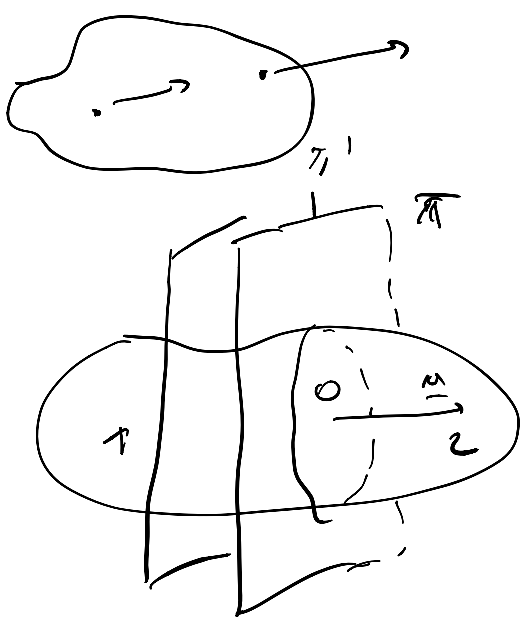
MECCANICA RAZIONALE

– LEZIONE DEL 25 MARZO 2020

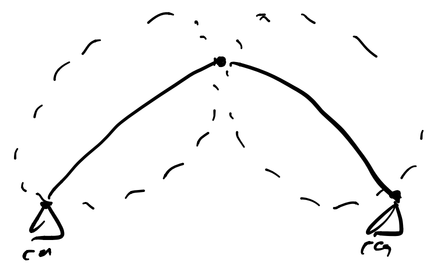
– PRIMA PARTE

Sforzi interni ad un corpo rigido

→ sollecitazioni interne al sistema che lo mantengono rigido



ECS



$$\begin{cases} \underline{R}_1^{(e)} + \underline{R}^{(i)} = \underline{0} \\ \underline{M}_1^{(e)}(O) + \underline{M}^{(i)}(O) = \underline{0} \end{cases}$$

$$N = \underline{R}^{(i)} \cdot \underline{n}$$

$$\underline{T} = \underline{R}^{i} - N\underline{n}$$

$$M_r(O) = \underline{M}^{(i)}(O) \cdot \underline{n}$$

$$\underline{M}_f(O) = \underline{M}^{i}(O) - M_r\underline{n}$$

Caso piano

$$\underline{x} = \underline{x}(s)$$

$$\underline{n} = \frac{d}{ds}\underline{x}(s)$$

$$\underline{t} = \underline{n} \wedge \underline{e}_3$$

3 funzioni scalari

$N(s)$ lungo $\underline{n}$

$T(s)$ lungo $\underline{t}$

$M_f(s)$ lungo $\underline{e}_3$

$$\underline{T}(s) = T(s)\,\underline{t}$$

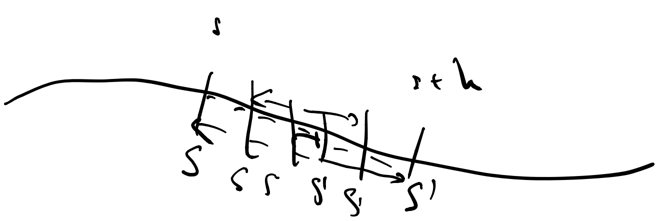
$$\underline{M}_f(s) = M_f\,\underline{e}_3$$

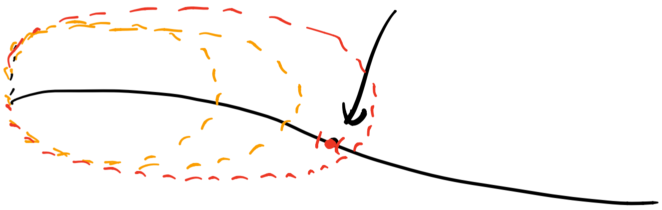
$$\begin{cases} \underline{R}_t^{(e)} + N\,\underline{n} + T\,\underline{t} = \underline{0} \\ \underline{M}^{e,1}(s) + M_f(s)\,\underline{e}_3 = \underline{0} \end{cases}$$

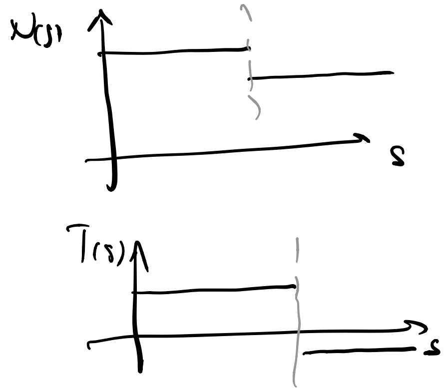
Relazioni differenziali

$$\begin{cases} \dfrac{d}{ds}\left(N\,\underline{n} + \underline{T}\right) = -\underline{f}(s) \\ \dfrac{d}{ds}\underline{M}_f(s) = \underline{T}(s) \wedge \underline{n}(s) \end{cases}$$

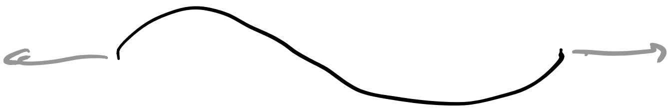
$h \longrightarrow 0$

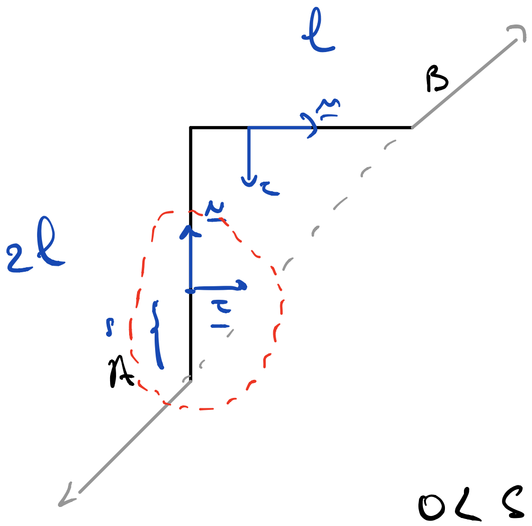






Archi scarichi





$0 < s < 3\ell$

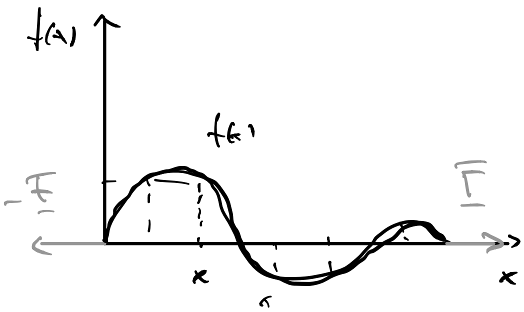
$\underline{n}$ verticale fino $s < 2\ell$ $(\underline{e}_2)$

$\underline{n}$ orizzontale da $s > 2\ell$ $(\underline{e}_1)$

$0 < s < 2\ell$ $\qquad -\underline{F} + N\overset{\underline{n}}{\underline{e}_2} + T\overset{\underline{\tau}}{\underline{e}_1} = \underline{0}$

$2\ell < s < 3\ell$ $\qquad -\underline{F} + N\underset{\underset{\underline{n}}{\uparrow}}{\underline{e}_1} + T\underset{\underset{\underline{\tau}}{\uparrow}}{(-\underline{e}_2)} = \underline{0}$

Esercizio



arco rigido determinato da una funzione $f(x)$

Calcoliamo gli sforzi interni al variare di x

$\underline{F} = F\underline{e}_1$

L lunghezza dell'arco

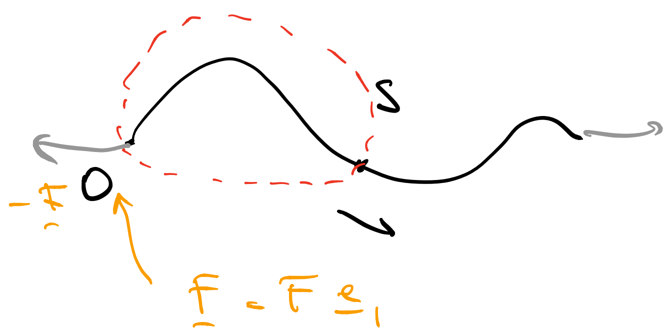
l'arco è dato

$$\{x, y : x \in [0, L], y = f(x)\}$$

Calcoliamo $\underline{n}(x)$: versore tangente alla curva nel punto $(x, f(x))$

$$\underline{n} = \frac{\underline{e}_1 + f' \underline{e}_2}{\sqrt{1 + (f')^2}}$$

$$\underline{\tau} = \underline{n} \wedge \underline{e}_3 = \frac{\underline{e}_1 + f' \underline{e}_2}{\sqrt{1+(f')^2}} \wedge \underline{e}_3 = \frac{f' \underline{e}_1 - \underline{e}_2}{\sqrt{1+(f')^2}}$$



consideriamo OS e scriviamo le eq di bilancio per OS

$$1) \quad -\underline{F} + N \underline{n} + T \underline{\tau} = \underline{0}$$

$$2) \quad M_f(x) + (\underline{x}_0 - \underline{x}_S) \wedge (-\underline{F}) \cdot \underline{e}_3 = 0$$

1) è un'equazione vettoriale

Per risolverla : moltiplichiamo scalarmente per $\underline{n}$ e per $\underline{\tau}$, usando $\underline{n} \cdot \underline{\tau} = 0$

$$\left(-\underline{F} + N\underline{n} + T\underline{\tau}\right)\cdot\underline{n} = 0$$

$$\rightarrow \quad -\underline{F}\cdot\underline{n} + N = 0$$

$$\Rightarrow \quad N = \underline{F}\cdot\underline{n} = \frac{F}{\sqrt{1+(f')^2}}$$

$$\left(-\underline{F} + N\underline{n} + T\underline{\tau}\right)\cdot\underline{\tau} = 0$$

$$-\underline{F}\cdot\underline{\tau} + T = 0$$

$$\Rightarrow \quad T = \underline{F}\cdot\underline{\tau} = \frac{F f'}{\sqrt{1+(f')^2}}$$

2) $$M_f(x) = (\underline{x}_0 - \underline{x}_s)\wedge \underline{F}\cdot\underline{e}_3$$

$$= -\left(x\underline{e}_1 + f(x)\underline{e}_2\right)\wedge F\underline{e}_1\cdot\underline{e}_3$$

$$= f(x)F$$

$$\frac{d}{ds}M_f(x(s)) = T(x(s))$$

$$\frac{d}{ds}M_f(x(s)) = \frac{d}{dx}M_f(x)\frac{dx}{ds} = \frac{1}{\sqrt{1+(f')^2}}\frac{dM_f}{dx}$$

$$= f' F \cdot \frac{1}{\sqrt{1+(f')^2}} = T(x(s))$$

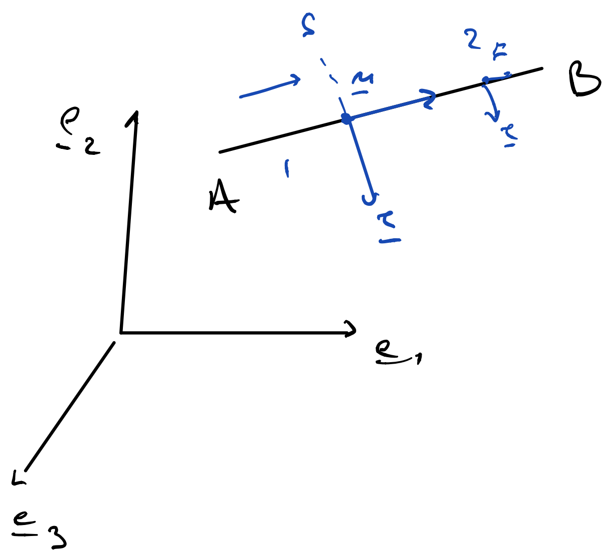
MECCANICA RAZIONALE

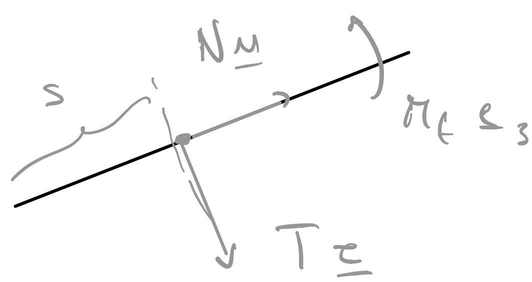
– LEZIONE DEL 25 MARZO 2020

– SECONDA PARTE

Aste

AB un'asta rettilinea, sottoposta a carichi continui e a carichi concentrati





Consideriamo la parte I

(sezione s non coincide con i punti di

applicazione dei carichi concentrati)

La parte 1 è in equilibrio sotto l' azione delle seguenti forze

1) Tutti i carichi concentrati agenti sulla parte 1

- risultante $\underline{R}_1^c$
- momento risultante rispetto al polo s : $\underline{M}_1^c(s)$

2) Carichi distribuiti su 1

$\underline{f}(s)$ la forza specifica

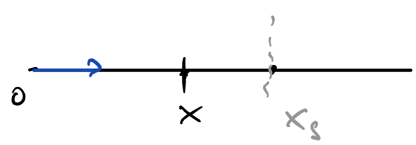
$$\underline{R}_1^d = \int_0^s \underline{f}(x)\,dx$$

$$\underline{M}_1^d(s) = \int_0^s (x-s)\,\underline{u} \wedge \underline{f}(x)\,dx$$

↑

$\underline{f}(x)$ è applicato nel punto x

$\underline{x}_s = s\,\underline{u}$



$\underline{x} = x\,\underline{u}$

$$\underline{x} - \underline{x}_S = x\,\underline{u} - s\,\underline{u} = (x - s)\,\underline{u}$$

$$(\underline{x} - \underline{x}_S) \wedge \underline{f}(x) = (x - s)\,\underline{u} \wedge \underline{f}(x)$$

3) Sforzi interni

applicate in S $\begin{cases} N(s)\,\underline{u} + T(s)\,\underline{\tau} \\ M_f(s)\,\underline{e}_3 \end{cases}$

In generale : bilanciamo 1) + 2) + 3)

- ricavare N & T dall' equazione vettoriale delle risultanti
- M_f dall' eq. dei momenti (dove la sezione S è scelta come polo)

Equivalentemente : usiamo le relazioni differenziali

$$\frac{d}{ds}\left(N\underline{u} + T\underline{\tau}\right) = \frac{dN}{ds}\,\underline{u} + \frac{dT}{ds}\,\underline{\tau} = -\underline{f}(s)$$

perché per asta u & τ sono costanti

→ due eq. scalari $\begin{cases} \dfrac{dN}{ds} = -\underline{f}(s)\cdot\underline{u} \\ \dfrac{dT}{ds} = -\underline{f}(s)\cdot\underline{\tau} \end{cases}$

$$\frac{d}{ds} M_f(s) = T(s) \longrightarrow \underline{M}_{f(s)} - \underline{M}_{f(0)} = \int_0^s T(x)\,dx$$

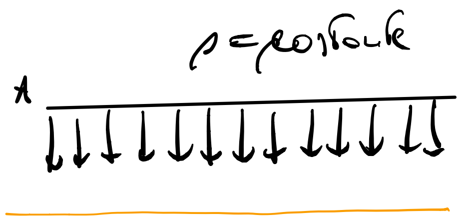
condizioni iniziali

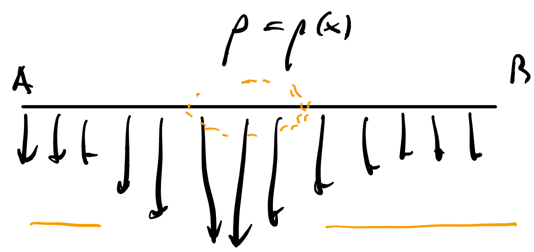
<u>Commenti</u>

•) Non bisogna fissare una sezione, ma calcolare gli sforzi al variare della sezione

•) non confondere forze specifiche e forze concentrate

→ forza peso non è carico concentrato nel centro di massa





•) carichi distribuiti non sono distribuiti in modo uniforme ($\neq$ costanti)

•) $\frac{d}{ds} M_{f(s)} = T(s)$ eq. differenziale

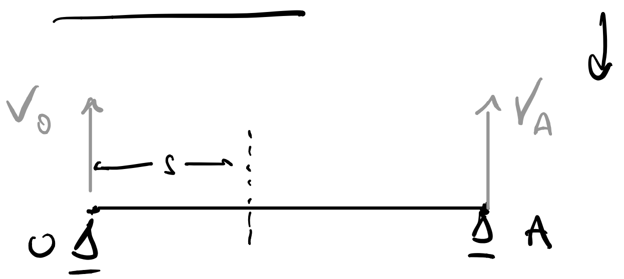
MECCANICA RAZIONALE

- LEZIONE DEL 25 MARZO 2020

- TERZA PARTE

Esercizio

$\overline{OA} = L$



$\underline{f}(s) = \rho(s)\,\underline{g}$

Bilanciare gli sforzi interni in funzione di s, con le forze esterne

Primo: calcolare le forze di reazione

Dopo: calcolare gli sforzi interni

1. Forze di reazione

Forza peso: applicata nel centro di massa

G $\qquad s_G = \frac{1}{M}\int_0^L s\,\rho(s)\,ds \qquad \left(x_G = \frac{\sum_p x_p \cdot m_p}{M}\right)$

$$M = \int_0^L \rho(s)\,ds$$

ECS → forze di reazione

$\underline{R}^e = \underline{0}$: $V_o + V_A - Mg = 0$

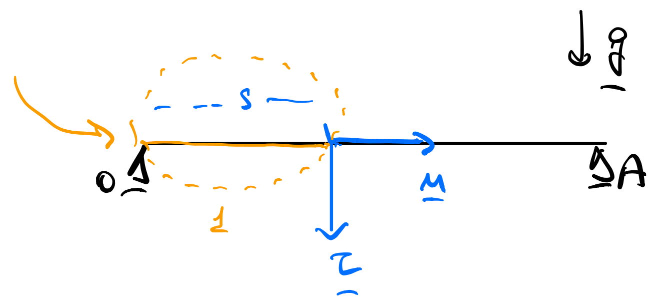
$V_A = Mg - V_o$

$\underline{M}^e(A) = \underline{0}$

$\Rightarrow V_o L = Mg(L - s_G)$

$V_o = Mg\left(1 - \frac{s_G}{L}\right)$

2\. Sforzi interni $0 < s < L$



$\underline{M} = \mathrm{vers}(\underline{x}_A - \underline{x}_o)$

$\underline{\tau} = \mathrm{vers}\ \underline{g}$

Allora : $N\underline{M} + T\underline{\tau} - V_o\underline{\tau} + m(s)g\,\underline{\tau} = \underline{0}$

$m(s) = \int_0^s \rho(x)\,dx$ massa del tratto OS

$N = 0$

$T = V_o - m(s)\,g$

momento flettente: $\frac{dM}{ds} = T$ sapendo ma $M_f \xrightarrow[s \to 0^+]{} 0$ $\qquad M_f(s) = \int_0^s T(x)\,dx$

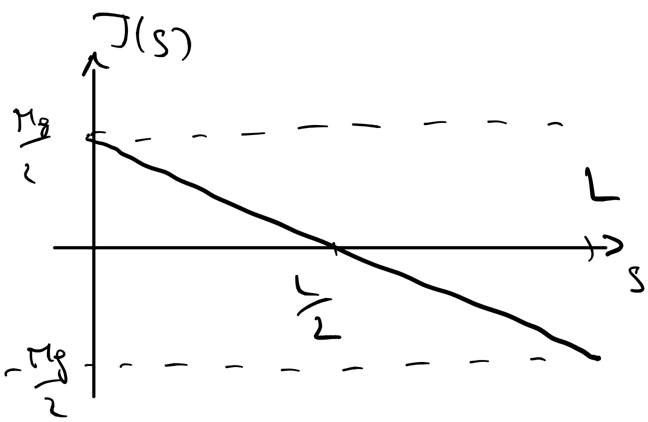
Esempio di $\rho = \rho(s)$

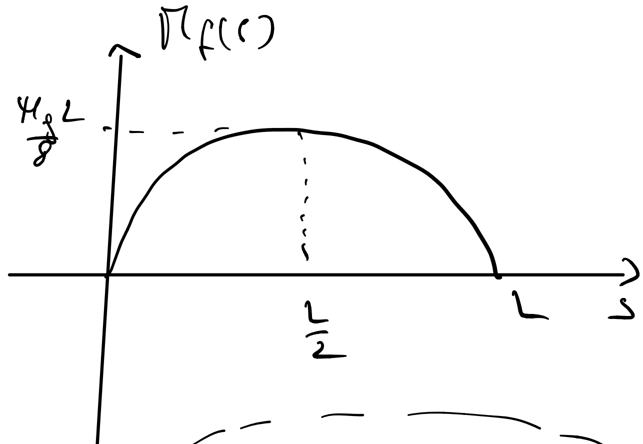
- densità costante $\rho = \frac{M}{L}$

$$V_0 = \frac{Mg}{2} = V_A$$

$$T_{(s)} = Mg\left(\frac{1}{2} - \frac{s}{L}\right) \qquad M_f = \frac{Mg}{2}\, s\left(1 - \frac{s}{L}\right) = \frac{Mg}{2}\left(s - \frac{s^2}{L}\right)$$

$$g\int_0^s \frac{M}{L} = \frac{M}{L} s g$$





$$M_f = \frac{Mg}{2} s - \frac{Mg}{2}\frac{s^2}{L}$$

$$\frac{d}{ds} M_f = \frac{Mg}{2} - Mg\frac{s}{L} = 0 \longrightarrow s = \frac{L}{2}, \quad M_f = \frac{MgL}{8}$$

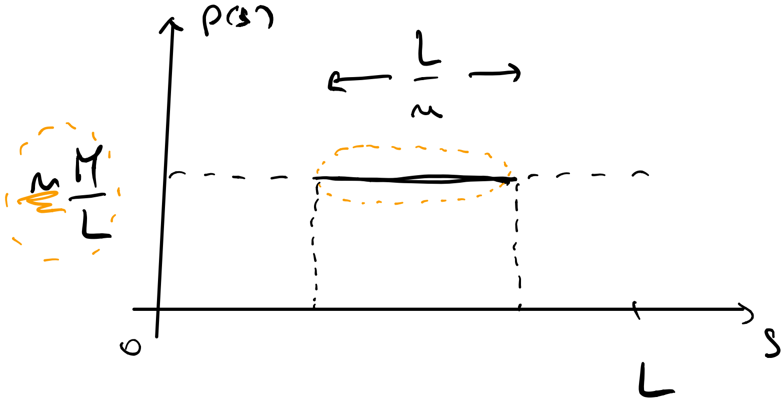
max $\longrightarrow \frac{Mg}{2} - Mg\frac{s}{L} = 0$

$$M_f\left(s = \frac{L}{2}\right) = \frac{Mg}{2}\frac{L}{2} - \frac{Mg}{2}\frac{L^2}{L}\frac{1}{4} = \frac{MgL}{8}$$

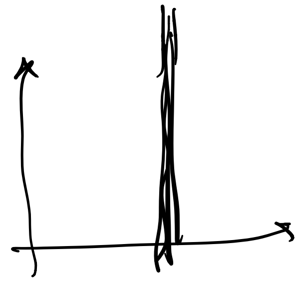
- densità

$$\rho(s) = \begin{cases} 0 & 0 < s < \frac{L}{2}\left(1 - \frac{1}{n}\right) \\ n\frac{M}{L} & \frac{L}{2}\left(1 - \frac{1}{n}\right) < s < \frac{L}{2}\left(1 + \frac{1}{n}\right) \\ 0 & \frac{L}{2}\left(1 + \frac{1}{n}\right) < s < L \end{cases}$$

$\boxed{n > 1}$



quando $n \to \infty$



→ massa totale, centro di massa, reazione V_0 rimangono uguali al caso precedente

$$m(s) = \begin{cases} 0 & 0 < s < \frac{L}{2}\left(1-\frac{1}{n}\right) \\ n\frac{M}{L}\left[s - \frac{L}{2}\left(1-\frac{1}{n}\right)\right] & \frac{L}{2}\left(1-\frac{1}{n}\right) < s < \frac{L}{2}\left(1+\frac{1}{n}\right) \\ M & \frac{L}{2}\left(1+\frac{1}{n}\right) < s < L \end{cases}$$

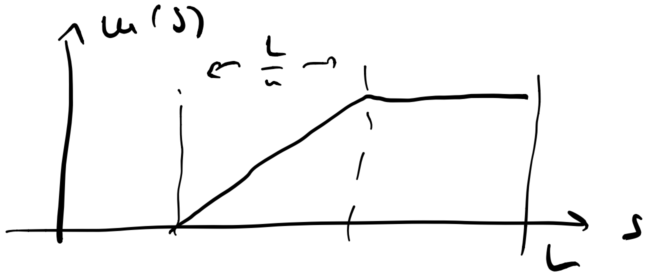
$\Downarrow$

$$\int_0^s \rho(x)\,dx$$

$$\int_0^s \rho(x)\,dx$$

$0 - \frac{L}{2}\left(1-\frac{1}{n}\right)\ ,\ \frac{L}{2}\left(1-\frac{1}{n}\right) - s$

Quindi



← M

Sforzi interni

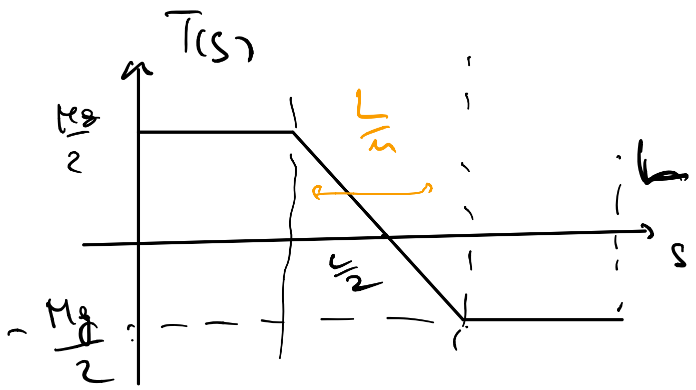
⟹ $T = V_0 - m(s)\,g$

$V_0 = \frac{Mg}{2}$

$$M_f(s) = \int_0^s T(x)\,dx$$

$$T(s) = \begin{cases} \dfrac{Mg}{2} & 0 < s < \dfrac{L}{2}\left(1 - \dfrac{1}{n}\right) \\ Mgn\left(\dfrac{1}{2} - \dfrac{s}{L}\right) & \dfrac{L}{2}\left(1 - \dfrac{1}{n}\right) < s < \dfrac{L}{2}\left(1 + \dfrac{1}{n}\right) \\ -\dfrac{Mg}{2} & \dfrac{L}{2}\left(1 + \dfrac{1}{n}\right) < s < L \end{cases}$$

(Verificatelo)



Momento flettente (verificatelo!)

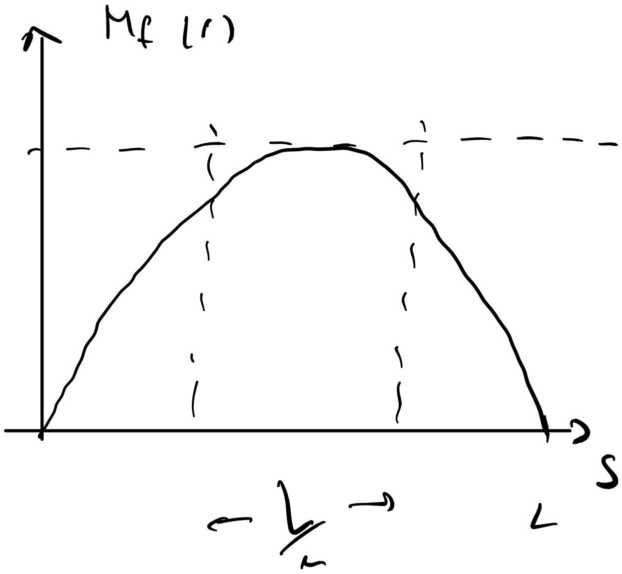
$$M_f(s) = \begin{cases} \dfrac{Mg}{2}\, s & 0 < s < \dfrac{L}{2}\left(1 - \dfrac{1}{n}\right) \\ Mgn\left(\dfrac{s}{2} - \dfrac{s^2}{2L}\right) + a & \dfrac{L}{2}\left(1 - \dfrac{1}{n}\right) < s < \dfrac{L}{2}\left(1 + \dfrac{1}{n}\right) \\ -\dfrac{Mg}{2}(L - s) & \dfrac{L}{2}\left(1 + \dfrac{1}{n}\right) < s < L \end{cases}$$

$$M_f = \int_0^s T(x)\, dx$$

Costante di integrazione

$$a = \frac{MgL}{4}\left(1 - \frac{n}{2} - \frac{1}{2n}\right)$$

imponendo che $M_f(s)$ sia una funzione continua



$$Mg\frac{L}{4}\left(1-\frac{1}{2n}\right)$$

$n \to \infty$