Exercise. Given $v : \overline{\Omega} \to \mathbb{R}$ of class C^6, find a function $C(x,y)$, $(x,y) \in \overline{\Omega}$, and a nonnegative constant M such that

$$\max_{(x,y)\in\Omega_h} \left| \Delta_h v|_{\overline{\Omega}_h} (x,y) - \Delta v (x,y) - C(x,y)h^2 \right| \le Mh^4.$$

Just above we have shown the expansion

$$\Delta_h v|_{\overline{\Omega}_h} (x,y) = \Delta v (x,y) + E_2(x,y),\ (x,y) \in \Omega,$$

where

$$\max_{(x,y)\in\Omega_h} |E_2(x,y)| \le \frac{h^2}{6} \max \left\{ \max_{(x,y)\in\overline{\Omega}} \left| \frac{\partial^4 v}{\partial x^4} (x,y) \right|, \max_{(x,y)\in\overline{\Omega}} \left| \frac{\partial^4 v}{\partial y^4} (x,y) \right| \right\}.$$

By this exercise , we can expand one more term:

$$\Delta_h v|_{\overline{\Omega}_h} (x,y) = \Delta v (x,y) + C(x,y)h^2 + E_4(x,y), (x,y) \in \Omega,$$

where

$$\max_{(x,y)\in\Omega_h} |E_4(x,y)| \le Mh^4.$$

For the second central difference

$$\frac{v(t-h) - 2v(t) + v(t+h)}{h^2} \approx v''(t)$$

approximating $v''(t)$, where v is a function of the real variable t, we have the following expansion of the error

$$\frac{v(t-h) - 2v(t) + v(t+h)}{h^2} - v''(t) = \frac{h^2}{24} \left(v^{(4)}(\alpha_h) + v^{(4)}(\beta_h) \right),$$

where $\alpha_h \in (t-h,t)$ and $\beta_h \in (t,t+h)$..

Now, we need a more refined expansion

$$\frac{v(t-h) - 2v(t) + v(t+h)}{h^2} - v''(t) = C(t)\,h^2 + E(t)$$

where

$$E(t) = o\left(h^2\right) = \text{higher order term in power of } h.$$

So, we assume v of class C^6 and then, by using the Taylor expansions

$$v(t-h) = v(t) - hv'(t) + \frac{h^2}{2}v''(t) - \frac{h^3}{6}v'''(t) + \frac{h^4}{24}v^{(4)}(t) - \frac{h^5}{120}v^{(5)}(t) + \frac{h^6}{720}v^{(6)}(\alpha_h)$$
$$v(t+h) = v(t) + hv'(t) + \frac{h^2}{2}v''(t) + \frac{h^3}{6}v'''(t) + \frac{h^4}{24}v^{(4)}(t) + \frac{h^5}{120}v^{(5)}(t) + \frac{h^6}{720}v^{(6)}(\beta_h)$$

where $\alpha_h \in (t-h,t)$ and $\beta_h \in (t,t+h)$. Thus, we obtain

$$v(t-h)+v(t+h)=2v(t)+h^2v''(t)+\frac{h^4}{12}v^{(4)}(t)+\frac{h^6}{720}\left(v^{(6)}(\alpha_h)+v^{(6)}(\beta_h)\right).$$

and then

$$\frac{v(t-h)-2v(t)+v(t+h)}{h^2}-v''(t)=\frac{h^2}{12}v^{(4)}(t)+\frac{h^4}{720}\left(v^{(6)}(\alpha_h)+v^{(6)}(\beta_h)\right).$$

Now, we pass to consider the five-point discretization of the Laplacian.
Let $v:\overline{\Omega}\to\mathbb{R}$ be of class C^6. We have, for $(x,y)\in\Omega_h$,

$$\begin{aligned}
&\Delta_h v|_{\overline{\Omega}_h}(x,y)-\Delta v(x,y)\\
=\ &\frac{v(x-h,y)-2v(x,y)+v(x+h,y)}{h^2}+\frac{v(x,y-h)-2v(x,y)+v(x,y+h)}{h^2}\\
&-\left(\frac{\partial^2 v}{\partial x^2}(x,y)+\frac{\partial^2 v}{\partial y^2}(x,y)\right)\\
=\ &\frac{h^2}{12}\frac{\partial^4 v}{\partial x^4}(x,y)+\frac{h^4}{720}\left(\frac{\partial^6 v}{\partial x^6}(\alpha_h,y)+\frac{\partial^6 v}{\partial x^6}(\beta_h,y)\right)\\
&+\frac{h^2}{12}\frac{\partial^4 v}{\partial y^4}(x,y)+\frac{h^4}{720}\left(\frac{\partial^6 v}{\partial y^6}(x,\gamma_h)+\frac{\partial^6 v}{\partial y^6}(x,\delta_h)\right)
\end{aligned}$$

with $\alpha_h \in (x-h,x)$, $\beta_h \in (x,x+h)$, $\gamma_h \in (y-h,y)$ and $\delta_h \in (y,y+h)$. Hence, we can write

$$\Delta_h v|_{\overline{\Omega}_h}(x,y)-\Delta v(x,y)=C(x,y)\,h^2+E_4(x,y),$$

where

$$C(x,y)=\frac{1}{12}\left(\frac{\partial^4 v}{\partial x^4}(x,y)+\frac{\partial^4 v}{\partial y^4}(x,y)\right)$$

and

$$E_4(x,y)=\frac{h^4}{720}\left(\frac{\partial^6 v}{\partial x^6}(\alpha_h,y)+\frac{\partial^6 v}{\partial x^6}(\beta_h,y)+\frac{\partial^6 v}{\partial y^6}(x,\gamma_h)+\frac{\partial^6 v}{\partial y^6}(x,\delta_h)\right).$$

We have

$$\begin{aligned}
|E_4(x,y)| &= \frac{h^4}{720}\left|\frac{\partial^6 v}{\partial x^6}(\alpha_h,y)+\frac{\partial^6 v}{\partial x^6}(\beta_h,y)+\frac{\partial^6 v}{\partial y^6}(x,\gamma_h)+\frac{\partial^6 v}{\partial y^6}(x,\delta_h)\right|\\
&\le \frac{h^4}{720}\left(\left|\frac{\partial^6 v}{\partial x^6}(\alpha_h,y)\right|+\left|\frac{\partial^6 v}{\partial x^6}(\beta_h,y)\right|+\left|\frac{\partial^6 v}{\partial y^6}(x,\gamma_h)\right|+\left|\frac{\partial^6 v}{\partial y^6}(x,\delta_h)\right|\right)\\
&\le \frac{h^4}{180}\max\left\{\max_{(x,y)\in\overline{\Omega}}\left|\frac{\partial^6 v}{\partial x^6}(x,y)\right|,\max_{(x,y)\in\overline{\Omega}}\left|\frac{\partial^6 v}{\partial y^6}(x,y)\right|\right\}\\
&= Mh^4,\ \text{where } M=\frac{1}{180}\max\left\{\max_{(x,y)\in\overline{\Omega}}\left|\frac{\partial^6 v}{\partial x^6}(x,y)\right|,\max_{(x,y)\in\overline{\Omega}}\left|\frac{\partial^6 v}{\partial y^6}(x,y)\right|\right\}.
\end{aligned}$$

Exercise. Write the 1D BVP given by the Poisson equation with Dirichlet boundary condition on $\Omega = (0,1)$. Solve analytically this problem. Propose a corresponding discrete problem and write the associated linear system. Prove that the discrete problem has a unique solution. Compute the computational cost for solving the linear system by gaussian elimination.

The 1D Poisson equation with Dirichlet boundary condition on $\Omega = (0,1)$ is

$$\begin{cases} u''(x) = f(x), \ x \in (0,1), \\ u(0) = g(0) \text{ and } u(1) = g(1). \end{cases}$$

1) Solve analytically this problem.
We have

$$u'(x) = u'(0) + \int_0^x u''(s)\,ds, \ x \in [0,1],$$

and then

$$\begin{aligned} u(x) &= u(0) + \int_0^x u'(s)\,ds \\ &= u(0) + \int_0^x \left(u'(0) + \int_0^s u''(t)\,dt \right) ds \\ &= u(0) + \int_0^x u'(0)\,ds + \int_0^x \int_0^s u''(t)\,dtds \\ &= u(0) + xu'(0) + \int_0^x \int_0^s u''(t)\,dtds, \ x \in [0,1]. \end{aligned}$$

By using the Poisson equation and the boundary condition $u(0) = g(0)$, we obtain

$$u(x) = g(0) + xu'(0) + \int_0^x \int_0^s f(t)\,dtds, \ x \in [0,1].$$

The unknown $u'(0)$ is determined by the other boundary condition $u(1) = g(1)$:

$$g(1) = u(1) = g(0) + u'(0) + \int_0^1 \int_0^s f(t)\,dtds$$

gives

$$u'(0) = g(1) - g(0) - \int_0^1 \int_0^s f(t)\,dtds.$$

The solution is

$$
\begin{aligned}
u(x) &= g(0) + xu'(0) + \int_0^x \int_0^s f(t)\,dtds \\
&= g(0) + x\left(g(1) - g(0) - \int_0^1 \int_0^s f(t)\,dtds\right) + \int_0^x \int_0^s f(t)\,dtds \\
&= (1-x)\,g(0) + xg(1) + \int_0^x \int_0^s f(t)\,dtds - x\int_0^1 \int_0^s f(t)\,dtds \\
x &\in [0,1].
\end{aligned}
$$

2) Propose a corresponding discrete problem and write the associated linear system.

Let $h = \frac{1}{N}$, where N is a positive integer. We introduce the mesh

$$\mathbb{R}_h = \{mh : m \in \mathbb{Z}\}$$

and the discretization of Ω given by

$$\Omega_h = \Omega \cap \mathbb{R}_h = \{mh : m \in \{1, \ldots, N-1\}\}.$$

By using the second central difference for approximating the second derivative, we obtain the discrete problem

$$\begin{cases} \frac{u_h(x-h) - 2u_h(x) + u_h(x+h)}{h^2} = f(x), \ x \in \Omega_h, \\ u_h(0) = g(0) \ \text{ and } u_h(1) = g(1) \end{cases}$$

where $u_h : \overline{\Omega}_h = \Omega_h \cup \{0,1\} \to \mathbb{R}$ approximates the solution u of the BVP Poisson equation with Dirichlet boundary condition.

By introducing

$$
\begin{aligned}
&u_i = u(ih), \ i \in \{0, 1, \ldots, N-1, N\}, \\
&f_i = f(ih), \ i \in \{1, \ldots, N-1\}, \\
&g_i = g(ih), \ i \in \{0, N\},
\end{aligned}
$$

we have the linear system

$$u_{i-1} - 2u_i + u_{i+1} = h^2 f_i, \ i \in \{1, \ldots, N-1\},$$

given by $N-1$ equations into the $N-1$ unknowns

$$u_i = u(ih), \ i \in \{1, \ldots, N-1\}.$$

Observe that $u_0 = g_0$ and $u_N = g_N$ are known. We can write the linear system in the form

$$\mathcal{A}U = b$$

as

$$\begin{bmatrix} -2 & 1 & 0 & . & . & . & 0 \\ 1 & -2 & 1 & . & & & . \\ 0 & 1 & . & . & . & & . \\ . & . & . & . & . & . & . \\ . & & . & . & . & . & 0 \\ . & & & . & . & -2 & 1 \\ 0 & & . & . & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_{N-2} \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} h^2 f_1 - g_0 \\ h^2 f_2 \\ . \\ . \\ . \\ h^2 f_{N-2} \\ h^2 f_{N-1} - g_N \end{bmatrix},$$

where $\mathcal{A}$ has order $n = N - 1$.

3) Prove that the discrete problem has a unique solution.

The matrix $\mathcal{A}$ is symmetric. Now, we prove that it is also negative definite. For $v \in \mathbb{R}^{N-1}$, we have

$$\langle v, \mathcal{A}v \rangle = v^T \mathcal{A} v = v^T \begin{bmatrix} -2 & 1 & 0 & . & . & . & 0 \\ 1 & -2 & 1 & . & & & . \\ 0 & 1 & . & . & . & & . \\ . & . & . & . & . & . & . \\ . & & . & . & . & . & 0 \\ . & & & . & . & -2 & 1 \\ 0 & & . & . & 0 & 1 & -2 \end{bmatrix} v = \sum_{i,j=1}^{N-1} v_i a_{ij} v_j$$

$$\begin{aligned}
&= -2v_1^2 + v_1 v_2 + v_2 v_1 - 2v_2^2 + v_2 v_3 + v_3 v_2 - 2v_3^2 \\
&\quad + \cdots \\
&\quad -2v_{N-2}^2 + v_{N-2} v_{N-1} + v_{N-1} v_{N-2} - 2v_{N-1}^2 \\
&= -2v_1^2 - 2v_2^2 - 2v_3^2 + \cdots - 2v_{N-2}^2 - 2v_{N-1}^2 \\
&\quad + 2v_1 v_2 + 2v_2 v_3 + \cdots + 2v_{N-2} v_{N-1} \\
&= -v_1^2 - (v_1^2 - 2v_1 v_2 + v_2^2) - (v_2^2 - 2v_2 v_3 + v_3^2) + \cdots - (v_{N-2}^2 - 2v_{N-2} v_{N-1} + v_{N-1}^2) - v_{N-1}^2 \\
&= -v_1^2 - (v_1 - v_2)^2 - (v_2 - v_3)^2 + \cdots - (v_{N-2} - v_{N-1})^2 - v_{N-1}^2 \leq 0.
\end{aligned}$$

We have proved that the matrix $\mathcal{A}$ is negative semi-definite. Now, we show that $\mathcal{A}$ is negative definite. For $v = (v_1, v_2, \ldots, v_{N-1}) \in \mathbb{R}^{N-1}$ such that

$$\langle v, \mathcal{A}v \rangle = 0$$

we have

$$-v_1^2 - (v_1 - v_2)^2 + \ldots - (v_{N-2} - v_{N-1})^2 - v_{N-1}^2 = 0.$$

Then

$$v_1 = 0, \ v_1 - v_2 = 0, \ \ldots, \ v_{N-2} - v_{N-1} = 0, \ v_{N-1} = 0,$$

and so $v_1 = v_2 = \ldots = v_{N-1} = 0$. The matrix $\mathcal{A}$ is negative definite.

Since the matrix $\mathcal{A}$ is negative definite, it is nonsingular because all the eigenvalues of $\mathcal{A}$ are negative and then non-zero. We conclude that the linear system has a unique solution.

4) Compute the computational cost for solving the linear system by gaussian elimination.

The matrix $\mathcal{A}$ of the system is tridiagonal. It is a band matrix with bandwidth $w = 1$ and then the computational cost for solving the linear system is

$$\text{number of arithmetic operations } = O(w^2 n) = O(n) = O(N-1).$$

In this 1D situation, we can solve the linear system by gaussian elimination.

Exercise. Consider the 3D BVP given by the Poisson equation with Dirichlet boundary condition on $\Omega = (0,1)^3$. Propose a discrete Laplacian and a consequent discrete problem. Describe, in the matrix of the linear system associated to the discrete problem, the row corresponding to a mesh point in Ω_h with nearest neighbors in Ω_h. Compute the computational cost for solving the linear system by gaussian elimination.

The 3D Poisson equation with Dirichlet boundary condition on $\Omega = (0,1)^3$ is

$$\begin{cases} \Delta u\,(x,y,z) = f\,(x,y,z)\,, & (x,y,z) \in \Omega, \\ u\,(x,y,z) = g\,(x,y,z)\,, & (x,y,z) \in \Gamma, \end{cases}$$

where Γ is the boundary of Ω.

1) Propose a discrete Laplacian and a consequent discrete problem.

Let N be a positive integer and let $h = \frac{1}{N}$. The space $\mathbb{R}^3$ is discretized by the mesh

$$\mathbb{R}^3_h := \{(mh, nh, ph) : m, n, p \in \mathbb{Z}\}\,.$$

Each mesh point $(mh, nh, ph) \in \mathbb{R}^3_h$ has six nearest neighbors in the mesh:

- $((m-1)h, nh, ph)$ left,
- $((m+1)h, nh, ph)$ right,
- $(mh, (n-1)h, ph)$ back,
- $(mh, (n+1)h, ph)$ front,
- $(mh, h, (p-1)\,h)$ down,
- $(mh, h, (p+1)\,h)$ up.

Now, we introduce

- $\Omega_h = \Omega \cap \mathbb{R}^3_h$ as a discretization of Ω.
- Γ_h, the set of the mesh points not in Ω_h but with a nearest neighbor in Ω_h, as a discretization of Γ.

- $\overline{\Omega}_h = \Omega_h \cup \Gamma_h$ as a discretization of $\overline{\Omega}$.

The discrete Laplacian is a seven-point discretization of the Laplacian: for a mesh function $v_h : \overline{\Omega}_h \to \mathbb{R}$, we have

$$\begin{aligned}
&\Delta_h v_h (x,y,z) \\
&= \frac{v_h (x-h,y,z) - 2v_h (x,y,z) + v_h (x+h,y,z)}{h^2} \\
&+ \frac{v_h (x,y-h,z) - 2v_h (x,y,z) + v_h (x,y+h,z)}{h^2} \\
&+ \frac{v_h (x,y,z-h) - 2v_h (x,y,z) + v_h (x,y,z+h)}{h^2} \\
&= \frac{1}{h^2} \\
&\quad \cdot (v_h (x,y,z-h) + v_h (x,y-h,z) + v_h (x-h,y,z) \\
&\quad\quad -6v_h (x,y,z) \\
&\quad\quad +v_h (x+h,y,z) + v_h (x,y+h,z) + v_h (x,y,z+h)) \\
&(x,y,z) \in \Omega_h.
\end{aligned}$$

The discrete problem is

$$\begin{cases} \Delta_h u_h (x,y,z) = f(x,y,z), \ (x,y,z) \in \Omega_h, \\ u_h (x,y,z) = g(x,y,z), \ (x,y,z) \in \Gamma_h, \end{cases}$$

where $u_h : \overline{\Omega}_h \to \mathbb{R}$ approximates the solution u of the BVP Poission equation with Dirichlet boundary condition.

2) Describe, in the matrix of the linear system associated to the discrete problem, the row corresponding to a mesh point in Ω_h with nearest neighbors in Ω_h.

By setting

$$\begin{aligned}
u_{ijk} &:= u_h (ih, jh, kh), \ (ih, jh, kh) \in \overline{\Omega}_h, \\
f_{ijk} &:= f(ih, jh, kh), \ (ih, jh, kh) \in \Omega_h, \\
g_{ijk} &:= g(ih, jh, kh), \ (ih, jh, kh) \in \Gamma_h,
\end{aligned}$$

the discrete Poisson problem becomes

$$\begin{cases} u_{ijk-1} + u_{ij-1k} + u_{i-1jk} - 6u_{ijk} + u_{i+1jk} + u_{ij+1k} + u_{ijk+1} = h^2 f_{ijk}, \ (i,j,k) \in \{1, \ldots, N-1\}^3, \\ u_{ijk} = g_{ijk}, \ (ih, jh, kh) \in \Gamma_h. \end{cases}$$

This is the system of the $(N-1)^3$ scalar linear equations

$$u_{ijk-1} + u_{ij-1k} + u_{i-1jk} - 6u_{ijk} + u_{i+1jk} + u_{ij+1k} + u_{ijk+1} = h^2 f_{ijk}, \ (i,j,k) \in \{1, \ldots, N-1\}^3,$$

into the $(N-1)^3$ scalar unknowns

$$u_{ijk}, \ (i,j,k) \in \{1,\ldots,N-1\}^3 .$$

We write the linear system in the form

$$\mathcal{A}U = b,$$

where

- $\mathcal{A}$ is the square matrix of the system of order $n = (N-1)^3$.
- U is the column vector of the unknowns of dimension n.
- b is the column vector of the known terms of dimension n.

The $(N-1)^3$ unknowns, which are arranged in a cube, need to be ordered in a column vector. We cut Ω_h by slices

$$\Omega_{h,k} = \left\{(ih, jh, kh) : (i,j) \in \{1,\ldots,N-1\}^2\right\}, \ k \in \{1,\ldots,N-1\},$$

where each slice is like Ω_h in the 2D case. The slices are ordered as $\Omega_{h,1},\ldots,\Omega_{h,N-1}$ from the bottom to the top. Inside each slice, the unknowns are ordered as in the 2D case. In this linear order of the unknowns, for an index $l \in \{1,\ldots,n\}$ corresponding to a mesh point in Ω_h with nearest neighbors in Ω_h, we have the equation

$$U_{l-(N-1)^2} + U_{l-(N-1)} + U_{l-1} - 6U_l + U_{l+1} + U_{l+N-1} + U_{l+(N-1)^2} = h^2 f_l.$$

3) Compute the computational cost for solving the linear system by gaussian elimination.

The matrix $\mathcal{A}$ has seven non-zero diagonals: the principal diagonal, the first lower and upper diagonals, the $(N-1)$-th lower and upper diagonals and $(N-1)^2$-th lower and upper diagonals. So, $\mathcal{A}$ is a band matrix with bandwidth $w = (N-1)^2$ and then the computational cost for solving the linear system is

$$\text{number of arithmetic operations } = O(w^2 n) = O((N-1)^4 (N-1)^3) = O((N-1)^7).$$

We cannot solve the linear system by gaussian elimination.