LESSON 8.

1. Dimension.

There are a few equivalent ways to give the definition of dimension for algebraic varieties. In this section we will first see a topological definition, then an algebraic characterization. In a later lesson, we will see a more geometrical interpretation.

Let X be a topological space.

Definition 1.1. The *topological dimension* of X is the supremum of the lengths of the chains of distinct irreducible closed subsets of X, where by definiton the following chain has length n:

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n.$$

The topological dimension of X is denoted by $\dim X$. It is also called combinatorial or Krull dimension.

Example 1.2. (1) $\dim \mathbb{A}^1 = 1$*: the maximal length chains have the form* $\{P\} \subset \mathbb{A}^1$.

(2) $\dim \mathbb{A}^n$*: a chain of length n is*

$$\{0\} = V(x_1, \dots, x_n) \subset V(x_1, \dots, x_{n-1}) \subset \cdots \subset V(x_1) \subset \mathbb{A}^n.$$

Note that $V(x_1, \dots, x_i)$ is irreducible for any $i \leq n$, because the ideal $\langle x_1, \dots, x_i \rangle$ is prime. Indeed $K[x_1, \dots, x_n]/\langle x_1, \dots, x_i \rangle \simeq K[x_{i+1}, \dots, x_n]$, which is an integral domain. Therefore we get that $\dim \mathbb{A}^n \geq n$*. We will see shortly that proving equality is non trivial. We note also that, from every chain of irreducible closed subsets of $\mathbb{A}^n$, passing to their ideals, we get a chain of the same length of prime ideals in $K[x_1, \dots, x_n]$.*

(3) *Let X be irreducible. Then* $\dim X = 0$ *if and only if X is the closure of every point of it.*

We prove now some useful relations between the dimensions of X and of its subspaces.

Proposition 1.3. *1. If $Y \subset X$, then* $\dim Y \leq \dim X$*. In particular, if* $\dim X$ *is finite, then also* $\dim Y$ *is finite. In this case, the number* $\dim X - \dim Y$ *is called the* **codimension** *of Y in X.*

2. If $X = \bigcup_{i \in I} U_i$ is an open covering, then $\dim X = \sup_i\{\dim U_i\}$.

3. If X is noetherian and $X_1, \dots, X_s$ are its irreducible components, then $\dim X = \sup_i \dim X_i$.

4. If $Y \subset X$ is closed, X is irreducible, $\dim X$ *is finite and* $\dim X = \dim Y$, *then* $Y = X$.

Proof. 1. Let $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ be a chain of irreducible closed subsets of Y. Then taking closures we get the following chain of irreducible closed subsets of X: $\overline{Y_0} \subseteq \overline{Y_1} \subseteq \cdots \subseteq \overline{Y_n}$. Note that, for any index i, $\overline{Y_i} \cap Y = Y_i$, because Y_i is closed into Y, so if $\overline{Y_i} = \overline{Y_{i+1}}$, then $Y_i = Y_{i+1}$. Therefore the two chains have the same length and we can conclude that $\dim Y \leq \dim X$.

2. Let $X_0 \subset X_1 \subset \cdots \subset X_n$ be a chain of irreducible closed subsets of X. Let $P \in X_0$ be a point: there exists an index $i \in I$ such that $P \in U_i$. So $\forall k = 0, \dots, n$ $X_k \cap U_i \neq \emptyset$: it is an irreducible closed subset of U_i, irreducible because open in X_k which is irreducible. Consider

$$X_0 \cap U_i \subset X_1 \cap U_i \subset \cdots \subset X_n \cap U_i;$$

it is a chain of length n, because $\overline{X_k \cap U_i} = X_k$: in fact $X_k \cap U_i$ is open in X_k hence dense. Therefore, for any chain of irreducible closed subsets of X, there exists a chain of the same length of irreducible closed subsets of some U_i. So $\dim X \leq \sup \dim U_i$. By 1., equality holds.

3. Any chain of irreducible closed subsets of X is completely contained in an irreducible component of X. The conclusion follows as in 2.

4. If $Y_0 \subset Y_1 \subset \cdots \subset Y_n$ is a chain of maximal length in Y, then it is a maximal chain in X, because $\dim X = \dim Y$. Hence $X = Y_n \subset Y$. □

Corollary 1.4. $\dim \mathbb{P}^n = \dim \mathbb{A}^n$.

Proof. The equality follows from $\mathbb{P}^n = U_0 \cup \cdots \cup U_n$, and the homeomorphism of U_i with $\mathbb{A}^n$ for all i. □

If X is noetherian and all its irreducible components have the same dimension r, then X is said to have *pure dimension* r. Note that the topological dimension is invariant by homeomorphism. By definition, a *curve* is an algebraic set of pure dimension 1; a *surface* is an algebraic set of pure dimension 2.

We want to study the dimensions of affine algebraic sets. The following definition results to be very important.

Definition 1.5. Let $X \subset \mathbb{A}^n$ be an algebraic set. The *coordinate ring* of X is

$$K[X] := K[x_1, \dots, x_n]/I(X).$$

It is a finitely generated reduced K–algebra, i.e. there are no non–zero nilpotents, because $I(X)$ is radical.

There is the canonical epimorphism $K[x_1, \dots, x_n] \to K[X]$ such that $F \to [F]$. The elements of $K[X]$ can be interpreted as *polynomial functions* on X: to a polynomial F, we can associate the function $f : X \to K$ such that $P(a_1, \dots, a_n) \to F(a_1, \dots, a_n)$.

Two polynomials F, G define the same function on X if, and only if, $F(P) = G(P)$ for every point $P \in X$, i.e. if $F - G \in I(X)$, which means exactly that F and G have the same image in $K[X]$.

$K[X]$ is generated as K–algebra by $[x_1], \dots, [x_n]$: they can be interpreted as *coordinate functions* on X. We will denote them by $t_1, \dots, t_n$. In fact $t_i : X \to K$ is the function which associates to $P(a_1, \dots, a_n)$ the coordinate a_i. Note that the function f can be interpreted as $F(t_1, \dots, t_n)$: the polynomial F evalued at the n– tuple of the coordinate functions.

In the projective space we can do an analogous construction. If $Y \subset \mathbb{P}^n$ is closed, then the *homogeneous coordinate ring* of Y is

$$S(Y) := K[x_0, x_1, \dots, x_n]/I_h(Y).$$

Also $S(Y)$ is a finitely generated reduced K–algebra, but its elements cannot be interpreted as functions on Y. They are functions on the cone $C(Y)$.

We note that, from the fact that $I_h(Y)$ is homogeneous it follows that also $S(Y)$ is a graded ring, with the graduation induced by the polynomial ring. Indeed, if $F - G \in I_h(Y)$, and $F = F_0 + \dots + F_d$, $G = G_0 + \dots + G_e$ are their decompositions in homogeneous components then it follows that $F_0 - G_0 \in I_h(Y)$, $F_1 - G_1 \in I_h(Y)$, and so on.

Definition 1.6. Let R be a ring. The *Krull dimension* of R is the supremum of the lengths of the chains of prime ideals of R

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \cdots \subset \mathcal{P}_r.$$

Similarly, the *heigth* of a prime ideal $\mathcal{P}$ is the sup of the lengths of the chains of prime ideals contained in $\mathcal{P}$: it is denoted ht$\mathcal{P}$.

Proposition 1.7. *Let K be an algebraically closed field. Let X be an affine algebraic set contained in $\mathbb{A}^n$. Then* $\dim X = \dim K[X]$. *In particular* $\dim \mathbb{A}^n = \dim K[x_1, \dots, x_n]$.

Proof. By the Nullstellensatz and by Proposition 1.6 of Lesson 7 the chains of irreducible closed subsets of X correspond bijectively to the chains of prime ideals of $K[x_1, \dots, x_n]$ containing $I(X)$, and therefore also to the chains of prime ideals of the quotient ring $K[X] = K[x_1, \dots, x_n]/I(X)$. □

The dimension theory for commutative rings contains some important theorems about the dimension of K–algebras. The following theorem states the basic properties in the case of integral domains and the algebraic characterization of dimension for affine varieties.

Theorem 1.8. *Let K be any field. Let A be a finitely generated K–algebra and an integral domain.*

1. $\dim A = tr.d.Q(A)/K$*, where $Q(A)$ is the quotient field of A. In particular* $\dim A$ *is finite.*

2. Let $\mathcal{P} \subset A$ be any prime ideal. Then $\dim A = \mathrm{ht}\mathcal{P} + \dim A/\mathcal{P}$.

Proof. We postpone the proof to next lesson. It relies on the Normalization Lemma and on the Cohen-Seidenberg theorems about the structure of prime ideals for integral extensions of K-algebras. □

Corollary 1.9. *Let K be an algebraically closed field.*

1. $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.

2. If X is an affine variety, then $\dim X = tr.d.K(X)/K$*, where $K(X)$ denotes the quotient field of $K[X]$.*

2. If $X \subset \mathbb{A}^n$ is closed and irreducible, then $\dim X = n - \mathrm{ht}I(X)$.

Proof. 1. $\dim K[x_1, \dots, x_n] = tr.d.K(x_1, \dots, x_n)/K = n$.

2. follows immediately from Theorem 1.8, 1.

3. is Theorem 1.8, 2, applied to the case $A = K[x_1, \dots, x_n]$ and $\mathcal{P} = I(X)$. □

The following is the characterization of the algebraic varieties of codimension 1 in $\mathbb{A}^n$.

Proposition 1.10. *Let $X \subset \mathbb{A}^n$ be an affine variety over an algebraically closed field. Then X is a hypersurface if and only if X is of pure dimension $n-1$.*

Proof. Let $X \subset \mathbb{A}^n$ be a hypersurface, with $I(X) = (F) = (F_1 \dots F_s)$, where $F_1, \dots, F_s$ are the irreducible factors of F all of multiplicity one. Then $V(F_1)$,. . ., $V(F_s)$ are the irreducible components of X, whose ideals are (F_1), . . ., (F_s). So it is enough to prove that $\mathrm{ht}(F_i) = 1$, for $i = 1, \dots, s$.

If $\mathcal{P} \subset (F_i)$ is a prime ideal, then either $\mathcal{P} = (0)$ or there exists $G \in \mathcal{P}$, $G \neq 0$. In the second case, let A be an irreducible factor of G belonging to $\mathcal{P}$: $A \in (F_i)$ so $A = HF_i$. Since A is irreducible, either H or F_i is invertible; F_i is irreducible, so H is invertible, hence $(A) = (F_i) \subset \mathcal{P}$. Therefore either $\mathcal{P} = (0)$ or $\mathcal{P} = (F_i)$, and $\mathrm{ht}(F_i) = 1$.

Conversely, assume that X is irreducible of dimension $n - 1$. Since $X \neq \mathbb{A}^n$, there exists $F \in I(X)$, $F \neq 0$, with irreducible factorization $F = F_1 \dots F_s$. Hence $X \subset V(F) =$

$V(F_1) \cup \ldots \cup V(F_s)$. By the irreducibility of X, some irreducible factor of F, call it F_i, also vanishes along X. Therefore $X \subset V(F_i)$, which is irreducible of dimension $n-1$, by the first part. So $X = V(F_i)$ (by Proposition 1.3, 3). □

This proposition does not generalise to higher codimension. There exist codimension 2 algebraic subsets of $\mathbb{A}^n$ whose ideal is not generated by two polynomials. An example in $\mathbb{A}^3$ is the curve X parametrised by (t^3, t^4, t^5). It is possible to show that a system of generators of $I(X)$ is formed by the three polynomials $x^3 - yz, y^2 - xz, z^2 - x^2y$. One can easily show that $I(X)$ cannot be generated by two polynomials. For a proof and a discussion of this example, and more generally of the ideals of the curves admitting a parametrization of the form $x = t^{n_1}, y = t^{n_2}, z = t^{n_3}$, see [Kunz], Chapter V.

Proposition 1.11. *Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be irreducible closed subsets. Then* $\dim X \times Y = \dim X + \dim Y$.

Proof. Let $r = \dim X$, $s = \dim Y$; let $t_1, \ldots, t_n$ (resp. $u_1, \ldots, u_m$) be coordinate functions on $\mathbb{A}^n$ (resp. $\mathbb{A}^m$). We can assume that $t_1, \ldots, t_r$ is a transcendence basis of $Q(K[X])$ and $u_1, \ldots, u_s$ a transcendence basis of $Q(K[Y])$. By definition, $K[X \times Y]$ is generated as K–algebra by $t_1, \ldots, t_n, u_1, \ldots, u_m$: we want to show that $t_1, \ldots, t_r, u_1, \ldots, u_s$ is a transcendence basis of $Q(K[X \times Y])$ over K. Assume that $F(x_1, \ldots, x_r, y_1, \ldots, y_s)$ is a polynomial which vanishes on $t_1, \ldots, t_r, u_1, \ldots, u_s$, i.e. F defines the zero function on $X \times Y$. Then, $\forall\ P \in X$, $F(P; y_1, \ldots, y_s)$ is zero on Y, i.e. $F(P; u_1, \ldots, u_s) = 0$. Since $u_1, \ldots, u_s$ are algebraically independent, every coefficient $a_i(P)$ of $F(P; y_1, \ldots, y_s)$ is zero, $\forall\ P \in X$. Since $t_1, \ldots, t_r$ are algebraically independent, the polynomials $a_i(x_1, \ldots, x_r)$ are zero, so $F(x_1, \ldots, x_r, y_1, \ldots, y_s) = 0$. So $t_1, \ldots, t_r, u_1, \ldots, u_s$ are algebraically independent. Since this is certainly a maximal algebraically free set, it is a transcendence basis. □

Exercises 1.12. 1*. Prove that a proper closed subset of an irreducible curve is a finite set. Deduce that any bijection between irreducible curves is a homeomorphism.

2*. Let $X \subset \mathbb{A}^2$ be the cuspidal cubic of equation: $x^3 - y^2 = 0$, let $K[X]$ be its coordinate ring. Prove that all elements of $K[X]$ can be written in a unique way in the form $f(x) + yg(x)$, where f, g are polynomial in the variable x. Deduce that $K[X]$ is not isomorphic to a polynomial ring.