \underline{x}_s) \wedge (-\underline{F}) = \underline{0}$$

moltiplichiamo per $\underline{e}_3$

$$M_f + (\underline{x}_A - \underline{x}_s) \wedge (-\underline{F}) \cdot \underline{e}_3 = 0$$

$\underline{x}_A = 0$



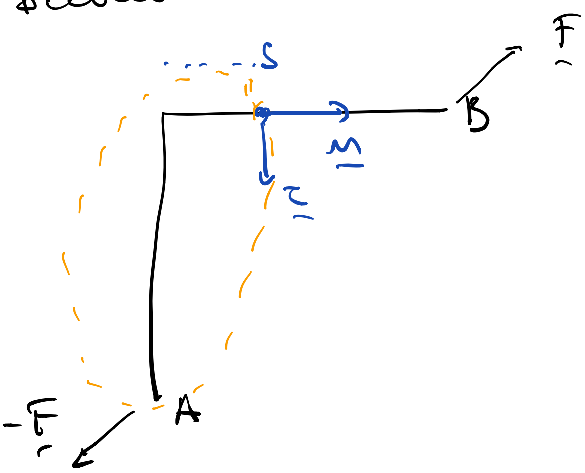
$$\underline{x}_s = s\,\underline{e}_2$$

$$M_f + s\,\underline{e}_2 \wedge \frac{F}{\sqrt{5}} (\underline{e}_1 + 2\underline{e}_2) \cdot \underline{e}_3$$

$$\rightarrow \quad M_f = \frac{F}{\sqrt{5}}\, s$$

vediamo esplicitamente $\quad \frac{d}{ds} M_f = \frac{F}{\sqrt{5}} = T$

Secondo caso $\quad 2l < s < 3l$



$$\underline{n} = \underline{e}_1$$

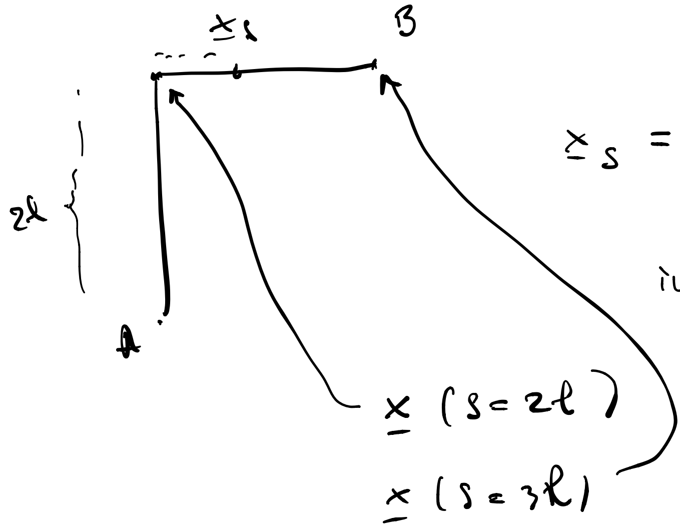
$$\underline{\tau} = -\underline{e}_2$$

Risultante: $\quad -\underline{F} + N \underbrace{\underline{e}_1}_{\underline{n}} - T \underbrace{\underline{e}_2}_{\underline{\tau} = -\underline{e}_2} = 0$

$$\longrightarrow \quad N = \frac{F}{\sqrt{5}}, \quad T = -\frac{2}{\sqrt{5}} F$$

$$\underline{F} = F \frac{\underline{e}_1 + 2\underline{e}_2}{\sqrt{5}}$$

$$M_f + (\underline{x}_A - \underline{x}_s) \wedge (-\underline{F}) \cdot \underline{e}_3 = 0$$



$$\underline{x}_s = (s-2\ell)\,\underline{e}_1 + 2\ell\,\underline{e}_2$$

in questo caso $2\ell < s < 3\ell$

$$M_f + (\underline{x}_A - \underline{x}_s)\wedge(-\underline{F})\cdot\underline{e}_3 = 0$$

$$M_f + \left[(s-2\ell)\,\underline{e}_1 + 2\ell\,\underline{e}_2\right]\wedge\frac{F}{\sqrt{5}}(\underline{e}_1+2\underline{e}_2)\cdot\underline{e}_3=0$$

$$M_f + \left[\left((s-2\ell)\,\underline{e}_1\wedge\frac{F}{\sqrt{5}}\,2\underline{e}_2\right)\cdot\underline{e}_3 + \underbrace{\left(2\ell\,\underline{e}_2\wedge\frac{F}{\sqrt{5}}\underline{e}_1\right)\cdot\underline{e}_3}\right]$$

$=0$

$$M_f + \left[2(s-2\ell)\frac{F}{\sqrt{5}}\underbrace{\underline{e}_1\wedge\underline{e}_2\cdot\underline{e}_3}_{=1} + 2\ell\frac{F}{\sqrt{5}}\underbrace{\underline{e}_2\wedge\underline{e}_1\cdot\underline{e}_3}_{=-1}\right]$$

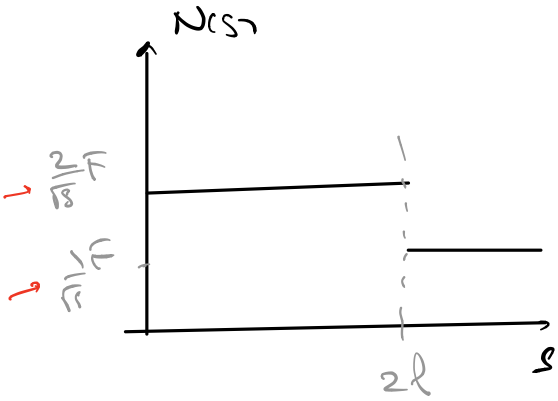
$$= M_f + \left[2(s-2\ell)\frac{F}{\sqrt{5}} - 2\ell\frac{F}{\sqrt{5}}\right] = 0$$

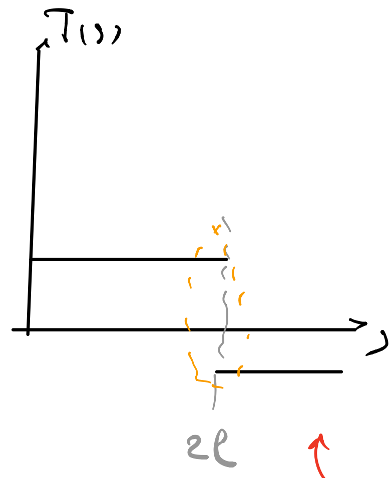
$$\Rightarrow M_f = -\frac{F}{\sqrt{5}}\left[2s-6\ell\right]$$

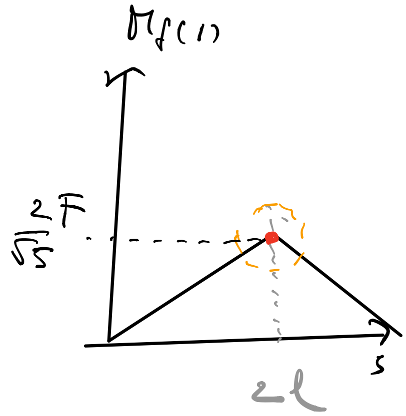
Controlliamo $\frac{d}{ds}M_f \overset{?}{=} T = -F\frac{2}{\sqrt{5}}$ ok

Riassumiamo

$$N(s) = \begin{cases} \frac{2}{\sqrt{5}}F & 0<s<2\ell \\ \frac{1}{\sqrt{5}}F & 2\ell<s<3\ell \end{cases} \qquad T(s) = \begin{cases} \frac{1}{\sqrt{5}}F \\ -\frac{2}{\sqrt{5}}F \end{cases} \qquad M_f = \begin{cases} \frac{F}{\sqrt{5}}s \\ -\frac{2}{\sqrt{5}}F(s-3\ell) \end{cases}$$







$$\frac{dM}{ds} = T$$