LESSON 10.

1. Regular and rational functions.

1.1. **Regular functions.** In this lesson, we will define the regular functions on algebraic varieties, not only on closed subsets of affine or projective space, but more in general on locally closed subsets. This will allow to associate to any algebraic variety an algebraic invariant, the ring of regular functions. An analogous construction will be given also for a more general class of functions, rational functions, that will bring to a second invariant, the field of rational functions.

Let $X \subset \mathbb{P}^n$ be a locally closed subset and P be a point of X. Let $\varphi : X \to K$ be a function.

Definition 1.1. φ is *regular at* P if there exists a suitable neighbourhood of P in which φ can be expressed as a quotient of homogeneous polynomials of the same degree; more precisely, if there exist an open neighbourhood U of P in X and homogeneous polynomials F, $G \in K[x_0, x_1, \dots, x_n]$ with $\deg F = \deg G$, such that $U \cap V_P(G) = \emptyset$ and $\varphi(Q) = F(Q)/G(Q)$, for all $Q \in U$. Note that the quotient $F(Q)/G(Q)$ is well defined.

φ is *regular on* X if φ is regular at every point P of X.

The set of regular functions on X is denoted by $\mathcal{O}(X)$: it contains K (identified with the set of constant functions), and can be given the structure of a K–algebra, by the definitions:

$$(\varphi + \psi)(P) = \varphi(P) + \psi(P)$$

$$(\varphi\psi)(P) = \varphi(P)\psi(P),$$

for $P \in X$. (Check that $\varphi + \psi$ and $\varphi\psi$ are indeed regular on X.)

Proposition 1.2. *Let $\varphi : X \to K$ be a regular function. Let K be identified with $\mathbb{A}^1$ with Zariski topology. Then φ is continuous.*

Proof. It is enough to prove that $\varphi^{-1}(c)$ is closed in X, $\forall\ c \in K$. For all $P \in X$, choose an open neighbourhood U_P and homogeneous polynomials F_P, G_P such that $\varphi|_P = F_P/G_P$. Then

$$\varphi^{-1}(c) \cap U_P = \{Q \in U_P | F_P(Q) - cG_P(Q) = 0\} = U_P \cap V_P(F_P - cG_P)$$

is closed in U_P. The proposition then follows from:

Lemma 1.3. *Let T be a topological space, $T = \cup_{i\in I} U_i$ be an open covering of T, $Z \subset T$ be a subset. Then Z is closed if and only if $Z \cap U_i$ is closed in U_i for all i.*

Proof. Assume that $U_i = X \setminus C_i$ and $Z \cap U_i = Z_i \cap U_i$, with C_i and Z_i closed in X.

Claim: $Z = \bigcap_{i\in I}(Z_i \cup C_i)$, hence it is closed.

In fact: if $P \in Z$, then $P \in Z \cap U_i$ for a suitable i. Therefore $P \in Z_i \cap U_i$, so $P \in Z_i \cup C_i$. If $P \notin Z_j \cap U_j$ for some j, then $P \notin U_j$ so $P \in C_j$ and therefore $P \in Z_j \cup C_j$.

Conversely, if $P \in \bigcap_{i\in I}(Z_i \cup C_i)$, then $\forall$ i, either $P \in Z_i$ or $P \in C_i$. Since $\exists j$ such that $P \in U_j$, hence $P \notin C_j$, so $P \in Z_j$, so $P \in Z_j \cap U_j = Z \cap U_j$. □

□

Corollary 1.4. *1. Let $\varphi \in \mathcal{O}(X)$: then $\varphi^{-1}(0)$ is closed. It is denoted $V(\varphi)$ and called the set of zeroes of φ.*

2. Let X be a quasi–projective variety and φ, $\psi \in \mathcal{O}(X)$. Assume that there exists U, open non –empty subset such that $\varphi|_U = \psi|_U$. Then $\varphi = \psi$.

Proof. $\varphi - \psi \in \mathcal{O}(X)$ so $V(\varphi - \psi)$ is closed. By assumption $V(\varphi - \psi) \supset U$, which is dense, because X is irreducible. So $V(\varphi - \psi) = X$.

□

If $X \subset \mathbb{A}^n$ is locally closed, we can use on X both homogeneous and non–homogeneous coordinates. In the second case, a regular function is locally represented as a quotient F/G, with F and $G \in K[x_1, \dots, x_n]$. In particular all polynomial functions are regular, so, if X is closed, $K[X] \subset \mathcal{O}(X)$.

If $\alpha \subset K[X]$ is an ideal, we can consider $V(\alpha) := \bigcap_{\varphi\in\alpha} V(\varphi)$: it is closed into X. Note that α is of the form $\alpha = \overline{\alpha}/I(X)$, where $\overline{\alpha}$ is the inverse image of α in the canonical epimorphism, it is an ideal of $K[x_1, \dots, x_n]$ containing $I(X)$, hence $V(\alpha) = V(\overline{\alpha}) \cap X = V(\overline{\alpha})$.

If K is algebraically closed, from the Nullstellensatz it follows that, if α is proper, then $V(\alpha) \neq \emptyset$. Moreover the following relative form of the Nullstellensatz holds: if $f \in K[X]$ and f vanishes at all points $P \in X$ such that $g_1(P) = \cdots = g_m(P) = 0$ ($g_1, \dots, g_m \in K[X]$), then $f^r \in \langle g_1, \dots, g_m\rangle \subset K[X]$, for some $r \geq 1$.

Theorem 1.5. *Let K be an algebraically closed field. Let $X \subset \mathbb{A}^n_K$ be closed in the Zariski topology. Then $\mathcal{O}(X) \simeq K[X]$. It is an integral domain if and only if X is irreducible.*

Proof. Let $f \in \mathcal{O}(X)$.

(i) Assume first that X is irreducible. For all $P \in X$ fix an open neighbourhood U_P of P and polynomials F_P, G_P such that $V_P(G_P) \cap U_P = \emptyset$ and $f|_{U_P} = F_P/G_P$. Let f_P, g_P be the functions in $K[X]$ defined by F_P and G_P. Then $g_P f = f_P$ holds on U_P, so it holds on X (by Corollary 1.4 (2), because X is irreducible). Let $\alpha \subset K[X]$ be the ideal $\alpha = \langle g_P\rangle_{P\in X}$;

α has no zeros on X, because $g_P(P) \neq 0$, so $\alpha = K[X]$. Therefore there exists $h_P \in K[X]$ such that $1 = \sum_{P \in X} h_P g_P$ (sum with finite support). Hence in $\mathcal{O}(X)$ we have the relation: $f = f \sum h_P g_P = \sum h_P(g_P f) = \sum h_P f_P \in K[X]$.

(ii) Let X be reducible: for any $P \in X$, there exists $R \in K[x_1, \ldots, x_n]$ such that $R(P) \neq 0$ and $R \in I(X \setminus U_P)$, so $r \in \mathcal{O}(X)$ is zero outside U_P. So $r g_P f = f_P r$ on X and we conclude as above by replacing g_P with $g_P r$ and f_P with $f_P r$.

□

The characterization of regular functions on projective varieties is completely different: we will see later that, if X is a projective variety, then $\mathcal{O}(X) \simeq K$, i.e. the unique regular functions are constant.

This gives the motivation for introducing the following weaker concept.

1.2. **Rational functions.**

Definition 1.6. Let X be a quasi–projective variety. A *rational function* on X is a germ of regular functions on some open non–empty subset of X.

Precisely, let $\mathcal{K}$ be the set $\{(U, f) | U \neq \emptyset, \text{open subset of } X, f \in \mathcal{O}(U)\}$. The following relation on $\mathcal{K}$ is an equivalence relation:

$$(U, f) \sim (U', f') \text{ if and only if } f|_{U \cap U'} = f'|_{U \cap U'}.$$

Reflexive and symmetric properties are quite obvious. Transitive property: let $(U, f) \sim (U', f')$ and $(U', f') \sim (U'', f'')$. Then $f|_{U \cap U'} = f'|_{U \cap U'}$ and $f'|_{U' \cap U''} = f''|_{U' \cap U''}$, hence $f|_{U \cap U' \cap U''} = f''|_{U \cap U' \cap U''}$. $U \cap U' \cap U''$ is a non–empty open subset of $U \cap U''$ (which is irreducible and quasi–projective), so by Corollary 1.4 $f|_{U' \cap U''} = f''|_{U' \cap U''}$.

Let $K(X) := \mathcal{K}/\sim$: its elements are by definition rational functions on X. $K(X)$ can be given the structure of a field in the following natural way.

Let $\langle U, f \rangle$ denote the class of (U, f) in $K(X)$. We define:

$$\langle U, f \rangle + \langle U', f' \rangle = \langle U \cap U', f + f' \rangle,$$
$$\langle U, f \rangle \langle U', f' \rangle = \langle U \cap U', f f' \rangle$$

(check that the definitions are well posed!).

There is a natural inclusion: $K \to K(X)$ such that $c \to \langle X, c \rangle$. Moreover, if $\langle U, f \rangle \neq 0$, then there exists $\langle U, f \rangle^{-1} = \langle U \setminus V(f), f^{-1} \rangle$: the axioms of a field are all satisfied.

There is also an injective map: $\mathcal{O}(X) \to K(X)$ such that $\varphi \to \langle X, \varphi \rangle$.

Proposition 1.7. *If $X \subset \mathbb{A}^n$ is affine, then $K(X) \simeq Q(\mathcal{O}(X)) = K(t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are the coordinate functions on X.*

Proof. The isomorphism is as follows:

(i) $\psi : K(X) \to Q(\mathcal{O}(X))$

If $\langle U, \varphi\rangle \in K(X)$, then there exists $V \subset U$, open and non–empty, such that $\varphi\mid_V = F/G$, where $F, G \in K[x_1, \dots, x_n]$ and $V(G) \cap V = \emptyset$. We set $\psi(\langle U, \varphi\rangle) = f/g$.

(ii) $\psi' : Q(\mathcal{O}(X)) \to K(X)$

If $f/g \in Q(\mathcal{O}(X))$, we set $\psi'(f/g) = \langle X \setminus V(g), f/g\rangle$.

It is easy to check that ψ and ψ' are well defined and inverse each other. □

Corollary 1.8. *If X is an affine variety, then $\dim X$ is equal to the transcendence degree over K of its field of rational functions.*

Proposition 1.9. *If X is quasi–projective and $U \neq \emptyset$ is an open subset, then $K(X) \simeq K(U)$.*

Proof. We have the maps: $K(U) \to K(X)$ such that $\langle V, \varphi\rangle \to \langle V, \varphi\rangle$, and $K(X) \to K(U)$ such that $\langle A, \psi\rangle \to \langle A \cap U, \psi\mid_{A\cap U}\rangle$: they are K–homomorphisms inverse each other. □

Corollary 1.10. *If X is a projective variety contained in $\mathbb{P}^n$, if i is an index such that $X \cap U_i \neq \emptyset$ (where U_i is the open subset where $x_i \neq 0$), then $\dim X = \dim X \cap U_i = tr.d.K(X)/K$.*

Proof. By Proposition 1.3, Lesson 8, $\dim X = \sup \dim(X \cap U_i)$. By Corollary 1.8 and Proposition 1.9, if $X \cap U_i$ is non–empty, $\dim(X \cap U_i) = tr.d.K(X \cap U_i)/K = tr.d.K(X)/K$ is independent of i. □

If $\langle U, \varphi\rangle \in K(X)$, we can consider all possible representatives of it, i.e. all pairs $\langle U_i, \varphi_i\rangle$ such that $\langle U, \varphi\rangle = \langle U_i, \varphi_i\rangle$. Then $\overline{U} = \bigcup_i U_i$ is the maximum open subset of X on which φ can be seen as a function: it is called the *domain of definition* (or of regularity) of $\langle U, \varphi\rangle$, or simply of φ. It is sometimes denoted domφ. If $P \in \overline{U}$, we say that φ *is regular at* P.

We can consider the set of rational functions on X which are regular at P: it is denoted by $\mathcal{O}_{P,X}$. It is a subring of $K(X)$ containing $\mathcal{O}(X)$, called the *local ring of X at P*. In fact, $\mathcal{O}_{P,X}$ is a local ring, whose maximal ideal, denoted $\mathcal{M}_{P,X}$, is the set of rational functions φ such that $\varphi(P)$ is defined and $\varphi(P) = 0$. To see this, observe that an element of $\mathcal{O}_{P,X}$ can be represented as $\langle U, F/G\rangle$: its inverse in $K(X)$ is $\langle U \setminus V_P(G), G/F\rangle$, which belongs to $\mathcal{O}_{P,X}$ if and only if $F(P) \neq 0$. We will see in §1.3 that $\mathcal{O}_{P,X}$ is the localization $K[X]_{I_X(P)}$.

As in Proposition 1.9 for the fields of rational functions, also for the local rings of points it can easily be proved that, if $U \neq \emptyset$ is an open subset of X containing P, then $\mathcal{O}_{P,X} \simeq \mathcal{O}_{P,U}$. So the ring $\mathcal{O}_{P,X}$ only depends on the local behaviour of X in the neighbourhood of P.

The *residue field* of $\mathcal{O}_{P,X}$ is the quotient $\mathcal{O}_{P,X}/\mathcal{M}_{P,X}$: it is a field which results to be naturally isomorphic to the base field K. In fact consider the evaluation map $\mathcal{O}_{P,X} \to K$ such that φ goes to $\varphi(P)$: it is surjective with kernel $\mathcal{M}_{P,X}$, so $\mathcal{O}_{P,X}/\mathcal{M}_{P,X} \simeq K$.

Example 1.11.

1. Let $Y \subset \mathbb{A}^2$ be the curve $V(x_1^3 - x_2^2)$. Then $F = x_2$, $G = x_1$ define the function $\varphi = x_2/x_1$ which is regular at the points $P(a_1, a_2)$ such that $a_1 \neq 0$. Another representation of the same function is: $\varphi = x_1^2/x_2$, which shows that φ is regular at P if $a_2 \neq 0$. If φ admits another representation F'/G', then $G'x_2 - F'x_1$ vanishes on an open subset of X, which is irreducible (see Exercise 2, Lesson 8), hence $G'x_2 - F'x_1$ vanishes on X, and therefore $G'x_2 - F'x_1 \in \langle x_1^3 - x_2^2 \rangle$. This shows that there are essentially only the above two representations of φ. So $\varphi \in K(X)$ and its domain of regularity is $Y \setminus \{0, 0\}$.

2. The stereographic projection.
Let $X \subset \mathbb{P}^2$ be the curve $V_P(x_1^2+x_2^2-x_0^2)$. Let $f := x_1/(x_0-x_2)$ denote the germ of the regular function defined by $x_1/(x_0-x_2)$ on $X \setminus V_P(x_0-x_2) = X \setminus \{[1, 0, 1]\} = X \setminus \{P\}$. On X we have $x_1^2 = (x_0 - x_2)(x_0 + x_2)$ so f is represented also as $(x_0 + x_2)/x_1$ on $X \setminus V_P(x_1) = X \setminus \{P, Q\}$, where $Q = [1, 0, -1]$. If we identify K with the affine line $V_P(x_2) \setminus V_P(x_0)$ (the points of the x_1–axis lying in the affine plane U_0), then f can be interpreted as the stereographic projection of X centered at P, which takes $A[a_0, a_1, a_2]$ to the intersection of the line AP with the line $V_P(x_2)$. To see this, observe that AP has equation $a_1x_0+(a_2-a_0)x_1-a_1x_2 = 0$; and $AP \cap V_P(x_2)$ is the point $[a_0 - a_2, a_1, 0]$.

1.3. **The algebraic characterization of the local ring $\mathcal{O}_{P,X}$.** Let us recall the construction of the *ring of fractions of a ring* A with respect to a multiplicative subset S.

Let A be a ring and $S \subset A$ be a multiplicative subset. The following relation in $A \times S$ is an equivalence relation:

$$(a, s) \simeq (b, t) \text{ if and only if } \exists u \in S \text{ such that } u(at - bs) = 0.$$

Then the quotient $A \times S/_{\simeq}$ is denoted $S^{-1}A$ or A_S and $[(a, s)]$ is denoted $\frac{a}{s}$. A_S becomes a commutative ring with unit with operations $\frac{a}{s} + \frac{b}{t} = \frac{at+bs}{st}$ and $\frac{a}{s}\frac{b}{t} = \frac{ab}{st}$ (check that they are well–defined). With these operations, A_S is called the ring of fractions of A with respect to S, or the *localization* of A in S.

There is a natural homomorphism $j : A \to S^{-1}A$ such that $j(a) = \frac{a}{1}$, which makes $S^{-1}A$ an A–algebra. Note that j is the zero map if and only if $0 \in S$. More precisely if $0 \in S$ then $S^{-1}A$ is the zero ring: this case will always be excluded in what follows. Moreover j is injective if and only if every element in S is not a zero divisor. In this case $j(A)$ will be identified with A.

Example 1.12.

1. Let A be an integral domain and set $S = A \setminus \{0\}$. Then $A_S = Q(A)$: the quotient field of A.

2. If $\mathcal{P} \subset A$ is a prime ideal, then $S = A \setminus \mathcal{P}$ is a multiplicative set and A_S is denoted $A_{\mathcal{P}}$ and called the localization of A at $\mathcal{P}$.

3. If $f \in A$, then the multiplicative set generated by f is

$$S = \{1, f, f^2, \dots, f^n, \dots\}:$$

A_S is denoted A_f.

4. If $S = \{x \in A \mid x \text{ is regular}\}$, then A_S is called the total ring of fractions of A: it is the maximum ring in which A can be canonically embedded.

It is easy to verify that the ring A_S enjoys the following *universal property*:

(i) if $s \in S$, then $j(s)$ is invertible;

(ii) if B is a ring with a given homomorphism $f : A \to B$ such that if $s \in S$, then $f(s)$ is invertible, then f factorizes through A_S, i.e. there exists a unique homomorphism $\overline{f}$ such that $\overline{f} \circ j = f$.

We will see now the relations between ideals of A_S and ideals of A.

If $\alpha \subset A$ is an ideal, then $\alpha A_S = \{\frac{a}{s} \mid a \in \alpha\}$ is called the *extension of* α in A_S and denoted also α^e. It is an ideal, precisely the ideal generated by the set $\{\frac{a}{1} \mid a \in \alpha\}$.

If $\beta \subset A_S$ is an ideal, then $j^{-1}(\beta) =: \beta^c$ is called the contraction of β and is clearly an ideal.

We have:

Proposition 1.13. *1.* $\forall \alpha \subset A : \alpha^{ec} \supset \alpha$*;*

2. $\forall \beta \subset A_S : \beta = \beta^{ce}$*;*

3. α^e *is proper if and only if* $\alpha \cap S = \emptyset$*;*

4. $\alpha^{ec} = \{x \in A \mid \exists s \in S \text{ such that } sx \in \alpha\}$.

Proof. 1. and 2. are straightforward.

3. if $1 = \frac{a}{s} \in \alpha^e$, then there exists $u \in S$ such that $u(s - a) = 0$, i.e. $us = ua \in S \cap \alpha$. Conversely, if $s \in S \cap \alpha$ then $1 = \frac{s}{s} \in \alpha^e$.

4.

$$\alpha^{ec} = \{x \in A \mid j(x) = \frac{x}{1} \in \alpha^e\} =$$
$$= \{x \in A \mid \exists a \in \alpha, t \in S \text{ such that } \frac{x}{1} = \frac{a}{t}\} =$$
$$= \{x \in A \mid \exists a \in \alpha, t, u \in S \text{ such that } u(xt - a) = 0\}.$$

Hence, if $x \in \alpha^{ec}$, then: $(ut)x = ua \in \alpha$. Conversely: if there exists $s \in S$ such that $sx = a \in \alpha$, then $\frac{x}{1} = \frac{a}{s}$, i.e. $j(x) \in \alpha^e$. □

If α is an ideal of A such that $\alpha = \alpha^{ec}$, α is called *saturated* with S. For example, if $\mathcal{P}$ is a prime ideal and $S \cap \mathcal{P} = \emptyset$, then $\mathcal{P}$ is saturated and $\mathcal{P}^e$ is prime. Conversely, if $\mathcal{Q} \subset A_S$ is a prime ideal, then $\mathcal{Q}^c$ is prime in A.

Therefore: *there is a bijection between the set of prime ideals of A_S and the set of prime ideals of A not intersecting S. In particular, if $S = A \setminus \mathcal{P}$, $\mathcal{P}$ prime, the prime ideals of $A_\mathcal{P}$ correspond bijectively to the prime ideals of A contained in $\mathcal{P}$, hence $A_\mathcal{P}$ is a local ring with maximal ideal $\mathcal{P}^e$, denoted $\mathcal{P}A_\mathcal{P}$, and residue field $A_\mathcal{P}/\mathcal{P}A_\mathcal{P}$.* Moreover $\dim A_\mathcal{P} = \mathrm{ht}\mathcal{P}$.

In particular we get the characterization of $\mathcal{O}_{P,X}$. Let $X \subset \mathbb{A}^n$ be an affine variety, let P be a point of X and $I(P) \subset K[x_1, \dots, x_n]$ be the ideal of P. Let $I_X(P) := I(P)/I(X)$ be the ideal of $K[X]$ formed by regular functions on X vanishing at P. Then we can construct the localization

$$\mathcal{O}(X)_{I_X(P)} = \{\frac{f}{g} | f, g \in \mathcal{O}(X), g(P) \neq 0\} \subset K(X):$$

it is canonically identified with $\mathcal{O}_{P,X}$. In particular: $\dim \mathcal{O}_{P,X} = \mathrm{ht}\ I_X(P) = \dim \mathcal{O}(X) = \dim X$.

There is a bijection between prime ideals of $\mathcal{O}_{P,X}$ and prime ideals of $\mathcal{O}(X)$ contained in $I_X(P)$; they also correspond to prime ideals of $K[x_1, \dots, x_n]$ contained in $I(P)$ and containing $I(X)$.

If X is affine, it is possible to define the local ring $\mathcal{O}_{P,X}$ also if X is reducible, simply as localization of $K[X]$ at the maximal ideal $I_X(P)$. The natural map j from $K[X]$ to $\mathcal{O}_{P,X}$ is injective if and only if $K[X] \setminus I_X(P)$ does not contain any zero divisor. A non-zero function f is a zero divisor in $K[X]$ if there exists a non-zero g such that $fg = 0$, i.e. $X = V(f) \cup V(g)$ is an expression of X as union of proper closed subsets. For j to be injective it is required that every zero divisor f belongs to $I_X(P)$, which means that all the irreducible components of X pass through P.

Exercises 1.14. 1. Prove that the affine varieties and the open subsets of affine varieties are quasi–projective.

2. Let $X = \{P, Q\}$ be the union of two points in an affine space over K. Prove that $\mathcal{O}(X)$ is isomorphic to $K \times K$.