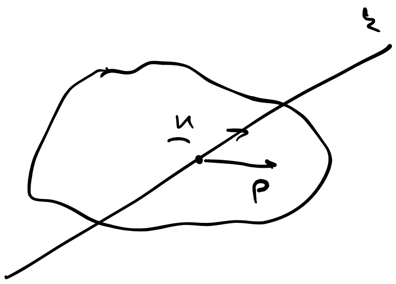
MECCANICA RAZIONALE

- LEZIONE DEL 3 APRILE 2020
- PRIMA PARTE

TRASFORMAZIONE DI INERZIA



momento di inerzia

$$I_r = \sum_{P \in R} m_P \, \| \underline{u} \wedge (\underline{x}_P - \underline{x}_O) \|^2$$

$$\sim \sum \text{massa} \, (\text{distanza})^2$$

$$I_O(\underline{y}) = \underbrace{\sum_{P \in R} m_P \, (\underline{x}_P - \underline{x}_O) \wedge \left[\underline{y} \wedge (\underline{x}_P - \underline{x}_O) \right]}_{\in \mathbb{R}^3}$$

$\underline{y} \in \mathbb{R}^3$

$$I_r = \underline{u} \cdot I_O(\underline{u}) = \underline{u} \, I_O \cdot u$$

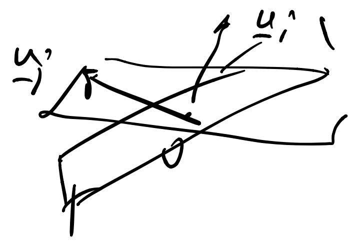
Trasformazione lineare e simmetrica

$$\to \quad I_O = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{12} & I_{22} & I_{23} \\ I_{13} & I_{23} & I_{33} \end{pmatrix}$$

$O \quad \underline{u}_1, \underline{u}_2, \underline{u}_3 \qquad I_{ij} = \underline{u}_i \cdot I_O(\underline{u}_j)$

I_{11}, I_{22}, I_{33} momenti di inerzia rispetto a $\underline{u}_1, \underline{u}_2, \underline{u}_3$

I_{12}, I_{13}, I_{23} momenti deviatori (associati a 2 piani π_0, π_0')

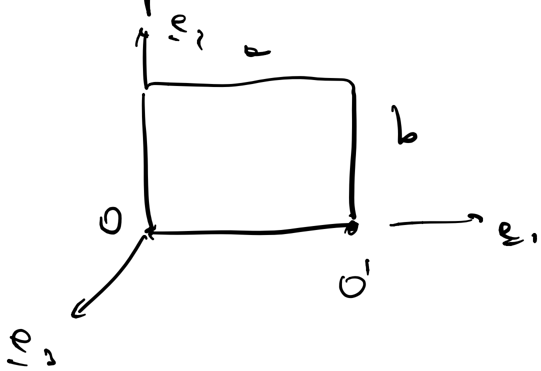


Rigido piano:
$$\begin{pmatrix} I_{11} & I_{12} & 0 \\ I_{12} & I_{22} & 0 \\ 0 & 0 & I_{11}+I_{22} \end{pmatrix}$$

I_O diagonale: $(\underline{u}_1, \underline{u}_2, \underline{u}_3)$
$$\begin{pmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{pmatrix} \rightarrow$$ Assi Principali di Inerzia

→ Tecniche di calcolo

→ esempi di I_O

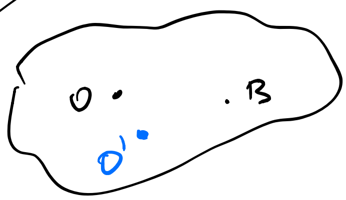


$$I_O = \begin{pmatrix} \frac{Mb^2}{3} & -\frac{Mab}{4} \\ -\frac{Mab}{4} & \frac{Ma^2}{3} \end{pmatrix}$$

Variazione di I_O con il polo O

Corpo rigido, $O, O' \in R$, I_O, $I_{O'}$

$$I_O(\underline{u}) = \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_O) \wedge [\underline{u} \wedge (\underline{x}_B - \underline{x}_O)]$$



$$(\underline{x}_B - \underline{x}_O) = (\underline{x}_B - \underline{x}_O) + \underline{x}_{O'} - \underline{x}_{O'}$$

$$= (\underline{x}_B - \underline{x}_{O'}) + \underbrace{(\underline{x}_{O'} - \underline{x}_O)}_{\text{non dipende da } B}$$

$$= \sum_{B \in R}' m_B [(\underline{x}_B - \underline{x}_{O'}) + (\underline{x}_{O'} - \underline{x}_O)] \wedge [\underline{u} \wedge$$

$$\wedge [(\underline{x}_B - \underline{x}_{O'}) + (\underline{x}_{O'} - \underline{x}_O)]]$$

$$= \sum_{B \in R}' m_B (\underline{x}_B - \underline{x}_{O'}) \wedge [\underline{u} \wedge (\underline{x}_B - \underline{x}_{O'})] +$$

$$+ \left(\sum_{B \in R} m_B\right) (\underline{x}_{O'} - \underline{x}_O) \wedge [\underline{u} \wedge (\underline{x}_{O'} - \underline{x}_O)]$$

$$+ \sum_{B \in R} m_B (\underline{x}_{O'} - \underline{x}_O) \wedge [\underline{u} \wedge (\underline{x}_B - \underline{x}_{O'})]$$

$$+ \sum_{B \in R} m_B (\underline{x}_B - \underline{x}_{O'}) \wedge [\underline{u} \wedge (\underline{x}_{O'} - \underline{x}_O)]$$

Abbastanza complicata : semplificazione

se scegliamo $O' = G$ centro di massa

$$\sum_{B \in R} m_B (\underline{x}_B - \underline{x}_{O'}) \xrightarrow{O' = G} \sum_{B \in R} m_B (\underline{x}_B - \underline{x}_G)$$

$$= \underbrace{\sum_{B \in R}' m_B \underline{x}_B}_{} - \underbrace{\left(\sum_{B \in R} m_B \right)}_{M} \underline{x}_G = M \underline{x}_G - M \underline{x}_G = 0$$

$$\boxed{\underline{x}_G := \frac{\sum_{B \in R} m_B \underline{x}_B}{M}}$$

$$\sum_{B \in R} m_B \underline{x}_B = M \underline{x}_G$$

Formula di Trasporto della matrice di inerzia

$$I_O(\underline{u}) = I_G(\underline{u}) + M (\underline{x}_G - \underline{x}_O) \wedge [\underline{u} \wedge (\underline{x}_G - \underline{x}_O)]$$

Teorema di Huygens - Steiner

Siano r_O e r_G due rette parallele per O e per G (centro di massa)

$$I_{r_O} = I_{r_G} + M \| \underline{u} \wedge (\underline{x}_G - \underline{x}_O) \|^2$$

$$= I_{r_G} + M (\text{distanza tra } r_O \text{ e } r_G)^2$$

$I_{z_0} > I_{z_G}$

MECCANICA RAZIONALE

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– SECONDA PARTE

$$I_{z_0} = I_{z_G} + M \, \| \underline{u} \wedge (\underline{x}_G - \underline{x}_0) \|^2$$

$$\underline{I}_0(\underline{u}) = \underline{I}_G(\underline{u}) + M \, (\underline{x}_G - \underline{x}_0) \wedge \left[\underline{u} \wedge (\underline{x}_G - \underline{x}_0) \right]$$

$$I_{z_0} = \underline{u} \cdot \underline{I}_0(\underline{u}) \qquad I_{z_G} = \underline{u} \cdot \underline{I}_G(\underline{u})$$

moltiplichiamo per $\underline{u}$

$$\underline{u} \cdot \underline{I}_0(\underline{u}) = \underline{u} \cdot \underline{I}_G(\underline{u}) + M \, \underline{u} \cdot \left\{ (\underline{x}_G - \underline{x}_0) \wedge \left[\underline{u} \wedge (\underline{x}_G - \underline{x}_0) \right] \right\}$$

$$\underline{u} \cdot \left\{ (\underline{x}_G - \underline{x}_0) \wedge \left[\underline{u} \wedge (\underline{x}_G - \underline{x}_0) \right] \right\}$$

$$\underline{a} \cdot \left(\underline{b} \wedge (\underline{a} \wedge \underline{b}) \right)$$

$$\left[\ \underline{a} \wedge (\underline{b} \wedge \underline{c}) = \underline{b}\,(\underline{c}\cdot\underline{a}) - \underline{c}\,(\underline{a}\cdot\underline{b})\ \right]$$

$$\underline{a}\cdot\Big(\underline{b}\wedge(\underline{a}\wedge\underline{b})\Big) = \underline{a}\cdot\Big[\underline{a}\,(\underline{b}\cdot\underline{b}) - \underline{b}\,(\underline{a}\cdot\underline{b})\Big]$$

$$= (\underline{a}\cdot\underline{a})(\underline{b}\cdot\underline{b}) - (\underline{a}\cdot\underline{b})(\underline{a}\cdot\underline{b})$$

$$= (\underline{a}\cdot\underline{a})(\underline{b}\cdot\underline{b}) - (\underline{a}\cdot\underline{b})^2$$

Identità : $(\underline{a}\cdot\underline{a})(\underline{b}\cdot\underline{b}) - (\underline{a}\cdot\underline{b})^2 \overset{?}{=} (\underline{a}\wedge\underline{b})\cdot(\underline{a}\wedge\underline{b})$

Identità generale.

Usiamo due identità:

1) $\underline{a}\wedge(\underline{b}\wedge\underline{c}) = \underline{b}\,(\underline{a}\cdot\underline{c}) - \underline{c}\,(\underline{a}\cdot\underline{b})$

2) $\underline{a}\cdot(\underline{b}\wedge\underline{c}) = \underline{b}\cdot(\underline{c}\wedge\underline{a}) = \underline{c}\cdot(\underline{a}\wedge\underline{b})$

Segue:

$$\underline{(\underline{a}\wedge\underline{b})\cdot(\underline{c}\wedge\underline{d})} \overset{2)}{=} \underline{c}\cdot\Big(\underline{d}\wedge(\underline{a}\wedge\underline{b})\Big)$$

$$\overset{1)}{=} \underline{c}\cdot\Big(\underline{a}\,(\underline{d}\cdot\underline{b}) - \underline{b}\,(\underline{d}\cdot\underline{a})\Big)$$

$$= \underline{(\underline{c}\cdot\underline{a})(\underline{d}\cdot\underline{b}) - (\underline{c}\cdot\underline{b})(\underline{d}\cdot\underline{a})}$$

Caso particolare

$$\underline{(\underline{a}\wedge\underline{b})}\cdot\underline{(\underline{a}\wedge\underline{b})} = (\underline{a}\cdot\underline{a})(\underline{b}\cdot\underline{b}) - (\underline{a}\cdot\underline{b})^2$$

$$M\ \underline{u}\cdot(\underline{x}_G-\underline{x}_0)\wedge\left[\underline{u}\wedge(\underline{x}_G-\underline{x}_0)\right]$$

$$= M\left(\underline{u}\wedge(\underline{x}_G-\underline{x}_0)\right)\cdot\left(\underline{u}\wedge(\underline{x}_G-\underline{x}_0)\right)$$

$$= M\ \|\underline{u}\wedge(\underline{x}_G-\underline{x}_0)\|^2$$

Mettendo tutto insieme

$$\underline{u}\cdot I_0(\underline{u}) = \underline{u}\cdot I_G(\underline{u}) + M\ \underline{u}\cdot(\underline{x}_G-\underline{x}_0)\wedge$$
$$\wedge\left[\underline{u}\wedge(\underline{x}_G-\underline{x}_0)\right]$$

$$= \underline{u}\cdot I_G(\underline{u}) + M\ \|\underline{u}\wedge(\underline{x}_G-\underline{x}_0)\|^2$$

$$\Rightarrow I_{r_0} = I_{r_G} + M\ \|\underline{u}\wedge(\underline{x}_G-\underline{x}_0)\|^2$$

c.v.d.

Corollario : Trasporto dei momenti deviatori

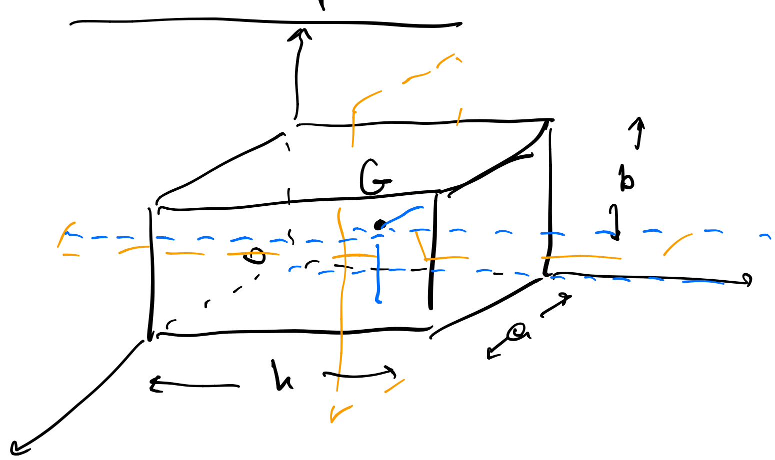
π_O , π_G 2 piani ortogonali a un vettore $\underline{u}$ e passanti per O e G

π'_O , π'_G 2 piani ortogonali a un vettore $\underline{v}$ e passanti per O e G

$$I_{\pi_O \pi'_O} = I_{\pi_G \pi'_G} + M \left[\underline{u} \cdot (\underline{x}_G - \underline{x}_O) \right] \cdot \left[\underline{v} \cdot (\underline{x}_G - \underline{x}_O) \right]$$

Esercizio determinare A.P.I. e momenti principali rispetto al centro di massa (corpi omogenei)

Parallelepipedo



I piani per G e paralleli alle facce sono piani di simmetria materiale ortogonale

c) le tre rette per G e parallele agli spigoli sono A.P.I.

$$I_{G,11} = ?$$

$$I_{O,11} = M \frac{a^2+b^2}{3}$$

Usiamo H-S :

$$I_{G,11} = M \frac{a^2+b^2}{3} - M\left(\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right)$$

$$I_{z_O} = \underline{I_{z_G}} + \underline{M(\text{distanza})^2}$$

$$= \frac{M}{12}(a^2+b^2)$$

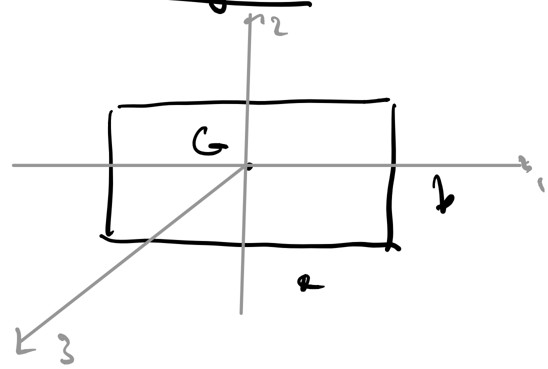
$$I_{G,11} = \frac{M}{12}(a^2+b^2), \quad I_{G,22} = \frac{M}{12}(a^2+h^2)$$

$$I_{G,33} = \frac{M}{12}(b^2+h^2)$$

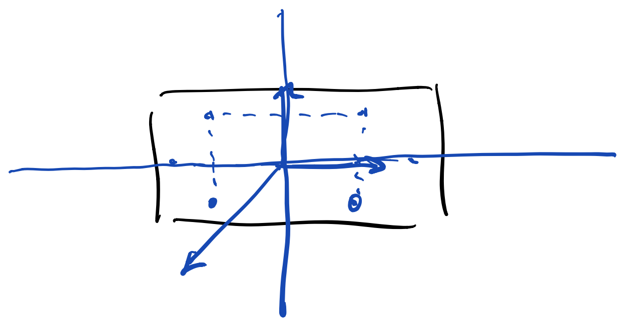
Caso particolare: per un cubo

$$I_G = \begin{pmatrix} \frac{Ml^2}{6} & 0 & 0 \\ 0 & \frac{Ml^2}{6} & 0 \\ 0 & 0 & \frac{Ml^2}{6} \end{pmatrix}$$

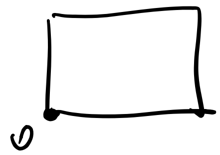
Rettangolo



È API per simmetria



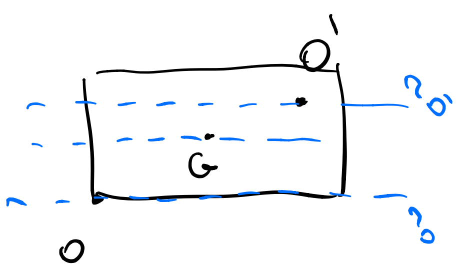
I_G ?



$$I_{G,11} = \underbrace{\left(\frac{M b^2}{3}\right)}_{I_{O,11}} - M\left(\frac{b}{2}\right)^2 = M\,\frac{b^2}{12}$$

$$I_{G,22} = M\frac{a^2}{3} - M\left(\frac{a}{2}\right)^2 = M\,\frac{a^2}{12}$$

$$I_{G,33} = I_{G,11} + I_{G,22} = \frac{M}{12}\left(a^2 + b^2\right)$$



$$I_{z_0} = I_{z_G} + M\,(\text{distanza})^2$$

$$I_{z_{0'}} = I_{z_G} + M\,(\text{distanza})^2$$

$I_O \longleftrightarrow I_G$ se A.P.I.