LESSON 3.

1. Examples of algebraic varieties.

1.1. **Points.** In the Zariski topology both in $\mathbb{A}^n$ and in $\mathbb{P}^n$ all points are closed.

If $P(a_1, \ldots, a_n) \in \mathbb{A}^n$, then $P = V(x_1 - a_1, \ldots, x_n - a_n)$.

But in the projective space, if $P[a_0, \ldots, a_n] \in \mathbb{P}^n$, the equations are different: $P = V_P(a_i x_j - a_j x_i)_{i,j=0,\ldots,n}$.

In this way the polynomials defining P as closed set are homogeneous. They can be seen as minors of order 2 of the matrix

$$\begin{pmatrix} a_0 & a_1 & \ldots & a_n \\ x_0 & x_1 & \ldots & x_n \end{pmatrix}$$

with entries in $K[x_0, x_1, \ldots, x_n]$. This expresses the fact that $x_0, \ldots, x_n$ are proportional to $a_0, \ldots, a_n$. Equations of the form $V_P(x_0 - a_0, \ldots, x_n - a_n)$ don't make sense.

1.2. **Affine and projective linear subspaces.** Generalizing the previous example, the linear subspaces, both in the affine and in the projective case, are examples of algebraic sets. As it is well known, they are defined by equations of degree 1.

1.3. **Hypersurfaces.** An affine hypersurface is an affine variety of the form $V(F)$, the set of zeroes of a unique polynomial F of **positive** degree. Similarly, in the projective space, a projective hypersurface is of the form $V_P(G)$, where $G \in K[x_0, x_1, \ldots, x_n]$ is a homogeneous non-constant polynomial.

Examples of hypersurfaces are the curves in the affine or projective plane, and the surfaces in a space of dimension 3, as for instance the quadrics.

Let us recall that the polynomial ring $K[x_1, \ldots, x_n]$ is a UFD (unique factorization domain), i. e. every non-constant polynomial F can be expressed in a unique way (up to the order and up to units) as $F = F_1^{r_1} F_2^{r_2} \ldots F_s^{r_s}$, where $F_1, \ldots, F_s$ are irreducible and two by two distinct polynomials, and $r_i \geq 1$ for any $i = 1, \ldots, s$. Hence the hypersurface of $\mathbb{A}^n$ defined by F is

$$X := V(F) = V(F_1^{r_1} F_2^{r_2} \ldots F_s^{r_s}) = V(F_1 F_2 \ldots F_s) = V(F_1) \cup V(F_2) \cup \cdots \cup V(F_s).$$

The equation $F_1 F_2 \ldots F_s = 0$ is called the reduced equation of X. Note that $F_1 F_2 \ldots F_s$ generates the radical $\sqrt{F}$. If $s = 1$, X is called an *irreducible* hypersurface; by definition its degree is the degree of its reduced equation. Therefore, any hypersurface is a finite union of irreducible hypersurfaces.

Assume now that $Z = V_P(G)$, with $G \in K[x_0, x_1, \dots, x_n]$, G homogeneous, is a projective hypersurface. Exercise 2 asks to prove that the irreducible factors of G are homogeneous. Therefore, as in the affine case, any projective hypersurface Z has a reduced equation (whose degree is, by definition, the degree of Z) and Z is a finite union of irreducible hypersurfaces.

If the field K is algebraically closed, the degree of a projective hypersurface has the following important geometrical meaning.

Proposition 1.1. *Let K be an algebraically closed field. Let $Z \subset \mathbb{P}^n$ be a projective hypersurface of degree d. Then any line in $\mathbb{P}^n$, not contained in Z, meets Z at exactly d points, counting multiplicities.*

In the proof we will see what we mean by saying "counting multiplicity".

Proof. Let G be the reduced equation of Z and $L \subset \mathbb{P}^n$ be any line.

We fix two points on L: $A = [a_0, \dots, a_n], B = [b_0, \dots, b_n]$. So L admits parametric equations of the form

$$\begin{cases} x_0 = \lambda a_0 + \mu b_0 \\ x_1 = \lambda a_1 + \mu b_1 \\ \dots \\ x_n = \lambda a_n + \mu b_n \end{cases}$$

The points of $Z \cap L$ are obtained from the homogeneous pairs $[\lambda, \mu]$ which are solutions of the equation $G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n) = 0$. If $L \subset Z$, then this equation is an identity. Otherwise, $G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n)$ is a non-zero homogeneous polynomial of degree d in the two variables λ, μ. Since K is algebraically closed, it can be factorized in linear factors:

$$G(\lambda a_0 + \mu b_0, \dots, \lambda a_n + \mu b_n) = (\mu_1 \lambda - \lambda_1 \mu)^{d_1} (\mu_2 \lambda - \lambda_2 \mu)^{d_2} \dots (\mu_r \lambda - \lambda_r \mu)^{d_r}$$

with $d_1 + d_2 + \dots + d_r = d$. Every factor corresponds to a point in $Z \cap L$, to be counted with the same multiplicity as the corresponding factor. $\square$

If K is not algebraically closed, considering the algebraic closure of K and using Proposition 1.1, we get that d is an upper bound on the number of points of $Z \cap L$.

1.4. **Product of affine spaces.** Let $\mathbb{A}^n$, $\mathbb{A}^m$ be two affine spaces over the field K. The cartesian product $\mathbb{A}^n \times \mathbb{A}^m$ is the set of pairs (P, Q), $P \in \mathbb{A}^n$, $Q \in \mathbb{A}^m$: it is in natural bijection with $\mathbb{A}^{n+m}$ via the map

$$\varphi : \mathbb{A}^n \times \mathbb{A}^m \longrightarrow \mathbb{A}^{n+m}$$

such that $\varphi((a_1,\dots,a_n),(b_1,\dots,b_m)) = (a_1,\dots,a_n,b_1,\dots,b_m)$.

From now on we will always identify $\mathbb{A}^n \times \mathbb{A}^m$ with $\mathbb{A}^{n+m}$. Therefore we have two topologies on $\mathbb{A}^n \times \mathbb{A}^m$: the Zariski topology and the product topology.

Proposition 1.2. *The Zariski topology is strictly finer than the product topology.*

Proof. Let us first observe that, if $X = V(\alpha) \subset \mathbb{A}^n$, $\alpha \subset K[x_1,\dots,x_n]$ and $Y = V(\beta) \subset \mathbb{A}^m$, $\beta \subset K[y_1,\dots,y_m]$, then $X \times Y \subset \mathbb{A}^n \times \mathbb{A}^m$ is Zariski closed, precisely $X \times Y = V(\alpha \cup \beta)$ where the union is made in the polynomial ring in $n+m$ variables $K[x_1,\dots,x_n,y_1,\dots,y_m]$. Now we consider $U = \mathbb{A}^n \setminus X$ and $V = \mathbb{A}^m \setminus Y$, open subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$ in the Zariski topology. Then $U \times V = \mathbb{A}^n \times \mathbb{A}^m \setminus ((\mathbb{A}^n \times Y) \cup (X \times \mathbb{A}^m))$: this is a set-theoretical fact that holds true in general. So it is open in $\mathbb{A}^n \times \mathbb{A}^m$ in the Zariski topology.

Conversely, we give an example to prove that the two topologies are different. Precisely we show that $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ contains some subsets which are Zariski open, but are not open in the product topology.

The proper open subsets in the product topology are of the form $\mathbb{A}^1 \times \mathbb{A}^1 \setminus \{$ finite unions of "vertical" and "horizontal" lines$\}$. See the figure.

Let $X = \mathbb{A}^2 \setminus V(x-y)$: it is Zariski open but does not contain any non-empty subset of the above form, so it is not open in the product topology. There are similar examples in $\mathbb{A}^n \times \mathbb{A}^m$ for any n, m. □

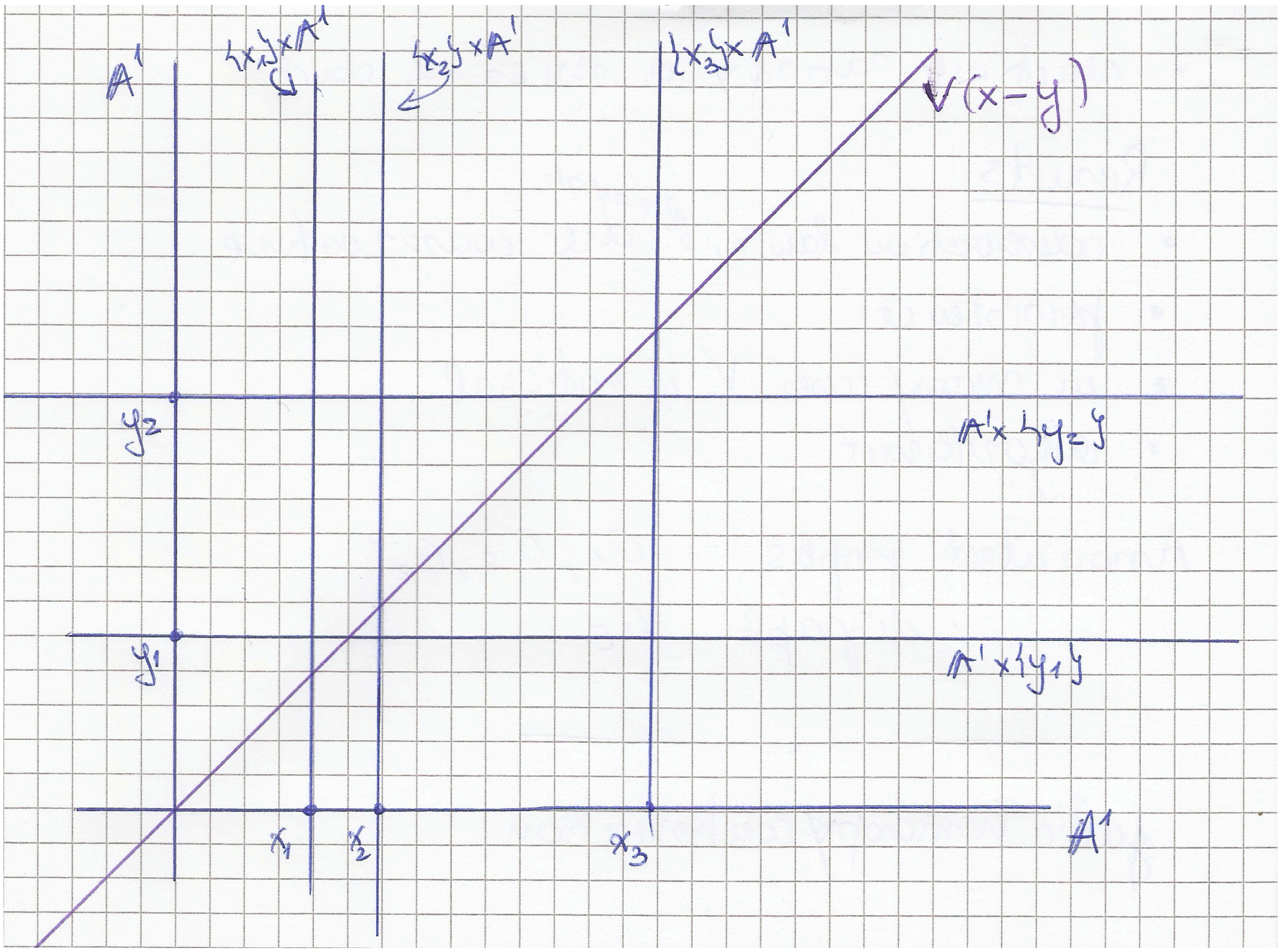
Note that there is no similar construction for $\mathbb{P}^n \times \mathbb{P}^m$. We will see later that there is an injective map of $\mathbb{P}^n \times \mathbb{P}^m$ to the projective space of dimension $(n+1)(m+1)-1$, whose image is a projective variety, the Segre map. This allows to give to the product of projective spaces a geometric structure. We see here only the first case.

1.5. $\mathbb{P}^1 \times \mathbb{P}^1$. The cartesian product $\mathbb{P}^1 \times \mathbb{P}^1$ is simply a set: we want to define an injective map σ from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^3$, so that the image is a projective variety, which will be identified with our product.

It is defined in the following way: $\sigma([x_0,x_1],[y_0,y_1]) = [x_0y_0, x_0y_1, x_1y_0, x_1y_1]$. Using coordinates $z_0,\dots,z_3$ in $\mathbb{P}^3$, σ is defined parametrically by

$$\begin{cases} z_0 = x_0y_0 \\ z_1 = x_0y_1 \\ z_2 = x_1y_0 \\ z_3 = x_1y_1 \end{cases}$$

It is easy to observe that σ is a well–defined map: the image is never $[0,0,0,0]$, and depends uniquely on the pair of points and not on the choice of their coordinates. Moreover


A¹
{x₁}×A¹
{x₂}×A¹
{x₃}×A¹
V(x−y)
y₂
A¹×{y₂}
y₁
A¹×{y₁}
x₁
x₂
x₃
A¹

σ is injective. Assume that $\sigma([x_0, x_1], [y_0, y_1]) = \sigma([x'_0, x'_1], [y'_0, y'_1])$. Then there exists a non-zero constant λ such that

$$\begin{cases} x_0y_0 = \lambda x'_0y'_0 \\ x_0y_1 = \lambda x'_0y'_1 \\ x_1y_0 = \lambda x'_1y'_0 \\ x_1y_1 = \lambda x'_1y'_1 \end{cases}$$

Now, if $y_0 \neq 0$, then $x_0 = (\lambda y'_0/y_0)x'_0$ and $x_1 = (\lambda y'_0/y_0)x'_1$; if $y_1 \neq 0$, then $x_0 = (\lambda y'_1/y_1)x'_0$ and $x_1 = (\lambda y'_1/y_1)x'_1$; in both cases $[x_0, x_1] = [x'_0, x'_1]$. Similarly one proves that $[y_0, y_1] = [y'_0, y'_1]$.

Let Σ denote the image $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$. It is the quadric $z_0z_3 - z_1z_2 = 0$. On one hand it is clear that $\sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset V_P(z_0z_3 - z_1z_2)$. Conversely, assume that $z_0z_3 = z_1z_2$ and $z_0 \neq 0$. Then, multiplying all coordinates by z_0 we get: $[z_0, z_1, z_2, z_3] = [z_0^2, z_0z_1, z_0z_2, z_0z_3]$; by assumption this coincides with $[z_0^2, z_0z_1, z_0z_2, z_1z_2]$, and is therefore equal to $\sigma([z_0, z_2], [z_0, z_1])$. If $z_0 = 0$, the argument is similar,using another non-zero coordinate.

The map σ is called the Segre map and Σ the **Segre variety**.

1.6. **Embedding of $\mathbb{A}^n$ in $\mathbb{P}^n$.** We will see now how to unify the two notions introduced so far of affine and projective variety. Precisely, after identifying $\mathbb{A}^n$ with the open subset $U_0 \subset \mathbb{P}^n$ (or with any U_i), we will prove that the Zariski topology on $\mathbb{A}^n$ coincides with the topology induced by the Zariski topology of $\mathbb{P}^n$.

Let H_i be the hyperplane of $\mathbb{P}^n$ of equation $x_i = 0$, $i = 0, \dots, n$; it is closed in the Zariski topology, and its complementar set U_i is open. So we have an open covering of $\mathbb{P}^n$: $\mathbb{P}^n = U_0 \cup U_1 \cup \dots \cup U_n$. Let us recall that for any i there is a bijection $\varphi_i : U_i \to \mathbb{A}^n$ such that $\varphi_i([x_0, \dots, x_i, \dots, x_n]) = (\frac{x_0}{x_i}, \dots, \hat{1}, \dots, \frac{x_n}{x_i})$. The inverse map is $j_i : \mathbb{A}^n \to U_i$ such that $j_i(y_1, \dots, y_n) = [y_1, \dots, 1, \dots, y_n]$.

Proposition 1.3. *The map φ_i is a homeomorphism, for $i = 0, \dots, n$.*

Proof. Assume $i = 0$ (the other cases are similar).

We introduce two maps:

(i) *dehomogeneization* of polynomials with respect to x_0.

It is a map $^a : K[x_0, x_1, \dots, x_n] \to K[y_1, \dots, y_n]$ such that

$$^a(F(x_0, \dots, x_n)) = {}^aF(y_1, \dots, y_n) := F(1, y_1, \dots, y_n).$$

Note that ^{a} is a ring homomorphism.

(ii) *homogeneization* of polynomials with respect to x_0.

It is a map ${}^h : K[y_1, \dots, y_n] \to K[x_0, x_1, \dots, x_n]$ defined by

$$^h(G(y_1, \dots, y_n)) = {}^hG(x_0, \dots, x_n) := x_0^{\deg G} G(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}).$$

${}^hG$ is always a homogeneous polynomial of the same degree as G. The map ^{h} is clearly not a ring homomorphism. Note that always ${}^a({}^hG) = G$ but in general ${}^h({}^aF) \neq F$; what we can say is that, if $F(x_0, \dots, x_n)$ is homogeneous, then there exists $r \geq 0$ such that $F = x_0^r({}^h({}^aF))$.

Let $X \subset U_0$ be closed in the topology induced by the Zariski topology of the projective space, i.e. $X = U_0 \cap V_P(I)$ where I is a homogeneous ideal of $K[x_0, x_1, \dots, x_n]$. Define ${}^aI = \{{}^aF \mid F \in I\}$: it is an ideal of $K[y_1, \dots, y_n]$ (because ^{a} is a ring homomorphism). We prove that $\varphi_0(X) = V({}^aI)$. Indeed, let $P[x_0, \dots, x_n]$ be a point of U_0; then $\varphi_0(P) = (\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) \in \varphi_0(X) \iff P[x_0, \dots, x_n] = [1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}] \in X = V_P(I) \iff F(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}) = 0 \ \forall \ {}^aF \in {}^aI \iff \varphi_0(P) \in V({}^aI)$.

Conversely: let $Y = V(\alpha)$ be a Zariski closed set of $\mathbb{A}^n$, where α ideal of $K[y_1, \dots, y_n]$. Let ${}^h\alpha$ be the homogeneous ideal of $K[x_0, x_1, \dots, x_n]$ generated by the set $\{{}^hG \mid G \in \alpha\}$. We prove that $\varphi_0^{-1}(Y) = V_P({}^h\alpha) \cap U_0$. Indeed $[1, x_0, \dots, x_n] \in \varphi_0^{-1}(Y) \iff (x_1, \dots, x_n) \in Y \iff G(x_1, \dots, x_n) = {}^hG(1, x_1, \dots, x_n) = 0 \ \forall \ G \in \alpha \iff [1, x_1, \dots, x_n] \in V_P({}^h\alpha)$. $\square$

From now on we will often identify $\mathbb{A}^n$ with U_0 via φ_0 (and similarly with U_i via φ_i). So if $P[x_0, \dots, x_n] \in U_0$, we will refer to $x_0, \dots, x_n$ as the homogeneous coordinates of P and to $\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}$ as the non–homogeneous or affine coordinates of P.

Exercises 1. It will useful to remember that any algebraically closed field is infinite.

(1) Assume that K is an algebraically closed field.

a) Prove that, if $n \geq 1$, then in $\mathbb{A}^n_K$ the complementar set of any hypersurface has infinitely many points.

b) Prove that, if $n \geq 2$, then also any hypersurface has infinitely many points.

(2) Prove that the Zariski topology on $\mathbb{A}^n$ is T_1.

(3) Let $F \in K[x_0, x_1, \dots, x_n]$ be a homogeneous polynomial. Check that its irreducible factors are homogeneous. (Hint: prove that a product of two polynomials not both homogeneous is not homogeneous.)

Solution of Exercise 1 (1).

Let the hypersurface in question be defined by $F(x_1, \dots, x_n) = 0$, F non constant. We can assume that the variable x_n occurs in F. So we have an expression

$$F = f_0 + f_1 x_n + \cdots + f_d x_n^d,$$

with $f_i \in K[x_1, \dots, x_{n-1}]$ $\forall i$, $d > 0$ and $f_d \neq 0$.

a) For this first part it is enough to assume that K is an infinite field. We proceed by induction on the number of variables. If $n = 1$, the statement is true because K is infinite. Let $n > 1$: by the inductive assumption, there exist infinitely many $(a_1, \dots, a_{n-1}) \in K^{n-1}$ such that $f_d(a_1, \dots, a_{n-1}) \neq 0$. Then for any such $(n-1)$-tuple $F(a_1, \dots, a_{n-1}, x_n)$ is a non zero polynomial of degree $d > 0$ in $K[x_n]$: it has finitely many zeros, so there are infinitely many $a_n \in K$ such that $F(a_1, \dots, a_{n-1}, a_n) \neq 0$.

b) As in a), there exist infinitely many $(a_1, \dots, a_{n-1}) \in K^{n-1}$ such that $f_d(a_1, \dots, a_{n-1}) \neq 0$. Since K is algebraically closed, for each of these $(a_1, \dots, a_{n-1})$ there is at least one $a_n \in K$ such that $F(a_1, \dots, a_{n-1}, a_n) = 0$.