

LESSON 8.

1. DIMENSION.

There are a few equivalent ways to give the definition of dimension for algebraic varieties. In this section we will first see a topological definition, then an algebraic characterization. In a later lesson, we will see a more geometrical interpretation.

Let X be a topological space.

Definition 1.1. The *topological dimension* of X is the supremum of the lengths of the chains of distinct irreducible closed subsets of X , where by definition the following chain has length n :

$$X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n.$$

The topological dimension of X is denoted by $\dim X$. It is also called combinatorial or Krull dimension.

Example 1.2. (1) $\dim \mathbb{A}^1 = 1$: the maximal length chains have the form $\{P\} \subset \mathbb{A}^1$.
(2) $\dim \mathbb{A}^n$: a chain of length n is

$$\{0\} = V(x_1, \dots, x_n) \subset V(x_1, \dots, x_{n-1}) \subset \cdots \subset V(x_1) \subset \mathbb{A}^n.$$

Note that $V(x_1, \dots, x_i)$ is irreducible for any $i \leq n$, because the ideal $\langle x_1, \dots, x_i \rangle$ is prime. Indeed $K[x_1, \dots, x_n]/\langle x_1, \dots, x_i \rangle \simeq K[x_{i+1}, \dots, x_n]$, which is an integral domain. Therefore we get that $\dim \mathbb{A}^n \geq n$. We will see shortly that proving equality is non trivial. We note also that, from every chain of irreducible closed subsets of \mathbb{A}^n , passing to their ideals, we get a chain of the same length of prime ideals in $K[x_1, \dots, x_n]$.

(3) Let X be irreducible. Then $\dim X = 0$ if and only if X is the closure of every point of it.

We prove now some useful relations between the dimensions of X and of its subspaces.

Proposition 1.3. 1. If $Y \subset X$, then $\dim Y \leq \dim X$. In particular, if $\dim X$ is finite, then also $\dim Y$ is finite. In this case, the number $\dim X - \dim Y$ is called the **codimension** of Y in X .

2. If $X = \bigcup_{i \in I} U_i$ is an open covering, then $\dim X = \sup_i \{\dim U_i\}$.

3. If X is noetherian and X_1, \dots, X_s are its irreducible components, then $\dim X = \sup_i \dim X_i$.

4. If $Y \subset X$ is closed, X is irreducible, $\dim X$ is finite and $\dim X = \dim Y$, then $Y = X$.

Proof. 1. Let $Y_0 \subset Y_1 \subset \dots \subset Y_n$ be a chain of irreducible closed subsets of Y . Then taking closures we get the following chain of irreducible closed subsets of X : $\overline{Y_0} \subseteq \overline{Y_1} \subseteq \dots \subseteq \overline{Y_n}$. Note that, for any index i , $\overline{Y_i} \cap Y = Y_i$, because Y_i is closed into Y , so if $\overline{Y_i} = \overline{Y_{i+1}}$, then $Y_i = Y_{i+1}$. Therefore the two chains have the same length and we can conclude that $\dim Y \leq \dim X$.

2. Let $X_0 \subset X_1 \subset \dots \subset X_n$ be a chain of irreducible closed subsets of X . Let $P \in X_0$ be a point: there exists an index $i \in I$ such that $P \in U_i$. So $\forall k = 0, \dots, n$ $X_k \cap U_i \neq \emptyset$: it is an irreducible closed subset of U_i , irreducible because open in X_k which is irreducible. Consider

$$X_0 \cap U_i \subset X_1 \cap U_i \subset \dots \subset X_n \cap U_i;$$

it is a chain of length n , because $\overline{X_k \cap U_i} = X_k$: in fact $X_k \cap U_i$ is open in X_k hence dense. Therefore, for any chain of irreducible closed subsets of X , there exists a chain of the same length of irreducible closed subsets of some U_i . So $\dim X \leq \sup \dim U_i$. By 1., equality holds.

3. Any chain of irreducible closed subsets of X is completely contained in an irreducible component of X . The conclusion follows as in 2.

4. If $Y_0 \subset Y_1 \subset \dots \subset Y_n$ is a chain of maximal length in Y , then it is a maximal chain in X , because $\dim X = \dim Y$. Hence $X = Y_n \subset Y$. \square

Corollary 1.4. $\dim \mathbb{P}^n = \dim \mathbb{A}^n$.

Proof. The equality follows from $\mathbb{P}^n = U_0 \cup \dots \cup U_n$, and the homeomorphism of U_i with \mathbb{A}^n for all i . \square

If X is noetherian and all its irreducible components have the same dimension r , then X is said to have *pure dimension* r . Note that the topological dimension is invariant by homeomorphism. By definition, a *curve* is an algebraic set of pure dimension 1; a *surface* is an algebraic set of pure dimension 2.

We want to study the dimensions of affine algebraic sets. The following definition results to be very important.

Definition 1.5. Let $X \subset \mathbb{A}^n$ be an algebraic set. The *coordinate ring* of X is

$$K[X] := K[x_1, \dots, x_n]/I(X).$$

It is a finitely generated reduced K -algebra, i.e. there are no non-zero nilpotents, because $I(X)$ is radical.

There is the canonical epimorphism $K[x_1, \dots, x_n] \rightarrow K[X]$ such that $F \rightarrow [F]$. The elements of $K[X]$ can be interpreted as *polynomial functions* on X : to a polynomial F , we can associate the function $f : X \rightarrow K$ such that $P(a_1, \dots, a_n) \rightarrow F(a_1, \dots, a_n)$.

Two polynomials F, G define the same function on X if, and only if, $F(P) = G(P)$ for every point $P \in X$, i.e. if $F - G \in I(X)$, which means exactly that F and G have the same image in $K[X]$.

$K[X]$ is generated as K -algebra by $[x_1], \dots, [x_n]$: they can be interpreted as *coordinate functions* on X . We will denote them by t_1, \dots, t_n . In fact $t_i : X \rightarrow K$ is the function which associates to $P(a_1, \dots, a_n)$ the coordinate a_i . Note that the function f can be interpreted as $F(t_1, \dots, t_n)$: the polynomial F evaluated at the n -tuple of the coordinate functions.

In the projective space we can do an analogous construction. If $Y \subset \mathbb{P}^n$ is closed, then the *homogeneous coordinate ring* of Y is

$$S(Y) := K[x_0, x_1, \dots, x_n]/I_h(Y).$$

Also $S(Y)$ is a finitely generated reduced K -algebra, but its elements cannot be interpreted as functions on Y . They are functions on the cone $C(Y)$.

We note that, from the fact that $I_h(Y)$ is homogeneous it follows that also $S(Y)$ is a graded ring, with the graduation induced by the polynomial ring. Indeed, if $F - G \in I_h(Y)$, and $F = F_0 + \dots + F_d, G = G_0 + \dots + G_e$ are their decompositions in homogeneous components then it follows that $F_0 - G_0 \in I_h(Y), F_1 - G_1 \in I_h(Y)$, and so on.

Definition 1.6. Let R be a ring. The *Krull dimension* of R is the supremum of the lengths of the chains of prime ideals of R

$$\mathcal{P}_0 \subset \mathcal{P}_1 \subset \dots \subset \mathcal{P}_r.$$

Similarly, the *height* of a prime ideal \mathcal{P} is the sup of the lengths of the chains of prime ideals contained in \mathcal{P} : it is denoted $\text{ht}\mathcal{P}$.

Proposition 1.7. Let K be an algebraically closed field. Let X be an affine algebraic set contained in \mathbb{A}^n . Then $\dim X = \dim K[X]$. In particular $\dim \mathbb{A}^n = \dim K[x_1, \dots, x_n]$.

Proof. By the Nullstellensatz and by Proposition 1.6 of Lesson 7 the chains of irreducible closed subsets of X correspond bijectively to the chains of prime ideals of $K[x_1, \dots, x_n]$ containing $I(X)$, and therefore also to the chains of prime ideals of the quotient ring $K[X] = K[x_1, \dots, x_n]/I(X)$. \square

The dimension theory for commutative rings contains some important theorems about the dimension of K -algebras. The following theorem states the basic properties in the case of integral domains and the algebraic characterization of dimension for affine varieties.

Theorem 1.8. *Let K be any field. Let A be a finitely generated K -algebra and an integral domain.*

1. $\dim A = \text{tr.d.}Q(A)/K$, where $Q(A)$ is the quotient field of A . In particular $\dim A$ is finite.
2. Let $\mathcal{P} \subset A$ be any prime ideal. Then $\dim A = \text{ht}\mathcal{P} + \dim A/\mathcal{P}$.

Proof. We postpone the proof to next lesson. It relies on the Normalization Lemma and on the Cohen-Seidenberg theorems about the structure of prime ideals for integral extensions of K -algebras. \square

Corollary 1.9. *Let K be an algebraically closed field.*

1. $\dim \mathbb{A}^n = \dim \mathbb{P}^n = n$.
2. If X is an irreducible affine variety, then $\dim X = \text{tr.d.}K(X)/K$, where $K(X)$ denotes the quotient field of $K[X]$.
3. If $X \subset \mathbb{A}^n$ is an irreducible affine variety, then $\dim X = n - \text{ht}I(X)$.

Proof. 1. $\dim K[x_1, \dots, x_n] = \text{tr.d.}K(x_1, \dots, x_n)/K = n$.

2. follows immediately from Theorem 1.8, 1.

3. is Theorem 1.8, 2, applied to the case $A = K[x_1, \dots, x_n]$ and $\mathcal{P} = I(X)$. \square

Note that the homogeneous coordinate ring of \mathbb{P}^n is $K[x_0, \dots, x_n]$, whose dimension is $n + 1$, strictly bigger than the dimension of \mathbb{P}^n . Similarly, if Y is a projective algebraic variety, then $\dim S(Y) = \dim C(Y)$, the affine cone over Y .

Corollary 1.9 tells us how to compute the dimension of an affine irreducible variety. If X is a reducible affine variety, and $X = X_1 \cup \dots \cup X_r$ is its decomposition as union of irreducible components, then $\dim X$ is the maximum of the dimensions $\dim X_i$.

The following is the characterization of the algebraic varieties of codimension 1 in \mathbb{A}^n .

Proposition 1.10. *Let $X \subset \mathbb{A}^n$ be an affine variety over an algebraically closed field. Then X is a hypersurface if and only if X is of pure dimension $n - 1$.*

Proof. Let $X \subset \mathbb{A}^n$ be a hypersurface, with $I(X) = (F) = (F_1 \dots F_s)$, where F_1, \dots, F_s are the irreducible factors of F all of multiplicity one. Then $V(F_1), \dots, V(F_s)$ are the irreducible components of X , whose ideals are $(F_1), \dots, (F_s)$. So it is enough to prove that $\text{ht}(F_i) = 1$, for $i = 1, \dots, s$.

If $\mathcal{P} \subset (F_i)$ is a prime ideal, then either $\mathcal{P} = (0)$ or there exists $G \in \mathcal{P}$, $G \neq 0$. In the second case, let A be an irreducible factor of G belonging to \mathcal{P} : $A \in (F_i)$ so $A = HF_i$. Since A is irreducible, either H or F_i is invertible; F_i is irreducible, so H is invertible, hence $(A) = (F_i) \subset \mathcal{P}$. Therefore either $\mathcal{P} = (0)$ or $\mathcal{P} = (F_i)$, and $\text{ht}(F_i) = 1$.

Conversely, assume that X is irreducible of dimension $n - 1$. Since $X \neq \mathbb{A}^n$, there exists $F \in I(X)$, $F \neq 0$, with irreducible factorization $F = F_1 \dots F_s$. Hence $X \subset V(F) = V(F_1) \cup \dots \cup V(F_s)$. By the irreducibility of X , some irreducible factor of F , call it F_i , also vanishes along X . Therefore $X \subset V(F_i)$, which is irreducible of dimension $n - 1$, by the first part. So $X = V(F_i)$ (by Proposition 1.3, 4). \square

This proposition does not generalise to higher codimension. There exist codimension 2 algebraic subsets of \mathbb{A}^n whose ideal is not generated by two polynomials. An example in \mathbb{A}^3 is the curve X parametrised by (t^3, t^4, t^5) . It is possible to show that a system of generators of $I(X)$ is formed by the three polynomials $x^3 - yz, y^2 - xz, z^2 - x^2y$. One can easily show that $I(X)$ cannot be generated by two polynomials. For a proof and a discussion of this example, and more generally of the ideals of the curves admitting a parametrization of the form $x = t^{n_1}, y = t^{n_2}, z = t^{n_3}$, see [Kunz], Chapter V.

Proposition 1.11. *Let $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ be irreducible closed subsets. Then $\dim X \times Y = \dim X + \dim Y$.*

Proof. Let $r = \dim X$, $s = \dim Y$; let t_1, \dots, t_n (resp. u_1, \dots, u_m) be coordinate functions on \mathbb{A}^n (resp. \mathbb{A}^m). We can assume that t_1, \dots, t_r is a transcendence basis of $Q(K[X])$ and u_1, \dots, u_s a transcendence basis of $Q(K[Y])$. By definition, $K[X \times Y]$ is generated as K -algebra by $t_1, \dots, t_n, u_1, \dots, u_m$: we want to show that $t_1, \dots, t_r, u_1, \dots, u_s$ is a transcendence basis of $Q(K[X \times Y])$ over K . Assume that $F(x_1, \dots, x_r, y_1, \dots, y_s)$ is a polynomial which vanishes on $t_1, \dots, t_r, u_1, \dots, u_s$, i.e. F defines the zero function on $X \times Y$. Then, $\forall P \in X$, $F(P; y_1, \dots, y_s)$ is zero on Y , i.e. $F(P; u_1, \dots, u_s) = 0$. Since u_1, \dots, u_s are algebraically independent, every coefficient $a_i(P)$ of $F(P; y_1, \dots, y_s)$ is zero, $\forall P \in X$. Since t_1, \dots, t_r are algebraically independent, the polynomials $a_i(x_1, \dots, x_r)$ are zero, so $F(x_1, \dots, x_r, y_1, \dots, y_s) = 0$. So $t_1, \dots, t_r, u_1, \dots, u_s$ are algebraically independent. Since this is certainly a maximal algebraically free set, it is a transcendence basis. \square

Exercises 1.12. 1*. Prove that a proper closed subset of an irreducible curve is a finite set. Deduce that any bijection between irreducible curves is a homeomorphism.

2*. Let $X \subset \mathbb{A}^2$ be the cuspidal cubic of equation: $x^3 - y^2 = 0$, let $K[X]$ be its coordinate ring. Prove that all elements of $K[X]$ can be written in a unique way in the form $f(x) + yg(x)$, where f, g are polynomial in the variable x . Deduce that $K[X]$ is not isomorphic to a polynomial ring.