FUNZIONALE GENERATORE [S.14.3]

Consideriamo

$$Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n \, \langle 0| T\{\hat\varphi(x_1)\dots\hat\varphi(x_n)\}|0\rangle \, J(x_1)\dots J(x_n)$$

$$\equiv \langle 0| T\left\{ \exp\left(i \int d^4x \, \hat\varphi(x) J(x) \right) |0\rangle \right.$$

FUNZIONALE GEN. DELLE FUNZIONI DI GREEN

le J sono funzioni di prova, non operatori.
Da $Z[J]$ posso estrarre una funzione di Green a n-punti prendendo n DERIVATE FUNZIONALI

$$\boxed{\langle 0| T\{\varphi(x_1)\dots\varphi(x_n)\}|0\rangle = \frac{1}{Z[0]} \frac{1}{i^n} \left. \frac{\delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)} \right|_{J(x)=0}}$$

DERIVATE FUNZIONALI:

$$\frac{\delta F[J]}{\delta J(x)} \equiv \lim_{\varepsilon\to 0} \frac{F[J(\cdot) + \varepsilon\,\delta(\cdot - x)] - F[J]}{\varepsilon}$$

esempio $F[J] = f(y)\,J(y) \;\to\; \dfrac{\delta F[J]}{\delta J(x)} = f(y)\,\delta(y-x)$

esempio: $F[J] = \displaystyle\int_{-\infty}^{+\infty} dy\, f(y)\, J(y)$

$$\frac{\delta F}{\delta J} = \lim_{\varepsilon\to 0} \frac{\int dy\, f(y)\,(J(y) + \varepsilon\,\delta(y-x)) - \int dy\, f(y)\, J(y)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\varepsilon f(x)}{\varepsilon} = f(x)$$

es: $F[J] = e^{\int (A + \varphi J)\, dy}$

$$\frac{\delta F}{\delta J(x)} = \lim_{\varepsilon\to 0} \frac{e^{\int (A+\varphi J) dy}\,(1 + \varepsilon\varphi(x)) - e^{\int(A+\varphi J)dy}}{\varepsilon} = \varphi(x)\, e^{\int (A+\varphi J)\,dy}$$

Dall'espressione di $\langle 0 | T\{\varphi(x_1)\dots\varphi(x_n)\} | 0 \rangle$ con il path integral possiamo ugualmente avere:

$$Z[J] = \langle 0 | T\left\{ e^{i\int d^4x\, \varphi(x) J(x)} \right\} | 0 \rangle = N \int D\varphi \; e^{iS[\varphi] + i\int d^4x\, \varphi(x) J(x)}$$

funzioni, quindi commutano

$$Z[0] = N \int D\varphi \; e^{iS[\varphi]}$$

Infatti abbiamo

$$\frac{1}{i^n} \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} = N \int D\varphi \; e^{iS[\varphi] + i\int d^4x\, \varphi(x) J(x)} \varphi(x_1) \dots \varphi(x_n)$$

Da cui

$$\boxed{\frac{1}{Z[0]} \frac{1}{i^n} \frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \bigg|_{J(x)=0} = \frac{\int D\varphi \; e^{iS[\varphi]} \varphi(x_1) \dots \varphi(x_n)}{\int D\varphi \; e^{iS[\varphi]}} = \langle 0 | T\{\varphi(x_1) \dots \varphi(x_n)\} | 0 \rangle}$$

$Z[J]$ è l'analogo della funzione di partizione in meccanica statistica e contiene in sè tutta la dinamica del sistema.

RISOLVIAMO LA TEORIA LIBERA

$$\mathcal{L} = -\frac{1}{2}\varphi(\Box + m^2)\varphi$$

$$Z_0[J] = \int D\varphi \; \exp\left\{ i \int d^4x \left(-\frac{1}{2}\varphi(\Box + m^2)\varphi + J(x)\varphi(x) \right) \right\}$$

Questo integrale non è ben definito in quanto è divergente. Per renderlo convergente possiamo aggiungere una componente immaginaria alla massa (del segno corretto!)

$$Z_0[J] = \lim_{\varepsilon \to 0^+} \int D\varphi \; \exp\left\{ \int d^4x \left(-\frac{1}{2}\varphi(\Box + m^2)\varphi + J(x)\varphi(x) \right) \right\} \exp\left(-\frac{\varepsilon}{2}\int d^4x \, \varphi^2 \right)$$

$$= \int D\varphi \; \exp\left\{ i \int d^4x \left(-\frac{1}{2}\varphi(\Box + m^2 - i\varepsilon)\varphi + J(x)\varphi(x) \right) \right\}$$

Possiamo risolverlo come l'integrale Gaussiano completando il quadrato

$$\int_{-\infty}^{+\infty} d\vec{p} \; \exp\left(-\frac{1}{2}\vec{p}^{\,T} A \vec{p} + \vec{J}^{\,T}\cdot\vec{p} \right) = \sqrt{\frac{(2\pi)^n}{\det A}} \exp\left(\frac{1}{2}\vec{J}^{\,T} A^{-1} \vec{J} \right), \qquad A = i(\Box + m^2 - i\varepsilon)$$

A^{-1} è l'inverso : $\quad (\Box_x + m^2 - i\varepsilon)\,\Pi(x-y) = -\delta(x-y)$

con $$\Pi(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\varepsilon} e^{ip(x-y)}$$

$$\Rightarrow Z_0[J] = N \exp\left\{ -i \int d^4x \, d^4y \; \frac{1}{2} J(x)\,\Pi(x-y)\,J(y) \right\}$$

$$\Rightarrow \langle 0 | T\{\varphi(x)\varphi(y)\} | 0 \rangle = (-i)^2 \frac{1}{Z_0[0]} \frac{\delta^2 Z[J]}{\delta J(x)\,\delta J(y)}\bigg|_{J=0} = i\,\Pi(x-y)$$

$$\equiv D_F(x-y) =$$ 

↖ Propagatore di Feynman

Calcoliamo la FUNZIONE DI GREEN A 4 PUNTI

$$\langle 0|T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}|0\rangle = (-i)^4 \frac{1}{Z_0[0]} \frac{\delta^4 Z[J]}{\delta J(x_1)\dots\delta J(x_4)}\bigg|_{J=0} =$$

$$= \frac{1}{Z_0[0]} \frac{\delta^4}{\delta J(x_1)\dots\delta J(x_4)} N e^{-\frac{1}{2}\int d^4x\, d^4y\, J(x) D_F(x-y) J(y)}\bigg|_{J=0} =$$

$$= \frac{\delta^3}{\delta J_1 \delta J_2 \delta J_3}\left(-D_{4z}J_z\right) e^{-\frac{1}{2} J_x D_{xy} J_y}\bigg|_{J=0}$$

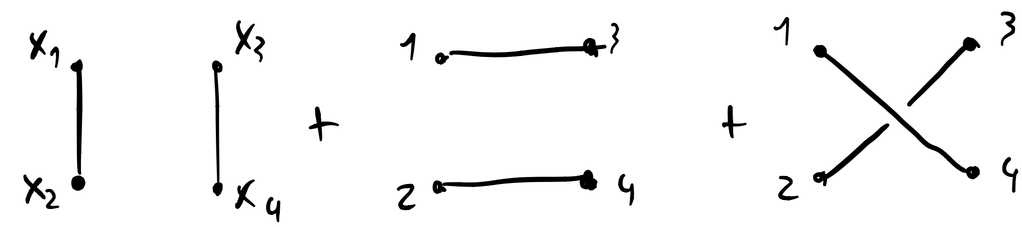
Notazione semplificata

$$= \frac{\delta^2}{\delta J_1 \delta J_2}\left(-D_{43} + D_{4z}J_z D_{3w}J_w\right) e^{-\frac{1}{2} J_x D_{xy} J_y}\bigg|_{J=0} =$$

$$= \frac{\delta}{\delta J_1}\left(D_{43}D_{2z}J_z + D_{42}D_{3w}J_w + D_{4z}J_z D_{32} - D_{4z}J_z D_{3w}J_w D_{2r}J_r\right)\overbrace{e^{-\frac{1}{2} J_x D_{xy} J_y}\Big|_{J=0}}$$

ϕ

$$= D_{34}D_{12} + D_{24}D_{13} + D_{14}D_{23}$$

$=$ 

tutti i diagrammi sono sconnessi!

INTERAZIONI [S.14.3.3]

Prendiamo $\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2 + \frac{g}{3!}\varphi^3$

ASSUMENDO DI AVER REGOLARIZZATO LE DIVERGENZE

$$Z[J] = \int D\varphi \, e^{i\int d^4x \left[\frac{1}{2}\varphi(-\Box - m^2)\varphi + J\varphi + \frac{g}{3!}\varphi^3\right]} =$$

$$= \int D\varphi \, e^{i\langle \frac{1}{2}\varphi_x(-\Box-m^2)\varphi_x + J_x\varphi_x\rangle} \, e^{i\frac{g}{3!}\varphi_x^3} =$$

ESPANSIONE PERTURBATIVA

$$= \int D\varphi \, e^{i\langle \frac{1}{2}\varphi_x(-\Box-m^2)\varphi_x + J_x\varphi_x\rangle}\left[1 + i\frac{g}{3!}\langle\varphi_z^3\rangle - \frac{1}{2}\left(\frac{g}{3!}\right)^2\langle\varphi_z^3\rangle\langle\varphi_w^3\rangle + \dots\right]$$

Possiamo scriverlo in termini dello $Z_0[J]$ della TEORIA LIBERA:

$$Z[J] = Z_0[J] + i\frac{g}{3!}\int d^4z \, (-i)^3 \frac{\delta^3 Z_0[J]}{(\delta J(z))^3} - \frac{1}{2}\left(\frac{g}{3!}\right)^2\int d^4z \, d^4w \, \frac{\delta^6 Z_0[J]}{\delta J(z)^3 \, \delta J(w)^3} + \dots$$

Calcoliamo:

$\varphi_0(x)$: campo libero

$$Z_0[J] = N e^{-\frac{1}{2}\langle J_x D_{xy} J_y\rangle}$$

$$\langle 0|T\{\hat\varphi(x_1)\hat\varphi(x_2)\}|0\rangle = \frac{1}{Z[0]}(-i)^2 \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} =$$

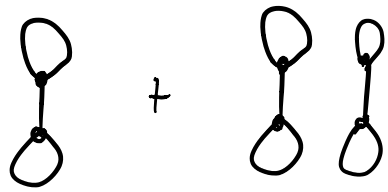
$$= \frac{Z_0[0]}{Z[0]}\langle 0|T\{\hat\varphi_0(x_1)\hat\varphi_0(x_2)\}|0\rangle + \frac{ig}{3!}\frac{Z_0[0]}{Z[0]}\int d^4z \, \langle 0|T\{\hat\varphi_0(x_1)\hat\varphi_0(x_2)\hat\varphi_0(z)^3\}|0\rangle + \dots$$

$$= \frac{\langle 0|T\{\hat\varphi_0(x_1)\hat\varphi_0(x_2)\, e^{i\int d^4z \frac{g}{3!}\hat\varphi_0(z)^3}\}|0\rangle}{\langle 0|T\{e^{i\int d^4z \frac{g}{3!}\hat\varphi_0(z)^3}\}|0\rangle}$$

Dove:

$$Z[0] = \int D\varphi \, e^{iS_0[\varphi]} \, e^{i\langle\frac{g}{3!}\varphi_z^3\rangle} = Z_0[0]\,\langle 0|T\{e^{i\frac{g}{3!}\int d^4z \, \varphi_0^3(z)}\}|0\rangle$$

$$\langle 0|\,T\{\, e^{i\int d^4z\,\frac{g}{3!}\hat{\varphi}_0(z)^3}\}\,|0\rangle = $$



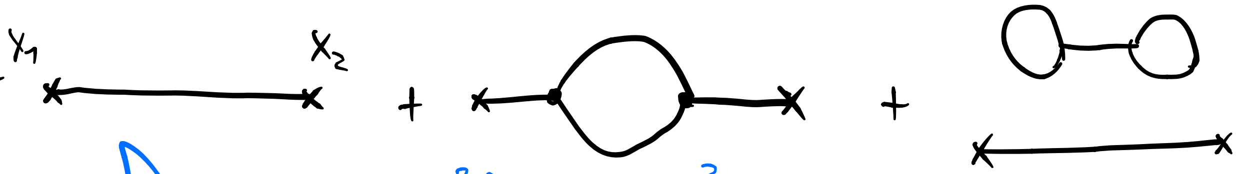
$+ \dots$

Diagrammi vuoto-vuoto

$\propto g^2 D_{zz} D_{zw} D_{ww}$

$$\langle 0|\,T\{\hat{\varphi}_0(x_1)\,\hat{\varphi}_0(x_2)\, e^{i\int d^4z\,\frac{g}{3!}\hat{\varphi}_0(z)^3}\}\,|0\rangle =$$

$=$



$+ \dots$

$\propto D_{12}$ $\qquad \propto g^2 D_{1z} D_{2w} D_{zw}^2$ $\qquad \propto g^2 D_{12} D_{zz} D_{zw} D_{ww}$

$\downarrow$

sconnesso

INTERAZIONI - 2 ← non nel programma

Consideriamo la Lagrangiana $\mathcal{L} = -\frac{1}{2}\varphi(\Box+m^2)\varphi - \frac{\lambda}{4!}\varphi^4$

$$Z[J] = \int \mathcal{D}\varphi\, \exp\left\{ i\int d^4x \left[-\frac{1}{2}\varphi(\Box+m^2)\varphi + J\varphi - \frac{\lambda}{4!}\varphi^4 \right]\right\} =$$

[assumendo di regolarizzare la teoria per levare le divergenze.]

$$= \int \mathcal{D}\varphi\, \exp\left\{ i\int d^4x \left[-\frac{1}{2}\varphi(\Box+m^2)\varphi + J\varphi\right]\right\} \exp\left\{ i\int d^4x \left(-\frac{\lambda}{4!}\right)\varphi^4 \right\} =$$

espansione perturbativa

$$= \int \mathcal{D}\varphi\, \exp\left\{ i\int d^4x \left[-\frac{1}{2}\varphi(\Box+m^2)\varphi + J\varphi\right]\right\} \times$$

$$\times \left[1 - i\frac{\lambda}{4!}\int d^4z\, \varphi(z)^4 + \frac{1}{2}\left(i\frac{\lambda}{4!}\right)^2 \int d^4z\, d^4w\, \varphi^4(z)\,\varphi^4(w) + \dots \right] =$$

$$= Z_0[J] - \frac{i\lambda}{4!}\int d^4z\, (-i)^4 \frac{\delta^4 Z_0[J]}{\delta J(z)^4} + \frac{1}{2}\left(\frac{i\lambda}{4!}\right)^2 \int d^4z\, d^4w\, (-i)^8 \frac{\delta^8 Z[J]}{\delta J(z)^4\, \delta J(w)^4} + \dots$$

dove $Z_0[J]$ è il funzionale generatore della teoria libera

Calcoliamo $\langle 0| T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}|0\rangle$ al primo ordine in λ.

$$\langle 0| T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}|0\rangle = (-i)^4 \frac{1}{Z_0[0]} \frac{\delta^4 Z[J]}{\delta J(x_1)\dots\delta J(x_4)}\bigg|_{J=0}$$

Il termine $Z_0[J]$ ci da il risultato della teoria libera ottenuto prima.

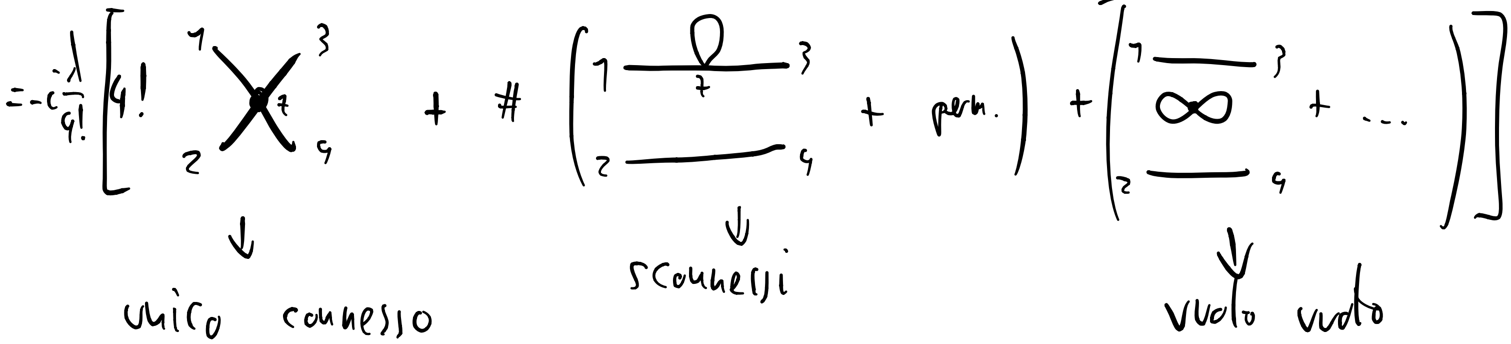
Il nuovo termine è:

provare a casa

$$-\frac{i\lambda}{4!}(-i)^4(-i)^4\frac{1}{Z_0[0]}\int d^4z \left.\frac{\delta^8 Z_0[J]}{\delta J_1\,\delta J_2\,\delta J_3\,\delta J_4\,(\delta J_z)^4}\right|_{J=0} = [\dots]$$

$$-\frac{i\lambda}{4!}\Big[4!\,D_{1z}D_{2z}D_{3z}D_{4z} + \#\left(D_{1z}D_{2z}D_{zz}D_{34} + \text{perm.}\right) +$$

$$+\left(D_{34}D_{12}+D_{24}D_{13}+D_{14}D_{23}\right)D_{zz}D_{zz}\Big] =$$



$$G_c^{(4)}(1,2,3,4) = -i\lambda\int d^4z\, D_{1z}D_{2z}D_{3z}D_{4z}$$

REGOLA DI Feynman.

$\downarrow$ LSZ

$$iM = -i\lambda$$

"elemento di matrice di scattering"

I diagrammi vuoto-vuoto si cancellano con $\frac{1}{Z[0]}$

Per ottenere elementi di matrice S ci interessano solo le funzioni di <u>Green connesse</u>