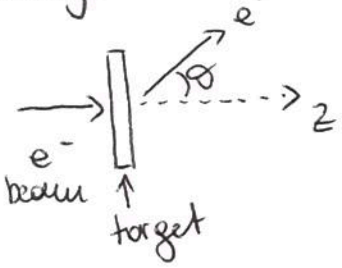
NUCLEAR PROPERTIES

How can we "measure" the size of a nucleus? For sure not with "light" since the dimension of the nucleus is much smaller than that of the visible light, but we can use "probes" with an adeguated wave lenght. We can for instance use electrons: A beam of e^- is going into an target, from the study of the number of scattered e^- we can infer some characteristic of the target, in our case the nucleus we whant to study.



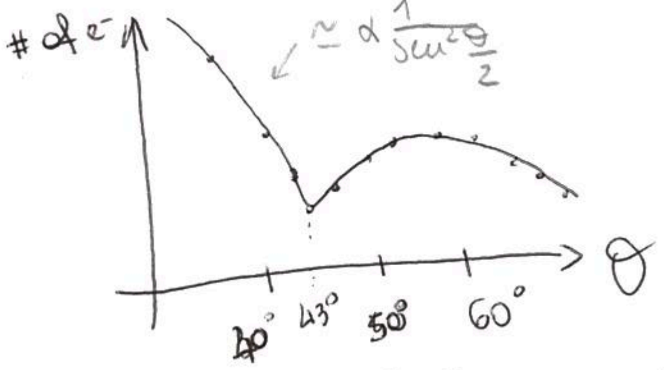
According to the theory of scattering from Rutherford, the # of scattered electrons should behave like

$$\#\text{ of scattered } e^- \propto \frac{1}{\sin^2\frac{\theta}{2}}$$

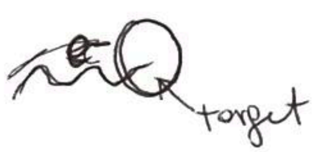
[Note that there are no relativistic corrections in here, but on a first approximation this is fine]

When the data are observed one see something like this:

Target = Oxigen (^{16}O)
beam: e^- with $E = 420$ MeV

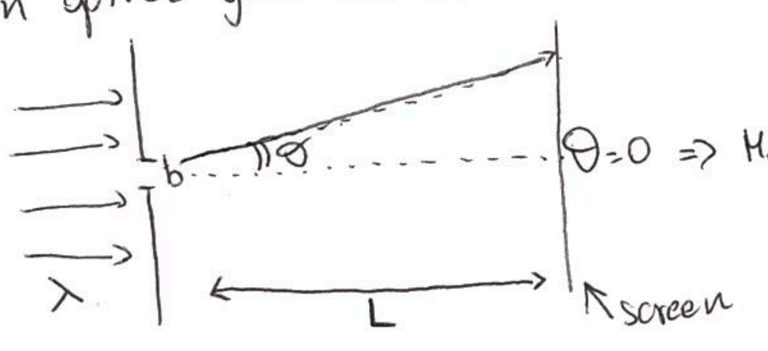


Why this happens? Because e^- behave like wave ⇒ they undergo into interference and diffraction processes.



→ There is something similar to the diffraction from a single slit

In optics you have (for mono cromatic wave)

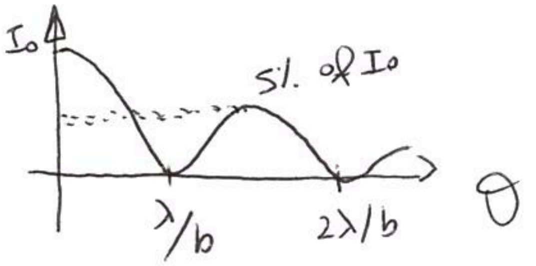


θ = 0 ⇒ MAX INTENSITY + Intensity minima and maxima

θ = diffraction angle

$$I = I_0 \frac{\sin^2\beta}{\beta} \qquad \beta = \frac{L}{\lambda}\, b \cdot \sin\theta$$

$\Rightarrow I = 0$ is $b \cdot \sin\theta = m\lambda$, for $m = \pm 1, \pm 2, \dots$



In case of a circular aperture

$$\sin\theta = \frac{1{,}22\,\lambda}{D} \qquad D = \text{diameter of the aperture.}$$

This is ANALOG to what we are looking at (of course not exactly the same)

$\Rightarrow$ Rought calculation for the observed case

$$E = 420 \text{ MeV} \qquad \lambda = \frac{hc}{E} = \frac{1240 \text{ MeV}\cdot\text{fm}}{420 \text{ MeV}} \simeq 3 \text{ fm}$$

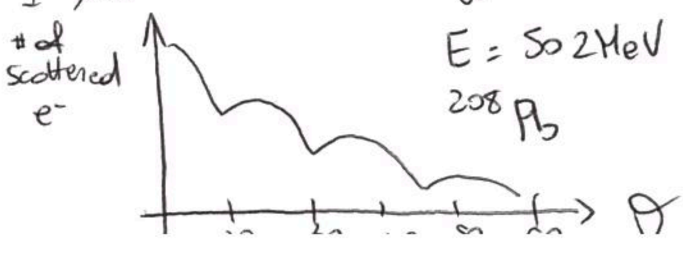
$$\sin\theta = 1{,}22\frac{\lambda}{D} \Rightarrow D = \frac{1{,}22\,\lambda}{\sin\theta} = \frac{1{,}22\cdot 3\text{ fm}}{\frac{1}{\sqrt{2}}}$$

Minima $I = 0$
$\Rightarrow \simeq \theta = 45°$

$\simeq 5$ fm

While from ACTUAL meament $r(^{16}O) \cong 2{,}6$ fm indicating that the ~~The cores~~ $\Rightarrow$ The meament is roughly Ok nucleus behaves like a disk.

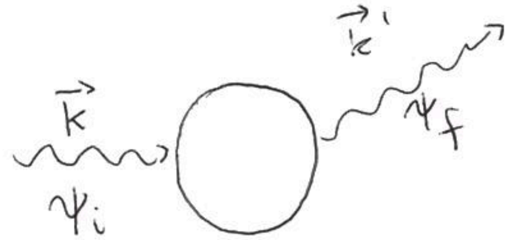
If I look for other minima, in case of ^{16}O I would end up to "unphysical" value $\Rightarrow$ the approximation is only "approximate", but if I look at other (bigger) nuclei, what can be seen is



$E = 502$ MeV
^{208}Pb

$\Rightarrow$ DIFFRACTIVE BEHAVIOUR!

Now, let's be more quantitative and let's try to evolute the effect of elastic scattering using wave functions and potential, and all the relevant stuff.



ELASTIC SCATTERING

· Energy is conserved and only the direction of $\vec{k}$ and $\vec{k}'$ is different

$$\psi_i = e^{i\vec{k}\cdot\vec{r}} \qquad \psi_f = e^{i\vec{k}'\cdot\vec{r}}$$

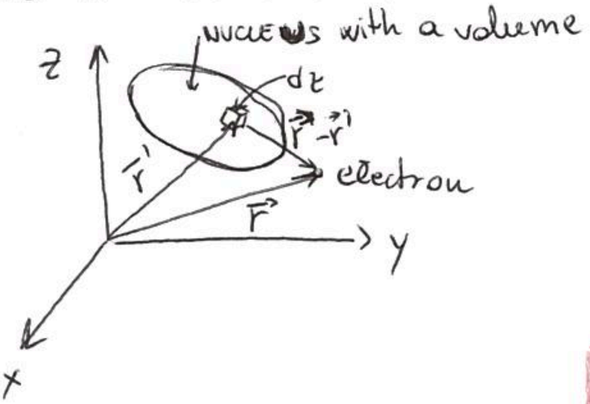
We are in presence of coulomb interaction => We have to write the interaction energy potential.
In case of Coulomb potential the potential $V(\vec{r})$ can be written as $V(\vec{r}) \dfrac{q_1 q_2}{4\pi\varepsilon_0 \vec{r}}$ hence the Hamiltonian operator can be written as:

$$H' = \int_{\text{VOLUME}} \frac{(\rho(r')\,d\tau')(-e)}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|}$$

$\rho(r')\,d\tau'$ is the charge of the "target" in a small volume which has to be integrated

$(-e)$ is the charge of incident electron

$d\tau'$ is volume element



The probability of $\psi_i \rightarrow \psi_f$ is proportional to

$$|\langle \psi_i | H' | \psi_f \rangle|^2$$

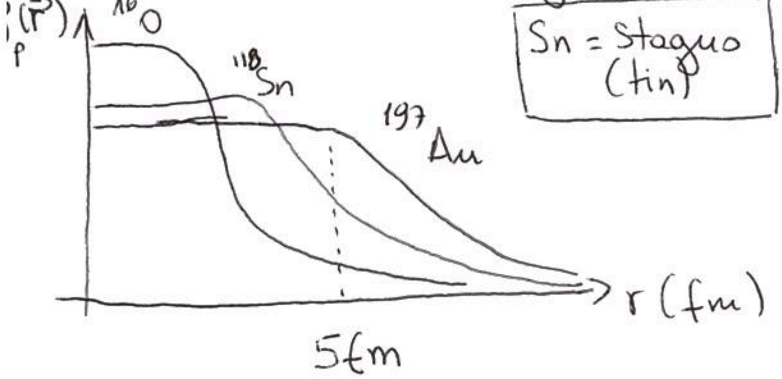
$$\int_{\text{Volume}} \psi_i^*(\vec{r}) H' \psi_f(\vec{r})\,d\tau = \boxed{e^{i(\vec{k}_i-\vec{k}_f)\vec{r}} \left[\int_{\text{Volume}} \frac{\rho(r')\,(-e)}{4\pi\varepsilon_0 |r-r'|} d\tau' \right.}$$

complex conjugate

$\equiv$ Probability of a particular θ

⇒ If you know the charge density, then you can get θ.
Now the question is how the charge inside a nucleus is distributed?

From electron scattering experiment what is obtained for charged density is shown in figure.
i.e. proton density



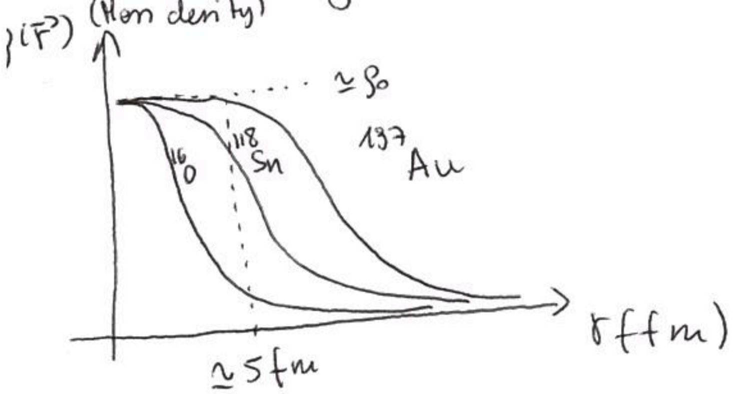
$\boxed{\text{Sn = Stagno (tin)}}$

- So the charge is $\simeq$ constant up to 5 fm and then it decreases.
- Different nuclei have different maxima

Now, we know that nuclei are formed by p and n do not contribute directly to the "charge density" but to the nucleon density they does

Z proton → proton density ρ_p
A nucleons → Mass density ρ ; $\rho = \rho_p \left(\frac{A}{Z}\right)$
Proton density

In this case, we know that for nuclei with small A $N_p \simeq N_n$ while for nuclei with large A $N_n > N_p$ ⇒ For ^{16}O : $\frac{A}{Z} = 2$ For ^{208}Pb $\frac{A}{Z} > 2$

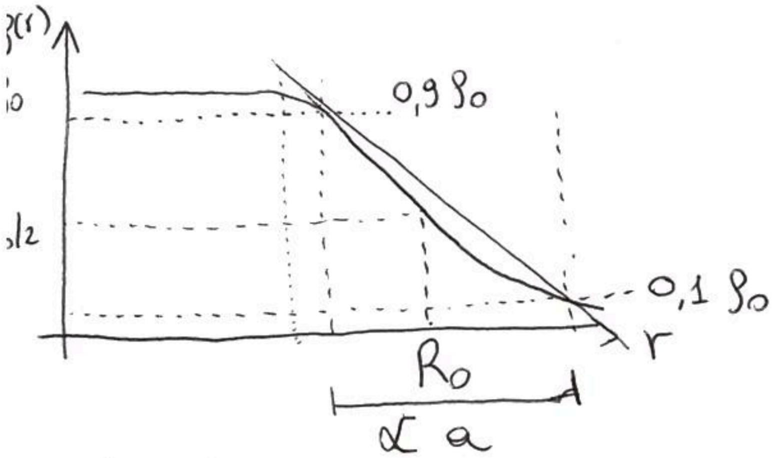


The mass density distributions are $\simeq$ equal for all the nuclei and can be described by a function which is called Woods-Saxon

$$\rho(r) = \frac{\rho_0}{1 + e^{(r-R)/a}}$$

where ρ_0 is the total mass density (i.e. the maximum of $\rho(r)$ distribution)
R is defined as the radius at which the density is equal to half of initial density ρ_0, and a represents the "sharpness" of the drop.

There are no sharp boundary because the nucleus is a quantum object!



How "large" is a

$$0{,}9\rho_0 = \frac{\rho_0}{1+e^{(r_1-R)/a}} \qquad \frac{1}{0{,}9} = 1+e^{(r_1-R)/a} \qquad \simeq \frac{1}{10} = e^{(r_1-R)/a}$$

$$\frac{r_1-R}{a} = \ln\left(\frac{1}{10}\right) = -2$$

Similarly for 0,1

$$0{,}1\,\rho_0 = \frac{\rho_0}{1+e^{(r_2-R)/a}} \qquad \frac{1}{0{,}1} = 1+e^{(r-R)/a}$$

$$r_1 = R - 2{,}3a$$

$$r_2 = R + 2{,}3a$$

$$\simeq \quad 10 = \cancel{1+}\,e^{(r-R)/a}$$

$$\frac{r-R}{a} = \ln(10) = 2{,}3$$

$$r_2 - r_1 = 2{,}3a + 2{,}3a = 4{,}6\,a$$

~~$r_2 - r_1 = 4{,}6\,a$~~

$r_2 - r_1 = 4{,}6\,a$. For all the nuclei $a \simeq 0{,}5$ fm $\Rightarrow$ The surface of the nucleus is $\simeq$ the same for ALL the nuclei.

From the measurement we also learn that ~~volume~~ density is $\simeq$ flat

So ~~the~~ the number of nucleons $\simeq$ constant

$$\frac{A}{\frac{4}{3}\pi R^3} \simeq \text{constant}$$

$\Rightarrow R \propto A^{1/3}$, and defining the proportionality constant

$$\boxed{R = R_0 A^{1/3}}$$

$A = Z + N$

$R_0 \simeq 1{,}2$ fm

$\rho_0 = 0{,}17$ nucleons / fm^3

$R(^{12}C) \cong 2{,}7 fm$
$R(^{40}Ni) \simeq 4{,}7 fm$
$R(^{100}A) \simeq 5{,}57 fm$
$R(^{235}U) \simeq 7{,}4 fm$

All the radii varies from 2-8 fm

FROM This observations we learn that nuclear force is attractive and dominant w.r.t. Coulomb.
One can also ask why the nucleus is not "more dense"...
This happens because "strong" force (the force that holds the nuclei) IS SHORT RANGE

Now lets study ~~charge density~~ RADIUS using other "techingues"
The nuclear charge density can be also be examined using atomic transition
Let's start from the Hydrogen atom H.
The potential energy V(r) can be written as

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r^2}$$

Where we ensure to have a "point" nucleus. In this case we can write the Schrodinger equation:

$$\underset{\text{Hamiltonian}}{H}\psi = \underset{\text{Energy}}{E}\psi$$

$$H = -\frac{\hbar^2}{2m}\underset{\text{Nabla (Partial derivative)}}{\nabla^2} - \frac{Ze^2}{4\pi\varepsilon_0 r}$$

Z = # of protons

At lowest energy $E = -13{,}6$ eV

$$\psi_{100} = \sqrt{\frac{1 \cdot Z^3}{\pi a_0^3}}\, e^{\frac{-Z \cdot r}{a_0}} \neq 0 \text{ at } r = 0$$

a_0 = Bohr Radius

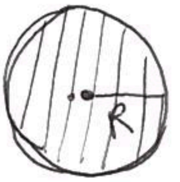
This ψ gives the probability to find an electron INSIDE the nucleus, assuming that Coulomb interaction is 0 at r = 0.

If the nucleus is extended ⇒ Energy is different hence we can measure the source radius (size).
Let's see if we can evaluate the radius of nucleus starting from the deviation from the simplest assumptions.
As first thing $V(r)$ have to change

$$U(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$$ Is the potential for a point-like nucleus.

In case of spherical nucleus with radius R and charge uniformly distributed. From electrostatic lessons for

$$r > R \,, \quad V_{out}(r) = \frac{Ze^2}{4\pi\varepsilon_0 r}$$ i.e. point nucleus form



For $r < R$

$$V(r) = +\frac{Ze^2}{4\pi\varepsilon R}\left\{\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right\}$$

$Q = Z \cdot e$

How to arrive to the potential:
Start from $\vec{E}$

$$E(\vec{r}) = \frac{Ze}{4\pi\varepsilon_0}\frac{r}{R^3}\hat{r}$$

$$V(r) - V(R) = -\int_R^r \vec{E}(\vec{r}\,')d\vec{r}\,' =$$

$$= -\int_R^r \frac{Ze}{4\pi\varepsilon_0}\frac{r}{R^3}dr = -\frac{Ze}{4\pi\varepsilon_0 R^3}\frac{r^2 - R^2}{2}$$

$$V_{ins}(r) = \frac{Ze}{4\pi\varepsilon_0 R} - \frac{Ze}{4\pi\varepsilon_0 R^3}\frac{r^2 - R^2}{2} =$$

$$= \frac{Ze}{4\pi\varepsilon_0 R}\left(1 - \frac{r^2}{2R^2} + \frac{\cancel{R^2}}{2\cancel{R^2}}\right) =$$

$$= \frac{Z \cdot e}{4\pi\varepsilon_0 R}\left\{\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right\}$$

FOR $r < R$

$$\Delta V(r) = V_{INS}(r) - V_{OUT}(r)$$

$$= \frac{Ze^2}{4\pi\varepsilon_0 R}\left\{\frac{3}{2} - \frac{1}{2}\left(\frac{r}{R}\right)^2\right\} - \left(\frac{Ze^2}{4\pi\varepsilon_0 r}\right)$$

$$\Delta U(r) = \underbrace{-\left[\frac{Ze^2}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)\right]}_{u^i(r)} - \underbrace{\left(-\frac{Ze^2}{4\pi\varepsilon_0 r}\right)}_{u^o(r)}$$

SPHERICAL + UNIFORM CHARGED SPHERE

This ΔU enters in the Hamiltonian and can be written as

H (Point like) $= H_0$
H (extended) $= H_0 + H_1$

$$H = -\frac{\hbar}{2m}\nabla^2 - \frac{Ze^2}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

$(H_0 + H_1)\psi = (E + \Delta E)\psi$

-> We have introduced a first order correction and we can interpret H_1 as a perturbation of H.
So, we can evaluted ψ_0 from H_0 and then use that ψ to evaluate ΔE.

$$\Delta E = \langle \psi_0 | H_1 | \psi_0 \rangle \quad , \quad H_1 = \frac{Ze^2}{4\pi\varepsilon_0 r} - \frac{Ze^2}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

$$\Delta E = \int \psi_0^*(\vec{r})\, \Delta U\, \psi(\vec{r})\, d\tau = \int |\psi_0(\vec{r})|^2 \Delta U\, d\tau$$

Volume element

$$\Delta E = \int \psi_0^*(\vec{r}) \left[\frac{Ze^2}{4\pi\varepsilon_0 r} - \frac{Ze^2}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)\right] \psi_0(\vec{r})\, d\tau$$

Hydrogen-like $\psi(1s)$ -> The $\bar{e}$ don't "disturb" the interaction.
(This is an $\boxed{\text{assumption}}$: the dimension of the nucleus do not change that much the wave-function)

$$\Rightarrow \psi_{1s}^{H}(\vec{r}) = \sqrt{\frac{1}{\pi a_0^3}}\, e^{-\frac{r}{a_0}} \xRightarrow{\text{(Hydrogenic)}} $$

Coulomb radial wave-function

$$\hookrightarrow R_{10} = 2\sqrt{\frac{Z^3}{\pi a_0^3}}\, e^{-\frac{Zr}{a_0}} \quad \text{or} \quad \sqrt{\frac{Z^3}{\pi a_0^3}}\, e^{-\frac{Zr}{a_0}}$$

$$\Delta E = \int_0^R \frac{Z^3}{\pi a_0^3} e^{-\frac{2Z\cdot r}{a_0}}\, d\tau \cdot \Delta U \qquad d\tau = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

Now let's see if we can simplify something...

$$e^{-\frac{2Z\cdot r}{a_0}}$$

$R \simeq 10^{-15}\, m$

$a_0 \simeq 10^{-10}\, m \; (0,5\, Å)$

$\frac{r}{a_0} \simeq 10^{-5}$

~~If~~ $Z < 100 \Rightarrow e^{-\frac{2Z\cdot r}{a_0}} \simeq 1$
($\simeq$ ALL NUCLEI)

$$\Delta E = \int_0^R \frac{Z^3}{\pi a_0^3}\left[\frac{Ze^2}{4\pi\varepsilon_0 r} - \frac{Ze^2}{8\pi\varepsilon_0 R}\left(3-\frac{r^2}{R^2}\right)\right]4\pi r^2 dr$$

$$= \frac{Z^3}{\pi a_0^3}\,\frac{Ze^2}{4\pi\varepsilon_0}\,4\pi\int_0^R\left[\frac{1}{r} - \frac{1}{2R}\left(3-\frac{r^2}{R^2}\right)\right]r^2 dr$$

$$= \frac{Z^4 e^2}{\varepsilon_0 \pi a_0^3}\int_0^R\left[r - \frac{r^2}{2R}\left(3-\frac{r^2}{R^2}\right)\right]dr =$$

$$= \frac{Z^4 e^2}{\varepsilon_0 \pi a_0^3}\int_0^R\left[r - \frac{3r^2}{2R} + \frac{r^4}{2R^3}\right]dr = \frac{Z^2 e^2}{\varepsilon_0 \pi a_0^3}\left[\frac{R^2}{2} - \frac{\cancel{3}}{2\cancel{R}}\cdot\frac{R^{\cancel{3}2}}{\cancel{3}} + \frac{1}{2\cancel{R^3}}\frac{R^{\cancel{5}}}{5}\right]$$

$$\Delta E = \frac{Z^4 e^2}{\varepsilon_0 \pi a_0^3}\,\frac{R^2}{10}$$

$E_{1S} = E_{1S}^{POINT} + \Delta E$

How can we study the effect of radius? By measuring ΔE. How can we measure ΔE? Via spectroscopy of $\boxed{\text{k alfa x-ray}}$ from electronic transitions

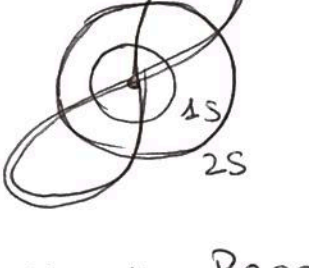
M

L ——— 1p

——— 2S

$\frac{hc}{\lambda} = E_{2p} - E_{1s}$ → k-α x-ray L → k

k ——— 1S



Why 1p and not 2s?

ψ_{2p} goes to 0 at $r=0$

⇒ It's the electron in 1p that emits the X-ray Because it overlap with the nucleus

How can the Energy of a k_α x-ray can be written?

$$E_{k_\alpha}^{(A_0)} = E_{2p} - E_{1s} = E_{2p}^{point} - E_{1s}^{point} - \frac{Z^4 e^2 R^2}{10\pi\varepsilon_0 a_0^3}$$

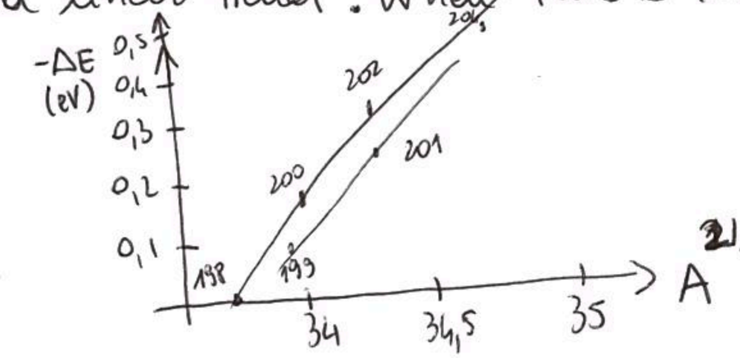
If we consider different isotopes E^{point} is the same because Z is constant. The meorement of $E_{k\alpha}$ can be used also to study the beahavior of $\neq$ ~~pta~~ isotopes. (Same Z, A different)

If we have 2 isopopes with $A = A$ an A_0

$$\Delta E_{k\alpha} = E_{k\alpha}^{(A)} - E_{k\alpha}^{(A_0)} = \frac{Z^4 e^2}{10\pi\varepsilon_0 a_0^3}\left(R^2 - R_0^2\right)$$

If $R = R^0 A^{1/3}$ then $\Delta E_{k\alpha}$ can be written as

$$\Delta E_{k\alpha} = \frac{Z^4 e^2}{10\pi\varepsilon_0 a_0^3}\left(R^0\right)^2\left(A_0^{2/3} - A^{2/3}\right)$$ and ΔE would follow a linear trend. When this is measured for instance for Hg



Note ~~Awe~~:

Ⓐ the scale is in eV $\Rightarrow$ 0,1 eV

For a "typical" x-ray we have

$E_{k\alpha}(Hg) \simeq 100$ keV

$\Delta E = 0{,}1$ eV $\Rightarrow \boxed{\frac{\Delta E}{E} \simeq 10^{-6}}$

Related to the difference between R and a (1 fm vs 0,5

Ⓑ There is an "even-odd" shift in radius of odd mon ptc, but both A-odd and A-even show a $A^{2/3}$ dependence. (This will be discussed later)

To overcome Ⓐ one can use μ instead of e^-. Muons have a mass which is 207 bigger than e^-, so the Bohr radius for muonic atoms would be:

$a_0^e = \frac{a}{M_e^2}$ $\quad a_0^\mu = \frac{a}{M_\mu^2}$ $\quad \frac{a_0^\mu}{a_0^e} = \frac{M_e^2}{M_\mu^2}$ $\quad a_0^\mu = \left(\frac{1}{207}\right)^2 a_0$

$$a_0^e = \frac{4\pi\varepsilon_0\hbar}{m_e^2} \rightarrow a_0^\mu = \frac{1}{(207)^2} a_0 \rightarrow a_0^\mu = \frac{0{,}5 \cdot 10^{-10}\,m}{4 \cdot 10^4} \simeq 10^{-5}\,m$$

$\rightarrow$ and the shift will be in keV