THE GAUSSIAN DISTRIBUTION

d-dimensional G.D $\mathcal{N}(x|\mu, \Sigma)$

μ = d-dim vector = MEAN

Σ : d×d matrix = COVARIANCE MATRIX
DEFINITE POSITIVE

$$\mathcal{N}(x|\mu,\Sigma) = \left((2\pi)^d \det(\Sigma)\right)^{-1/2} \cdot \exp\left(-\frac{1}{2}(x-\mu)^T \underbrace{\Sigma^{-1}}_{} (x-\mu)\right)$$

$\Sigma^{-1} = A$ <u>PRECISION</u>

$(x-\mu)^T \Sigma^{-1} (x-\mu)$ MAHALANOBIS distance

($\Sigma = I \rightsquigarrow$ EUCLIDEAN DISTANCE)

$$\Sigma = E \Lambda E^T, \quad \underline{\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_d)}, \quad E^T E = I.$$

$$y = \Lambda^{-\frac{1}{2}} E^T (x-\mu) \Rightarrow \boxed{y \sim \mathcal{N}(y|0, I)}$$

$$(x-\mu)^T \Sigma^{-1} (x-\mu) = \overbrace{(x-\mu)^T E \Lambda^{-1/2}}^{y^T} \overbrace{\Lambda^{-1/2} E^T (x-\mu)}^{y} = y^T y$$

COMPLETING THE SQUARE

$$p(x) = c \cdot \exp\left(-\frac{1}{2}x^T A x + b^T x\right)$$ is a Gaussian distribution

$$\frac{1}{2}x^T A x - b^T x = \frac{1}{2}(x - A^{-1}b)^T A (x - A^{-1}b) - \overbrace{\frac{1}{2} b^T A^{-1} b}^{\text{const.}}$$

prove it

$$c \cdot \exp\left(-\frac{1}{2}x^T A x + b^T x\right) = \mathcal{N}(x \mid A^{-1}b, A^{-1}) \underbrace{\sqrt{(2\pi)^d \det(A^{-1})} \cdot \exp\left(\frac{1}{2} b^T A^{-1} b\right) \cdot c}_{1}$$

$$p(x) \text{ is } \mathcal{N}(x \mid A^{-1}b, A^{-1})$$

$$\log p(x) \propto \boxed{-\frac{1}{2}x^T A x + b^T x}$$

- LINEAR TRANSFORMATION

$$y = M \cdot x + \eta \qquad x \sim \mathcal{N}(\mu_x, \Sigma_x) \quad \eta \sim \mathcal{N}(\mu, \Sigma) \quad x \perp\!\!\!\perp \eta$$

$$y \sim \mathcal{N}\left(M\mu_x + \mu,\ M \Sigma_x M^T + \Sigma\right)$$

- MARGINALS and CONDITIONALS

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \qquad z \sim \mathcal{N}(\mu, \Sigma) \quad \mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \quad \Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix}$$

$$\Sigma_{yx} = \Sigma_{xy}^T$$

$$x \sim \mathcal{N}(\mu_x, \Sigma_{xx})$$

$$p(x|y) = \mathcal{N}\left(x \mid \mu_x + \Sigma_{xy} \Sigma_{yy}^{-1} (y - \mu_y),\ \Sigma_{xx} - \Sigma_{xy} \Sigma_{yy}^{-1} \Sigma_{yx}\right)$$

PRODUCT.

$$\mathcal{N}(x \mid \mu_1, \Sigma_1)\,\mathcal{N}(x \mid \mu_2, \Sigma_2) = \mathcal{N}(x \mid \mu, \Sigma) \cdot K$$

$$\begin{cases} S = \Sigma_1 + \Sigma_2 \\ \mu = \Sigma_1 S^{-1} \mu_2 + \Sigma_2 S^{-1} \mu_1 \quad , \quad \Sigma = \Sigma_1 S^{-1} \Sigma_2 \end{cases}$$

$$K = \frac{\exp\left(-\frac{1}{2}(\mu_1 - \mu_2)^T S^{-1} (\mu_1 - \mu_2)\right)}{\sqrt{\det(2\pi S)}}$$

BAYESIAN THEOREM

$$p(x) = \mathcal{N}(x, \mu, \Lambda^{-1}) \qquad p(y|x) = \mathcal{N}(y | Ax+b, L^{-1})$$

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \ln p(z) = \ln p(x) + \ln p(y|x) = \text{const} +$$

$$-\frac{1}{2}(x-\mu)^T \Lambda (x-\mu) - \frac{1}{2}(y-Ax-b)^T L (y-Ax-b)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ A\mu+b \end{pmatrix}, R^{-1}\right) \qquad R = \begin{pmatrix} \Lambda + A^T L A & -A^T L \\ -LA & L \end{pmatrix}$$

$$R^{-1} = \begin{pmatrix} \Lambda^{-1} & \Lambda^{-1} A^T \\ A\Lambda^{-1} & L^{-1} + A\Lambda^{-1}A^T \end{pmatrix}$$

$$p(x|y) = \frac{p(y|x)p(x)}{p(y)}$$

$$p(y) = \mathcal{N}(y | A\mu+b, L^{-1} + A\Lambda^{-1}A^T)$$

$$p(x|y) = \mathcal{N}(x | m, (\Lambda + A^T L A)^{-1}) \qquad m = (\Lambda + A^T L A)^{-1} [A^T L (y-b) + \Lambda\mu]$$