LESSON 7.

1. Irreducible components.

The aim of this lesson is to introduce the irreducible components of the affine varieties, the "building blocks" of the algebraic varieties. The idea is that the irreducible varieties are a generalization in any dimensions of the irreducible hypersurfaces: any hypersurface is a finite union of irreducible hypersurfaces, similarly any algebraic variety (affine or projective) is a finite unione of irreducible varieties. The notion of irreducible topological space is typical of algebraic geometry and is interesting in this context, although it is not so for Hausdorff topological spaces.

Definition 1.1. Let X be a topological space. X is *irreducible* if it is not empty and the following condition holds: if $X = X_1 \cup X_2$ with X_1, X_2 closed subsets of X, then either $X = X_1$ or $X = X_2$.

Equivalently, passing to the complementar sets, X is irreducible if it is non empty and, for all pair of non–empty open subsets U, V, we have $U \cap V \neq \emptyset$.

Note that, by definition, $\emptyset$ is not irreducible.

Proposition 1.2. *X is irreducible if and only if any non–empty open subset U of X is dense in X.*

Proof. Let X be irreducible, let P be a point of X and let I_P be an open neighbourhood of P in X. I_P and U are non–empty and open, so $I_P \cap U \neq \emptyset$, therefore $P \in \overline{U}$. This proves that $\overline{U} = X$.

Conversely, assume that all open subsets are dense. Let U, $V \neq \emptyset$ be open subsets. Let $P \in U$ be a point. By assumption $P \in \overline{V} = X$, so $V \cap U \neq \emptyset$ (U is an open neighbourhood of P). □

Example 1.3. *1. If $X = \{P\}$ is a unique point, then X is irreducible.*

2. Let K be an infinite field. Then $\mathbb{A}^1$ is irreducible, because proper closed subsets are finite sets. The same holds for $\mathbb{P}^1$.

3. Let $f : X \to Y$ be a continuous map of topological spaces. If X is irreducible and f is surjective, then Y is irreducible.

4. Let $Y \subset X$, $Y \neq \emptyset$, be a subset endowed with the induced topology. Then Y is irreducible if and only if the following holds: if $Y \subset Z_1 \cup Z_2$, with Z_1 and Z_2 closed in X, then either

$Y \subset Z_1$ or $Y \subset Z_2$; equivalently: if $Y \cap U \neq \emptyset$, $Y \cap V \neq \emptyset$, with U, V open subsets of X, then $Y \cap U \cap V \neq \emptyset$.

Proposition 1.4. *Let X be a topological space, Y a subset of X. Y is irreducible if and only if $\overline{Y}$ is irreducible.*

Proof. Note first that if $U \subset X$ is open and $U \cap Y = \emptyset$ then $U \cap \overline{Y} = \emptyset$. Otherwise, if $P \in U \cap \overline{Y}$, let A be an open neighbourhood of P: then $A \cap Y \neq \emptyset$. In particular, U is an open neighbourhood of P so $U \cap Y \neq \emptyset$.

Let Y be irreducible. If U and V are open subsets of X such that $U \cap \overline{Y} \neq \emptyset$, $V \cap \overline{Y} \neq \emptyset$, then $U \cap Y \neq \emptyset$ and $V \cap Y \neq \emptyset$ so $Y \cap U \cap V \neq \emptyset$ by the irreducibility of Y. Hence $\overline{Y} \cap (U \cap V) \neq \emptyset$. So $\overline{Y}$ is irreducible. If $\overline{Y}$ is irreducible, we get the irreducibility of Y in a completely analogous way. □

Corollary 1.5. *Let X be an irreducible topological space and let U be a non–empty open subset of X. Then U is irreducible.*

Proof. By Proposition 1.2 $\overline{U} = X$, which is irreducible. By Proposition 1.4 U is irreducible. □

For algebraic sets (both affine and projective) irreducibility can be expressed in a purely algebraic way.

Proposition 1.6. *Let $X \subset \mathbb{A}^n$ (resp. $\mathbb{P}^n$) be an algebraic variety equipped with the Zariski topology, i.e. the induced topology by the Zariski topology of the affine (or projective) space. X is irreducible if and only if $I(X)$ (resp. $I_h(X)$) is prime.*

Proof. Assume first that X is irreducible, $X \subset \mathbb{A}^n$. Let F, G be polynomials in $K[x_1, \dots, x_n]$ such that $FG \in I(X)$: then

$$V(F) \cup V(G) = V(FG) \supset V(I(X)) = X,$$

hence either $X \subset V(F)$ or $X \subset V(G)$. In the former case, if $P \in X$ then $F(P) = 0$, so $F \in I(X)$, in the second case $G \in I(X)$; hence $I(X)$ is prime.

Assume now that $I(X)$ is prime. Let $X = X_1 \cup X_2$ be the union of two closed subsets. Then $I(X) = I(X_1) \cap I(X_2)$ (see Lesson 4). Assume that $X_1 \neq X$, then $I(X_1)$ strictly contains $I(X)$, otherwise, if $I(X) = I(X_1)$, it would follow $X_1 = V(I(X_1)) = V(I(X)) = X$ because both are closed. So there exists $F \in I(X_1)$ such that $F \notin I(X)$. But for every $G \in I(X_2)$, $FG \in I(X_1) \cap I(X_2) = I(X)$, which is prime: since $F \notin I(X)$, then $G \in I(X)$. So $I(X_2) \subset I(X)$, and we conclude that $I(X_2) = I(X)$, so $X_2 = X$.

If $X \subset \mathbb{P}^n$, the proof is similar, taking into account the following Lemma.

Lemma 1.7. *Let $\mathcal{P} \subset K[x_0, x_1, \dots, x_n]$ be a homogeneous ideal. Then $\mathcal{P}$ is prime if and only if, for every pair of homogeneous polynomials F, G such that $FG \in \mathcal{P}$, either $F \in \mathcal{P}$ or $G \in \mathcal{P}$.*

Proof of the Lemma. Let H, K be any polynomials such that $HK \in \mathcal{P}$. Let $H = H_0 + H_1 + \dots + H_d$, $K = K_0 + K_1 + \dots + K_e$ (with $H_d \neq 0 \neq K_e$) be their expressions as sums of homogeneous polynomials. Then $HK = H_0K_0 + (H_0K_1 + H_1K_0) + \dots + H_dK_e$: the last product is the homogeneous component of degree $d + e$ of HK. $\mathcal{P}$ being homogeneous, $H_dK_e \in \mathcal{P}$; by assumption either $H_d \in \mathcal{P}$ or $K_e \in \mathcal{P}$. In the former case, $HK - H_dK = (H - H_d)K$ belongs to $\mathcal{P}$ while in the second one $H(K - K_e) \in \mathcal{P}$. So in both cases we can proceed by induction. □

We list now some consequences of Proposition 1.6.

1. Let K be an infinite field. Then $\mathbb{A}^n$ and $\mathbb{P}^n$ are irreducible, because $I(\mathbb{A}^n) = I_h(\mathbb{P}^n) = (0)$.

2. Let $Y \subset \mathbb{P}^n$ be closed. Y is irreducible if and only if its affine cone $C(Y)$ is irreducible.

3. Let $Y = V(F) \subset \mathbb{A}^n$, be a hypersurface over an algebraically closed field K. If F is irreducible, then Y is irreducible.

4. Let K be algebraically closed. There is a bijection between prime ideals of $K[x_1, \dots, x_n]$ and irreducible algebraic subsets of $\mathbb{A}^n$. In particular, the maximal ideals correspond to the points. Similarly, there is a bijection between homogeneous non–irrelevant prime ideals of $K[x_0, x_1, \dots, x_n]$ and irreducible algebraic subsets of $\mathbb{P}^n$.

Our next task is to prove that any algebraic variety can be written as a **finite** union of irreducible varieties.

Definition 1.8. A topological space X is called *noetherian* if it satisfies the following equivalent conditions:

(i) the ascending chain condition for open subsets;
(ii) the descending chain condition for closed subsets;
(iii) any non–empty set of open subsets of X has maximal elements;
(iv) any non–empty set of closed subsets of X has minimal elements.

The proof of the equivalence is standard (compare with the properties defining noetherian rings).

Example 1.9. *$\mathbb{A}^n$ is noetherian: if the following is a descending chain of closed subsets of $\mathbb{A}^n$*

$$Y_1 \supset Y_2 \supset \cdots \supset Y_k \supset \dots,$$

then

$$I(Y_1) \subset I(Y_2) \subset \cdots \subset I(Y_k) \subset \dots$$

is an ascending chain of ideals of $K[x_1, \dots, x_n]$, hence it is stationary from a suitable m on; therefore $V(I(Y_m)) = Y_m = V(I(Y_{m+1})) = Y_{m+1} = \dots$.

Proposition 1.10. *Let X be a noetherian topological space and Y be a non–empty closed subset of X. Then Y can be written as a finite union $Y = Y_1 \cup \cdots \cup Y_r$ of irreducible closed subsets. The maximal Y_i's in the union are uniquely determined by Y and are called the "irreducible components" of Y. They are the maximal irreducible subsets of Y.*

Proof. By contradiction. Let S be the set of the non–empty closed subsets of X which are not a finite union of irreducible closed subsets: assume $S \neq \emptyset$. By noetherianity S has minimal elements, fix one of them Z. Z is not irreducible, so $Z = Z_1 \cup Z_2$, $Z_i \neq Z$ for $i = 1, 2$. So $Z_1, Z_2 \notin S$, hence Z_1, Z_2 are both finite unions of irreducible closed subsets, so such is Z: a contradiction.

Now assume that $Y = Y_1 \cup \cdots \cup Y_r$, with $Y_i \not\subseteq Y_j$ if $i \neq j$ and Y_i irreducible closed for all i. If there is another similar expression $Y = Y'_1 \cup \cdots \cup Y'_s$, $Y'_i \not\subseteq Y'_j$ for $i \neq j$, then $Y'_1 \subset Y_1 \cup \dots Y_r$, so $Y'_1 = \bigcup_{i=1}^r (Y'_1 \cup Y_i)$, hence $Y'_1 \subset Y_i$ for some i, and we can assume $i = 1$. Similarly, $Y_1 \subset Y'_j$, for some j, so $Y'_1 \subset Y_1 \subset Y'_j$, so $j = 1$ and $Y_1 = Y'_1$. Now let $Z = \overline{Y - Y_1} = Y_2 \cup \cdots \cup Y_r = Y'_2 \cup \cdots \cup Y'_s$ and proceed by induction. □

Corollary 1.11. *Any algebraic variety in $\mathbb{A}^n$ (resp. in $\mathbb{P}^n$) can be written in a unique way as the finite union of its irreducible components.*

Note that the irreducible components of X are its maximal algebraic subsets. They correspond to the minimal prime ideals over $I(X)$. Since $I(X)$ is radical, these minimal prime ideals coincide with the primary ideals appearing in the primary decomposition of $I(X)$.

Often the irreducible closed subsets of $\mathbb{A}^n$ are called *affine varieties*, i.e. the term variety is reserved to the irreducible ones. Similarly for the irreducible closed subsets of $\mathbb{P}^n$.

Definition 1.12. A locally closed subset in $\mathbb{P}^n$ is the intersection of an open and a closed subset. An irreducible locally closed subset of $\mathbb{P}^n$ is called a *quasi–projective variety.*

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We conclude this section with the (non-trivial) proof of the irreducibility of the product of irreducible affine varieties.

Proposition 1.13. *Let $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ be irreducible affine varieties. Then $X \times Y$ is an irreducible subvariety of $\mathbb{A}^{n+m}$.*

Proof. Let $X \times Y = W_1 \cup W_2$, with W_1, W_2 closed. For any $P \in X$, the map $\{P\} \times Y \to Y$ which takes (P, Q) to Q is a homeomorphism, so $\{P\} \times Y$ is irreducible. $\{P\} \times Y = (W_1 \cap (\{P\} \times Y)) \cup (W_2 \cap (\{P\} \times Y))$, so $\exists i \in \{1, 2\}$ such that $\{P\} \times Y \subset W_i$. Let $X_i = \{P \in X \mid \{P\} \times Y \subset W_i\}$, $i = 1, 2$. Note that $X = X_1 \cup X_2$.

Claim. X_i is closed in X.
Let $X^i(Q) = \{P \in X \mid (P, Q) \in W_i\}$, $Q \in Y$. We have: $(X \times \{Q\}) \cap W_i = X^i(Q) \times \{Q\} \simeq X^i(Q)$; $X \times \{Q\}$ and W_i are closed in $X \times Y$, so $X^i(Q) \times \{Q\}$ is closed in $X \times Y$ and also in $X \times \{Q\}$, so $X^i(Q)$ is closed in X. Note that $X_i = \bigcap_{Q \in Y} X^i(Q)$, hence X_i is closed, which proves the Claim.

Since X is irreducible, $X = X_1 \cup X_2$ implies that either $X = X_1$ or $X = X_2$, so either $X \times Y = W_1$ or $X \times Y = W_2$. □

Exercises 1.14. 1. Let $X \neq \emptyset$ be a topological space. Prove that X is irreducible if and only if all non–empty open subsets of X are connected.

2*. Prove that the *cuspidal cubic* $Y \subset \mathbb{A}^2_{\mathbb{C}}$ of equation $x^3 - y^2 = 0$ is irreducible. (Hint: express Y as image of $\mathbb{A}^1$ in a continuous map...)

3. Give an example of two irreducible subvarieties of $\mathbb{P}^3$ whose intersection is reducible.

4. Find the irreducible components of the following algebraic sets over the complex field:
a) $V(y^4 - x^2, y^4 - x^2y^2 + xy^2 - x^3) \subset \mathbb{A}^2$;
b) $V(y^2 - xz, z^2 - y^3) \subset \mathbb{A}^3$.

5*. Let Z be a topological space and let $\{U_\alpha\}_{\alpha \in I}$ be an open covering of Z such that $U_\alpha \cap U_\beta \neq \emptyset$ for $\alpha \neq \beta$ and that all U_α's are irreducible. Prove that Z is irreducible.